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AUXILIARY FIELDS AND A SUPERSPACE LAGRANGIAN
FOR LINEARIZED TEN - DIMENSIONAL SUPERGRAVITY

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A B S T R A C T

A set of auxiliary fields for linearized ten-dimensional supergravity is presented. The total number of off-shell field components, including those of the physical fields, is $65,664 + 65,664$. This set of fields has a simple description in terms of two superfields. A superspace Lagrangian which leads to the correct equations of motion is given.

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The problem of finding auxiliary fields for extended supergravities with $N \geq 3$ has turned out to be more difficult than was originally thought. The auxiliary fields for $N = 4$ conformal supergravity have been found via the $N = 4$ supercurrent constructed from $N = 4$ Maxwell theory ¹⁾, but the problem of finding compensating multiplets has proved to be rather severe [see, e.g., Ref. 2)]. By studying the supercurrent in higher dimensions one makes some progress, since Maxwell theory is only conformally invariant in four dimensions. The five-dimensional $N = 4$ supercurrent has been constructed ³⁾ and the corresponding supergravity multiplet is a Poincaré multiplet in five dimensions, even though it has the same number of components as the four-dimensional conformal supercurrent. It has been shown that this five-dimensional multiplet can be combined with five off-shell Maxwell multiplets ⁴⁾ to yield a complete set of off-shell fields for 5-d $N = 4$ Poincaré supergravity ⁵⁾ (at least at the linearized level), but the theory involves an off-shell central charge coming from the Maxwell sector. The presence of a central charge rules out the possibility of expressing the theory in terms of unconstrained superfields and is therefore not a desirable feature from the point of view of extended supergraphs.

In this letter we attack the auxiliary field problem in ten dimensions and find a solution which has no off-shell central charge and no auxiliary gauge fields, another feature which could cause problems at the non-linear level. We recall that the physical fields of $d = 10$ supergravity are the graviton h_{ab} , the gravitino ψ_a , a scalar A , a spinor λ and a two-form gauge field A_{ab} which is equivalent on-shell to a six-form gauge field A_{abcdef} . $d = 10$ supergravity therefore reduces to $N = 4$ supergravity together with six $N = 4$ Maxwell multiplets, so that we expect to be able to find off-shell $N = 4$ 4-d Poincaré supergravity from our results.

Our starting point is again the supercurrent. The Maxwell supercurrent has been studied previously ^{6),7)} and has been shown to have a rich structure. Specifically it consists of two irreducible submultiplets - a conformal multiplet with $128 + 128$ components and a constrained scalar superfield with $5632 + 5632$ components ⁷⁾. The currents of the conformal multiplet are the traceless conserved energy-momentum tensor $\hat{T}_{ab}(44)$, the Γ traceless conserved spinor current $\hat{\Sigma}_a(128)$ and a four-form current $\hat{J}_{abcd}(84)$ whose curl vanishes. $\hat{T}_{ab}$ and $\hat{\Sigma}_a$ involve non-local projection operators, so that the division of the supercurrent into its irreducible parts is also non-local. The difficulty with this supercurrent from the point of view of auxiliary fields lies in the scalar superfield sector, however. This is because this superfield contains many more conserved

quantities which transform under complicated representations of $O(10)$. The contragredient supergravity multiplet will therefore contain exotic auxiliary gauge fields which would undoubtedly lead to difficulties at the non-linear level. The origin of this problem is the fact that these exotic conserved currents correspond to invariances of a free-field theory, i.e., they are Zilches. Hence we do not expect them to be conserved in the supercurrent constructed from non-Abelian Yang-Mills, and this is the version we shall use here.

On-shell non-Abelian Yang-Mills is described by a scalar superfield χ whose independent components are a spinor, again denoted χ , and the Yang-Mills field strength tensor F_{ab} . The equations governing the system are *)

$$\begin{aligned} F_{\alpha\beta} &= 0 \\ F_{\alpha a} &= \frac{1}{2}(\Gamma_a)_{\alpha}{}^{\beta} \chi_{\beta} \\ \partial_{\alpha} \chi_{\beta} &= \frac{1}{4}(\Gamma_{ab})_{\alpha\beta} F_{ab} \\ \partial_{\alpha} F_{ab} &= -(\Gamma_{[a} \partial_{b]} \chi)_{\alpha} \\ \Gamma^a \partial_a \chi &= 0 \\ \partial^a F_{ab} &= i \bar{\chi} \Gamma_b \chi \end{aligned} \tag{1}$$

where F_{AB} is the superspace field-strength tensor ($F_{AB} = F_{AB}^r X_r$; $[X_r, X_s] = i f_{rst} X_t$, where X_r are the generators of the Yang-Mills group and f_{rst} its structure constants). $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_{\alpha})$ is the covariant derivative with respect to Yang-Mills and rigid supersymmetry. The supercurrent superfield, X_{abc} , is bilinear in the superfield χ and has the same form as in the Maxwell case 6)-8), except that there is now a trace over the Yang-Mills indices :

$$X_{abc} = \text{Tr } \bar{\chi} \Gamma_{abc} \chi \tag{2}$$

Using (1), one can show that the supercurrent again consists of the conformal multiplet $(\hat{T}_{ab}, \hat{\Sigma}_a, \hat{J}_{abcd})$ and a scalar superfield :

$$X_{abc} = -\frac{1}{18} \bar{\mathcal{D}} \Gamma_{abc} \mathcal{D} \cdot \frac{T}{\square} + 16 \frac{\partial^d}{\square} \tilde{J}_{abcd} \tag{3}$$

*) An equation such as $\Gamma^a \mathcal{D}_a \chi = 0$ can be interpreted either as a superfield equation or as a component equation if one just considers its $\theta = 0$ term.

where

$$T \equiv - 6 \operatorname{Tr} F_{ab} F^{ab}$$

$$\tilde{J}_{abcd} = \operatorname{Tr} F_{[ab} F_{cd]} + \frac{1}{6} \partial_{[a} X_{bcd]} \quad (4)$$

For our supercurrent, however, the superfield T is unconstrained. For example, in the θ^4 sector of T (θ^2 sector of X_{abc}) there is a current in the 1050 representation of $SO(10)$ which has the explicit form ^{*)}

$$X_{abcde}^2 = \operatorname{Tr} \left\{ \bar{\chi} \Gamma_{abcde} \mathcal{D}_{\frac{f}{f}} \chi - \bar{\chi} \Gamma_{[abcde} \mathcal{D}_{\frac{f}{f}} \chi + \right. \\ \left. + \frac{10}{3} \delta_{\frac{f}{f}[a} \bar{\chi} \Gamma_{bcd} \mathcal{D}_{e]} \chi \right\} \quad (5)$$

and is only conserved in the free case, $\mathcal{D}_f \rightarrow \partial_f$. Furthermore, the presence of interactions implies that the higher components of the supercurrent are in general multilinear in the Yang-Mills fields, and hence those components of the scalar superfield which can be seen to vanish by inspection in the Maxwell case do not do so here. An alternative way to describe the supercurrent is to specify the constraints it satisfies ; there are two independent ones :

$$D_{\alpha} X_{abc} = (\Gamma_{[a} X'_{bc])_{\alpha} + (\Gamma_{ab} X'_{c])_{\alpha} + (\Gamma_{abc} X')_{\alpha}$$

$$\bar{D} \Gamma_{cd[a} D X_{b]cd} = 28 \partial^c X_{abc} \quad (6)$$

Equations (3) and (6) are completely equivalent, so that one may define the supercurrent abstractly as a superfield X_{abc} satisfying Eq. (6). We now list the first few components of X_{abc} (fields with odd superscripts are fermions and those with even superscripts bosons) :

$$X_{abc} = \left\{ X_{abc}^0 ; X'_{ab}, X'_a, X'; X^2, X_a^2, X_{ab}^2, X_{cd}^2, X_{cd}^2, X_{cd}^2, \right. \\ \left. X_{abcde}^2 ; X_{ab}^3, X_{abcde}^3 ; \dots \right\} \quad (7)$$

^{*)} Numerical superscripts on fields indicate the θ sector of the basic superfield in which they occur. We use a version of the Torino notation ⁹⁾ in which a field in an irreducible representation of $SO(10)$ has its indices arranged in the same pattern as the corresponding Young tableau rotated clockwise through 90° .

Here, all components are in irreducible representations of $SO(10)$ so that the spinor current and the energy momentum tensor are given by

$$\begin{aligned}\Sigma_a &= \frac{1}{3}(X'_a + \frac{27}{2}\Gamma_a X^1) \\ T_{ab} &= -\frac{8}{3}(X^2_{ab} + \frac{27}{2}\delta_{ab}X^2)\end{aligned}\quad (8)$$

The dual of X^2_{abcd} ($\propto \tilde{J}_{abcd}$) is a six-index conserved current. (It is not a Zilch, however, but can be thought of as a topological current.) The transformation laws for the components can be read off either from (6) or from the explicit form of X_{abc} (3). One finds

$$\begin{aligned}\delta X^0_{abc} &= \bar{\epsilon}\Gamma_{[a}X'_{bc]} + \bar{\epsilon}\Gamma_{[ab}X'_{c]} + \bar{\epsilon}\Gamma_{abc}X^1 \\ \delta X^1_{ab} &= \Gamma_{cd}\epsilon \cdot X^2_{ab} - \Gamma_{cdefg}\Gamma_a\epsilon X^2_{cdefg} + \\ &\quad + \frac{1}{32}(\Gamma_{acde} - 9\delta_{ac}\Gamma_{de})\epsilon(\partial X^0)_{cde} + \\ &\quad + (\Gamma_{abcdef} + 10\delta_{ac}\Gamma_{bdef} - 30\delta_{cd}^{ab}\Gamma_{ef})\epsilon\left\{\frac{1}{192}(\partial X^0)_{cdef} - \frac{1}{18}X^2_{cdef}\right\} \\ &\quad - \frac{1}{96}(\Gamma_{abcd} + 14\delta_{ac}\Gamma_{bd} - 56\delta_{cd}^{ab})\epsilon \cdot (\partial X^0)_{cd} \text{ with } [ab] \\ \delta X^1_a &= -\frac{1}{32}\Gamma_{bcd}\epsilon(\partial X^0)_{bcd} - \frac{1}{32}(-2\delta_{ab}\Gamma_c + \frac{1}{5}\Gamma_a\Gamma_{bc})\epsilon(\partial X^0)_{bc} \\ &\quad + \Gamma_b\epsilon X^2_a + (\Gamma_{bcd}\delta_{ea} + \frac{1}{10}\Gamma_a\Gamma_{bcde})\epsilon\left\{\frac{1}{32}(\partial X^0)_{bcde} - X^2_{bcde}\right\} \\ \delta X^1 &= \Gamma_{abcd}\epsilon\left\{\frac{1}{480}(\partial X^0)_{abcd} - \frac{1}{90}X^2_{abcd}\right\} + \frac{1}{60}\Gamma_{bc}\epsilon(\partial X^0)_{bc} + \epsilon X^2 \\ \delta X^2_{abcd} &= \frac{1}{4}\bar{\epsilon}\Gamma_{[ab}\partial_c X^1_{d]} + \frac{3}{8}\bar{\epsilon}\Gamma_{[abc}\partial_d X^1 \\ \delta X^2_a &= -\frac{1}{4}\bar{\epsilon}\Gamma_{[ac}\partial_c X^1_{b]} - \frac{27}{8}\bar{\epsilon}\Gamma_{[a}\partial_b X^1 \\ \delta X^2 &= \frac{1}{4}\bar{\epsilon}\not{\partial} X^1 \\ \text{etc.}\end{aligned}\quad (9)$$

with

$$\partial_a X^0_{bcd} \equiv (\partial X^0)_{abcd} + (\partial X^0)_{bcd}_a + \delta_{a[b}(\partial X^0)_{cd]}$$

To summarize, the new supercurrent consists of the conformal multiplet $(128 + 128)$ and an unconstrained scalar superfield $(2^{15} + 2^{15})$ and has currents ranging in (mass) dimension ^{*}) from 3 to 10. The supergravity multiplet to which it couples therefore has fields ranging in dimension from 2 to -5. This multiplet is described by an unconstrained superfield (prepotential) which couples to the supercurrent via Noether coupling :

$$I = \kappa \int d^4x d^{16}\theta V_{abc} X_{abc} \quad (10)$$

Since X_{abc} is constrained, I is invariant under the following gauge transformations on V_{abc} :

$$\delta V_{abc} = \bar{D} \Lambda_{abc} + \bar{D} \Gamma_{d[ab} D L_{c]d} + 28 \partial_{[a} L_{bc]} \quad (11)$$

where the parameters $\Lambda_{abc\alpha}$ and L_{ab} are again in irreducible representations of $O(10)$. These parameters contain Wess-Zumino gauges as well as the gauge transformations for the x space gauge fields, the graviton h_{ab} , the gravitino ψ_a , and the six-form potential A_{abcdef} which couple to T_{ab} , Σ_a and $\tilde{X}_{abcdef}^2$, respectively. We define the components of V_{abc} in a Wess-Zumino gauge by the following equation :

$$I = \kappa \int d^4x \{ V_{abc}^{16} X_{abc}^0 + V_{ab}^{15} X_{ab}^1 + V_a^{15} X_a^1 + V^{15} X^1 + \dots \} \quad (12)$$

The fields V^n defined this way are not, strictly speaking, the components of V_{abc} in the $n\theta$ sector ; for example, V_{abc}^{16} will have contributions from all even θ sectors of V_{abc} . The transformation laws of the V 's can be obtained by requiring that (6) is invariant under supersymmetry. For example,

$$\begin{aligned} \delta V_{abc}^{16} &= \frac{3}{8} (2 \bar{E} \partial_{[a} V_{bc]}^{15} - \bar{E} \Gamma_{[a} \not{\partial} V_{bc]}^{15}) - \\ &\quad - \frac{1}{32} \bar{E} \Gamma_{abc} \partial^d V_d^{15} - \frac{1}{480} \bar{E} \Gamma_{abc} \not{\partial} V^{15} \\ \delta V_{abcd}^{14} &= 4 \bar{E} \Gamma_{[ab} V_{cd]}^{15} + \bar{E} \Gamma_{[abc} V_{d]}^{15} + \frac{1}{90} \bar{E} \Gamma_{abcd} V^{15} + \\ &\quad + \text{possible } \partial \cdot V^{13} \text{ - terms }^{**}) \end{aligned} \quad (13)$$

(V_{abcd}^{14} is the dual of the six-form gauge field).

^{*}) We employ 4-d dimensional units.

^{**}) These derivative terms drop out in the variation of the field strength.

We now have a supergravity multiplet which contains the physical gauge fields together with a host of auxiliary fields. The missing physical fields are the scalar A and the spinor λ . However, we know that on-shell 10-d supergravity is described by a scalar superfield satisfying¹⁰⁾

$$S^4_{abcde} = 0 \quad (14)$$

Its independent components are : $S^0 \propto A$, $S^1 \propto \lambda$, $S^2_{abc} \propto F_{abc} (A_{ab})$, $S^3_{ab} \propto (\partial_{[a} \psi_{b]} - \Gamma \text{ traces})$ and $S^4_{abcd} \propto C_{abcd}$, the Weyl tensor. Therefore, if we can find a Lagrangian involving V_{abc} and a new unconstrained scalar superfield S such that the V_{abc} equation implies that S satisfies (14) while the S equation sets the auxiliary fields in V_{abc} to zero, we will have solved the problem. The Lagrangian should take the form

$$\mathcal{L} = \frac{1}{2} V_{abc} G_{abc} - V_{abc} \bar{D} \Gamma_{abc} D S \quad (15)$$

where

$$G_{abc} = \Delta_{abc, def} V_{def} \quad (16)$$

with Δ being a linear operator containing terms of the form D^{16} , ∂D^{14} , etc., down to ∂^8 . Since $\mathcal{L}$ has to be invariant under (11), G_{abc} must be gauge invariant and satisfy the same constraints as X_{abc} [Eq. (6)]. Since it is non-trivial to construct Δ explicitly we shall instead make an ansatz for the first component of G_{abc} and show that the superfield thus defined does indeed satisfy the required constraints. Gauge invariance will be automatic if the components of G only involve the V^n 's in (x space) gauge invariant combinations. The only possible choice which ensures that $G^1_{abc} = 0$ is

$$G^0_{abc} = \tilde{F}_{abc} - \frac{8}{3} V^{16}_{abc} \quad (17)$$

where $\tilde{F}_{abc}$ is the dual of the field strength of the six-form potential. Using the transformation laws for the V 's derived from (16), we have verified that the second constraint in (6) is also satisfied. Hence we conclude that a superfield G_{abc} of the form (16) satisfying the required constraints does indeed exist and is furthermore unique. The equations of motion following from (15) are

$$\bar{D} \Gamma_{abc} D V_{abc} = 0 \quad (18)$$

$$G_{abc} = \bar{D} \Gamma_{abc} D S \quad (19)$$

But $\bar{D} \Gamma_{abc} D V_{abc}$ is essentially the scalar superfield contained in V_{abc} as can be seen by inserting the form (3) for the supercurrent into (10) [its last two components are $\Gamma^{ab} \partial_a \psi_b$ and $R(h)$]. Therefore (18) sets all the auxiliary fields in V_{abc} to zero. Having imposed (18), the components of G_{abc} become

$$\begin{aligned} G^0_{abc} &= \tilde{F}_{abc} \\ G^1_{ab} &= \frac{1}{2} \partial_{[a} \psi_{b]} - \Gamma\text{-traces} \\ G^1_a &= -\frac{3}{16} \Gamma_{abc} \partial^b \psi^c \\ G^2_{ab} &= \frac{3}{32} R_{ab}(h) \\ G^2_{ab\,cd} &= \frac{1}{16} C_{ab,cd}(h) \\ G^2_{abcd} &= \frac{3}{16} \partial_{[a} \tilde{F}_{bcd]} \end{aligned} \quad (20)$$

We immediately see that G^4_{abcde} is zero, so that (19) tells us that S^4_{abcde} vanishes. Hence S is the on-shell supergravity superfield. On the other hand, there are no components on the right-hand side of (19) which correspond to the representation G^2_{abcd} , so we obtain the equation of motion for the six-index gauge field. The equations for G^1_a and G^2_{ab} become the standard equations of motion for the gravitino and the graviton only after a suitable field redefinition. Finally, since $G^1 = -\frac{1}{90} \Gamma^{ab} \partial_a \psi_b$ and $G^2 = -\frac{1}{360} R(h)$ are zero from (18), we find $\square S^0 = \not{g} S^1 = 0$.

To summarize, we have found a solution to the auxiliary field problem for linearized $d = 10$ supergravity with a set of fields which consists of a $(128 + 128)$ component multiplet together with two unconstrained scalar superfields. The total number of component fields is therefore $128 + 128 + 2 \times (2^{15} + 2^{15}) = 65,664 + 65,664$. We have simultaneously found the linearized superfield form of the Lagrangian which may serve as a basis for developing extended supergraph Feynman rules. We have not yet investigated the dimensional reduction to 4-d so that it is not clear that one can separate off the six Maxwell multiplets from $N = 4$ supergravity within the framework of this set of auxiliary fields. If such a separation were possible, we would still not have solved the $N = 4$ Yang-Mills

problem. This is because an inspection of the physical fields reveals that 15 of the 36 scalars in the Maxwell sector occur as two-form gauge fields so that one could not generalize the result to the non-Abelian case. Furthermore, they will occur as an $SU(4)$ 15 so that one would be able to construct a singlet off-shell Maxwell multiplet. This shows that we have circumvented the the counting argument ¹¹⁾ which states that there can be no $SU(4)$ invariant off-shell Lagrangian for $N = 4$ Yang-Mills involving paired auxiliary fermions, since this argument is only applicable to an odd number of Maxwell or Yang-Mills multiplets.

An obvious question is whether or not our results can be generalized to the non-linear case. From the point of view of components there seems to be no immediate objection in principle since we have no auxiliary gauge fields and no central charges. On the other hand we do have auxiliary fields of high negative dimensions which cannot be fitted into the superspace field-strength tensors in a straightforward way. This might indicate that there are genuine problems with negative dimension auxiliaries, or it may just mean that the conventional superspace approach is not the most convenient one for this type of representation. Finally, it is natural to ask if our result has any implications for $d = 11$ ($N = 8$). The answer is that it is difficult to say, since there is no $d = 11$ supercurrent to build on and since, as we have just remarked, a straightforward analysis of superspace constraints is unlikely to be very helpful. However, the mere existence of auxiliary fields for $d = 10$ supergravity is an encouraging sign that there may indeed exist a complete off-shell $N = 8$ theory.

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