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LINEARIZATION OF Z_2 LATTICE GAUGE THEORY

IN FOUR DIMENSIONS

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ABSTRACT

Order-disorder loops of four-dimensional Z_2 gauge theory are shown to obey linear equations of motion. We argue that these equations are solved in terms of a particular ensemble of random surfaces.

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Recently it was shown that mixed order-disorder variables in the three-dimensional Z_2 gauge theory obey linear equations of motion that appear to be those of an ensemble of non-interacting random surfaces^{1)*}. In the present note we extend this work to four-dimensional Z_2 gauge theory.

The partition function of the model is

$$Z = \sum_{\{\sigma_\ell\}} e^{\beta \sum_p \sigma_{\partial p}}, \quad (1)$$

where $\sigma_\ell \in \{+1, -1\}$ are spin variables located on the links of a four-dimensional simple cubic lattice (Z^4) that are multiplied along the boundary of each elementary plaquette p to form

$$\sigma_{\partial p} = \prod_{\ell \in \partial p} \sigma_\ell. \quad (2)$$

The system (1) is self-dual³⁾; up to a trivial multiplicative factor, one has

$$Z = \sum_{\{\mu_{\ell^*}\}} e^{\beta^* \sum_{p^*} \mu_{\partial p^*}}, \quad (3)$$

where $\sinh 2\beta^* \sinh 2\beta = 1$, $\mu_{\ell^*} \in \{+1, -1\}$ and ℓ^* and p^* are links and plaquettes of the dual lattice obtained by shifting the original lattice by $1/2$ unit in all four directions simultaneously. We shall refer to variables in the representations (1) and (3) as order and disorder variables, respectively

The equations of motion of the model are found in standard fashion by exploiting the invariance of the statistical measure. They are

$$\left\langle \prod_{p, \partial p \ni \ell} e^{-2\beta \sigma_{\partial p}} \right\rangle = 1, \quad (4)$$

where $\langle \cdot \rangle$ denotes the statistical average with respect to the measure in (1) and the product in (4) is taken over all plaquettes sharing a given link ℓ . Equation (4) and similar equations for correlations of loops are non-linear. In contrast, certain combinations of order and disorder loops satisfy linear equations, as we will show below.

^{*}) See Ref. 2) for recent related works.

In order to find these equations, we use the following tautology:

$$1 = \frac{1}{\cosh 2\beta} e^{-2\beta \sigma_{ap}} + \tanh 2\beta \sigma_{ap}, \quad (5)$$

valid for every plaquette p . The term $e^{-2\beta \sigma_{ap}}$, when inserted into the statistical average, represents a flipped coupling $\beta \rightarrow -\beta$ for the plaquette p . Its physical content is best described in terms of the magnetic current

$$m_c = \prod_{p \in \partial c} \text{sign } \beta_p$$

defined for each three-dimensional cube c where $\beta_p = \pm\beta$ with the $-$ sign for flipped coupling. We find that $m_c = -1$ on the four cubes sharing the plaquette p . Replacing the cubes by the corresponding links on the dual lattice, one then finds that $e^{-2\beta \sigma_{ap}}$ represents the disorder loop around the plaquette p^* dual to p ⁴⁾. Thus the tautology (5) may be written in the following pictorial and suggestive way:

$$1 = \frac{1}{\cosh 2\beta} \text{ (diagram of a square with a dashed diamond inside, labeled } p \text{ and } p^*) \text{ } + \tanh 2\beta \text{ (diagram of a square with a solid diamond inside, labeled } p \text{ and } p^*) \text{ } . \quad (5')$$

Now it remains to find a set of pairs of order and disorder loops which is closed under multiplication with the right-hand side of (5'). Since the two plaquettes in (5') are mutually dual, we need loop operators which are built up of pairs of order and disorder links residing on mutually dual plaquettes. For example, the order link $\ell_x = \{(0,0,0,0), (1,0,0,0)\}$ and the disorder link $\ell_z^* = \{(1/2, 1/2, -1/2, -1/2), (1/2, 1/2, 1/2, -1/2)\}$ is such a pair belonging to the mutually dual plaquettes $P_{xy} = \{(0,0,0,0), (1,0,0,0), (1,1,0,0), (0,1,0,0)\}$ and $P_{zt}^* = \{(1/2, 1/2, -1/2, -1/2), (1/2, 1/2, 1/2, -1/2), (1/2, 1/2, 1/2, 1/2), (1/2, 1/2, -1/2, 1/2)\}$ which intersect each other at $(1/2, 1/2, 0, 0)$. There is actually a second pair of plaquettes associated with these links: $P_{xt} = \{(0,0,0,0), (1,0,0,0), (1,0,0,-1), (0,0,0,-1)\}$ and $P_{yz}^* = \{(1/2, 1/2, -1/2, -1/2), (1/2, 1/2, 1/2, -1/2), (1/2, -1/2, -1/2, -1/2), (1/2, -1/2, 1/2, -1/2)\}$ with intersection point $(1/2, 0, 0, -1/2)$. This two-fold ambiguity is quite general and the reason can be seen from Fig. 1^{*)}; the second pair is obtained from the first pair by simply interchanging the two extra directions ($\hat{t}$ and $\hat{y}$ for this example) that are orthogonal to the order and disorder links.

*) For clearer perspective, we choose the direction of lattice axes appropriately in each drawing.

Now consider an order-disorder loop made up of order-disorder link pairs of the geometry described above. In a standard fashion⁴⁾, the disorder loop implies that the gauge coupling is flipped $\beta \rightarrow -\beta$ for those plaquettes dual to a surface on the dual lattice that is appended to the disorder loop. Consider a particular pair of links on this order-disorder loop. Multiply by 1 and replace 1 by the right-hand side of (5') where the mutually dual plaquettes in (5') must contain the order-disorder link pair selected. Using the mod 2 property of Z_2 variables, we find diagrammatically,

Diagrammatic equation (6) shows the replacement of a link pair. On the left, a dashed line with an arrow labeled 1^* and a solid line with an arrow labeled 1 meet at a vertex. From this vertex, a solid line with an arrow labeled 2 goes right, then up, then right again to a vertex labeled 3 . From 3 , a solid line with an arrow labeled 3^* goes up, then right, then up again. A dashed line also goes from the first vertex up, then right, then up again. A central equals sign is followed by a fraction $\frac{1}{\cosh 2\beta}$ and a diagram. This diagram is identical to the left one but with an additional solid line segment (a backtracking piece) added to the path between the two vertices. Below this is a plus sign followed by $\tanh 2\beta$ and a second diagram. This second diagram is identical to the previous one but with a different backtracking piece added to the path. The entire equation is labeled (6) on the right.

To have Eq. (6) closed in the set of operators considered, we must add an extra backtracking piece to the loop that is not displaced in each term of (6). Using the mod 2 property once more, we may do this by aid of the replacement.

Diagrammatic equation (7) shows a replacement of a backtracking piece. On the left, a dashed line with an arrow labeled 1^* and a solid line with an arrow labeled 1 meet at a vertex. From this vertex, a solid line with an arrow labeled 2 goes up, then right, then up again to a vertex labeled 3 . From 3 , a solid line with an arrow labeled 3^* goes up, then right, then up again. A dashed line also goes from the first vertex up, then right, then up again. A central equals sign is followed by a diagram. This diagram is identical to the left one but with an additional solid line segment (a backtracking piece) added to the path between the two vertices. The entire equation is labeled (7) on the right.

which restores the one-to-one correspondence of the order-disorder link pairs. This concludes the derivation of the linear^{*)} equations up to the sign factor on the right-hand side of (6) which depend on the particular realization of the disorder loop, i.e., the choice of appended surface on the dual lattice across which the couplings are flipped ($\beta \rightarrow -\beta$).

We now attempt to obtain a better understanding of the equations by introducing the concept of a "superloop" which amalgamates the pair of order and disorder loops into a single loop. First choose an orientation for each loop of the pair. This gives an orientation to every pair of order-disorder links. To each pair of links, associate an oriented line segment which runs from the midpoint of their initial points to that of their end points and call it a "superlink", see Fig. 2a. Connecting the superlinks thus generated, one obtains the superloop.

To gain some insight into the geometry of superlinks, consider a pair of mutually dual plaquettes p and p^* . Construct a 4-cell^{**)} consisting of the 16 midpoints between the vertices of p and p^* . The superlinks are the diagonals across the 2-cells of such a 4-cell, see Fig. 2b. Of the 24 2-cells of this 4-cell, eight are essentially copies of p and p^* so that only the 16 remaining ones qualify to carry superlinks. These 16 2-cells form a two-dimensional torus which is generated by parallel translation of one 2-cell corresponding to p around another 2-cell corresponding to p^* (or, equivalently, vice versa). Recall that there are two pairs of mutually dual plaquettes associated with a superlink. The corresponding two 4-cells turn out to be reflections of each other across the 2-cell containing the superlink, followed by an interchange of the two directions orthogonal to the 2-cell, see Fig. 3. Repeated application of such a twisted reflection starting from a 4-cell (or the two-dimensional torus described above, to be more precise) generates the lattice for the superlinks. Equation (6) may now be drawn in terms of superlinks on the associated 4-cell. In Fig. 4 we show the term corresponding to the right-hand side of (7). Amusingly, the initial superlink and the displaced superlinks together form a non-trivial path winding around the two-dimensional torus, with the two terms on the right-hand side of (6) corresponding to the two topologically inequivalent windings.

*) Actually the equations are not absolutely linear since the displacement may split a loop into two (and vice versa) when the loop has segments spatially close together. Such processes give rise to surfaces with non-trivial topology, i.e., with handles.

**) To avoid confusion, we use the terminology n -cells for the n -dimensional cubes if they are related to superlinks.

We now examine what type of surfaces are generated by repeated displacement of a superloop. As in Ref. 1), our aim is to argue that the loop equations are solved by an ensemble of random surfaces. This actually requires a specification of the ensemble and a comparison of recursive equations for the ensemble with the present loop equations. To examine the class of surfaces swept out by successive displacement is just a first step in such an identification.

To understand the structure of the surface generated by the loop equations, one first has to specify the rule for iterating them. Recall that each displacement corresponds to a pair of mutually dual plaquettes of a 4-cell and that each 4-cell is surrounded by 16 4-cells related by reflections across its 2-cells. We may thus characterize a sequence of displacements by recording the 4-cells used in the sequence. In this language, our convention will be that no 4-cells are used twice in succession. In other words, a superloop, after entering a 4-cell, cannot be displaced within that cell but rather has to move on to the neighbouring 4-cells in successive iterations.

Now focus on the quadrilateral $A_1 A_2 A_3 A_4$ in Fig. 4. To understand the structure of a surface made up of such quadrilaterals, connect the centre O of the 4-cell with the vertices A_i . This generates four right triangles $OA_i A_{i+1}$ ($i = 1, \dots, 4$) each subtending an angle of $\pi/2$ at the centre O . Next imagine displacing the superlink $A_1 A_2$ into the neighbouring 4-cell with the centre at O_1 . One can easily check that $A_1 A_2$ and OO_1 both have length one, are orthogonal and intersect each other at the midpoint. Thus the two triangles $OA_1 A_2$ and $O_1 A_1 A_2$ together form a square of size $\sqrt{1/2}$, see Fig. 5. Repeating this construction for the remaining superlinks, one finds that the surface is made up of such squares. The lattice for these squares is tilted with respect to the original order-disorder lattice. The unit vectors of the tilted lattice $\vec{OA}_i$ and their inverse are given by $\sqrt{1/2} (\pm \hat{t} \pm \hat{x} \pm \hat{y} \pm \hat{z})$ with an even number of $+$ signs. They are thus all perpendicular and the vertex at O is a genuine four-dimensional vertex as befits those of a surface immersed in four-space. In Fig. 6 we show a simple example of a closed surface on the tilted lattice. It corresponds to the motion of a disorder loop sweeping around the faces of a three-dimensional cube, while the order loop repeats the backtracking motion on the link dual to the cube.

It is clear from the above that a concise description of the surface is provided by an enumeration of the centres of the 4-cells that it contains. Each centre is surrounded by four neighbouring centres and forms a vertex of the shape depicted in Fig. 5. Each vertex carries a factor $\tanh 2\beta$ or $(\cosh 2\beta)^{-1}$ depending on whether it corresponds to an order or disorder displacement,

respectively. Each vertex also carries a sign [the sign left out in (6)] arising from a consistent convention for realizing the disorder loop, e.g., by appending a surface in the $-\hat{t}$ direction to it and flipping the coupling $\beta \rightarrow -\beta$ on all plaquettes dual to it.

The preceding discussion suggests that the superloop correlation functions may be computed as sums over random surfaces made up of squares on the tilted lattice with appropriate weight factors and signs. We have worked out an algebraic rule for computing the weight. It still remains to be seen whether there exists a geometrical characterization for the sign.

A troublesome aspect of the four-dimensional Z_2 theory is the first-order phase transition reported in Monte Carlo studies of the model⁵⁾ which might exclude the existence of a continuum limit. Up to now, no direct measurements of the mass gap have been carried out, however, and it may be worthwhile to measure superloop correlations via Monte Carlo techniques.

A very intriguing feature of our equation is the initial local twofold ambiguity for the choice of the loop equations; each link of the initial superloop may be displaced into either of the two 4-cells associated with this superlink. Once the choice is made, however, further iterations are free of such ambiguities and the net result should of course be independent of the initial choice. This looks like a (fermionic?) gauge symmetry^{*)}. It is tempting to speculate that this gauge symmetry might be the lattice precursor of a local supersymmetry of a fermionic random surface in the continuum limit^{**)}.

Another interesting possibility concerns the complete integrability of the model. Remember that the two-dimensional Ising model is the simplest member of an entire class of solvable models, the Baxter class⁷⁾. There may exist a similar class of four-dimensional models of which the present one might be the simplest example.

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^{*)} The possible existence of such a fermionic symmetry on a lattice has been discussed extensively with P. Windey in the context of the lower-dimensional Z_2 models.

^{**)} A very interesting attempt to define a fermionic string in $d = 4$ is due to Polyakov⁶⁾.

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FIGURE CAPTIONS

- Fig. 1 : Two pairs of mutually dual plaquettes (P_{xy}, P_{zt}^*) and (P_{xt}, P_{yz}^*) associated with a pair of order-disorder links (ℓ_x, ℓ_z^*) .
- Fig. 2 : (a) Superlink S for the link pair (ℓ, ℓ^*) . P and P^* represent a pair of mutually dual plaquettes associated with (ℓ, ℓ^*) .
(b) 4-cell associated with the pair of plaquettes (P, P^*) of (a). The two shaded 2-cells and six others of similar type do not carry superlinks.
- Fig. 3 : The two 4-cells associated with a superlink S . They are twisted reflections of each other (see text). The shading has the same meaning as in Fig. 2b.
- Fig. 4 : The loop equation displaces the superlink $A_1 A_4$ to $A_1 A_2 - A_2 A_3 - A_3 A_4$.
- Fig. 5 : The four right triangles $OA_i A_{i+1}$ ($i = 1, \dots, 4$) joined at the centre O of the 4-cell. O_i is the centre of the neighbouring 4-cell reflected across the superlink $A_i A_{i+1}$. $OA_i O_i A_{i+1}$ is a square of size $\sqrt{1/2}$ ($i = 1, \dots, 4$).
- Fig. 6 : A simple example of a closed surface swept out by a superloop. Little blobs represent the centres of 4-cells used in successive iterations.

FIG. 1

(a)

(b)

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6