TOPOLOGICAL CHARGE ON THE LATTICE: THE $O(3)$ CASE

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ABSTRACT

We discuss the topological classification of the field configurations on the lattice. The regulator may act as a source of topological defects of the order of the lattice spacing a . To deal with them we introduce a family of operators Q_L (block topological charge) that discriminates against instantons of sizes less than or equal to L . We prove that the correct definition of the topological charge implies first taking the limit $a \rightarrow 0$ with L fixed and then $L \rightarrow 0$. Numerical simulations à la Monte Carlo for $O(3)$ on two dimensions are discussed. The observed density of small instantons agrees very well with theoretical calculations in the continuum theory.

1. INTRODUCTION

In this paper we face the problem of whether or not topology plays any role in quantum field theory and if so how one can compute its effects (such as the θ vacuum).

We put our field theory on a lattice. We introduce a family of operators measuring the topological structure of the field configurations and discuss under which conditions this structure reproduces the usual definition of the topological charge in the continuum limit.

The dynamical consequences of the topological defects will be the object of a future study. Here we simply want to emphasize that there is a way to obtain the correct continuum limit for the topological charge starting from its definition on a lattice. Our assertion is confirmed by Monte Carlo simulations. The answer turns out to be surprisingly simple. If one imagines the lattice embedded in continuum space-time, then every configuration on the lattice can be continuously interpolated between the lattice points. The topological charge for a certain configuration on the lattice is defined to be the one of the interpolating field. Of course there is a lot of ambiguity involved in the interpolation. However, we show that there is in general a natural way to solve this ambiguity. We simply force the topology, below a certain scale L (which is in general greater than the lattice spacing a), to be trivial. We show that the correct continuum limit is obtained by making $a \rightarrow 0$ at L (measured in units of the correlation length ξ) fixed. This operation defines a new regularized operator in the continuum limit which measures holes and defects at a scale larger than L . This is a well-defined operator even for theories like $O(3)$ where the dilute gas of small instantons diverges¹⁾ or for theories which are not asymptotically free. One can eventually take $L/\xi \rightarrow 0$ and thereby define a local operator (this will correspond to the usual topological charge of the continuum limit). However, even in this case, we cannot interchange the two limits $a \rightarrow 0$ with L/ξ fixed and $L/\xi \rightarrow 0$ because holes and defects near the scale of the cut-off may depend on the ultra-violet regulator a . We discuss in some detail the $O(3)$ non-linear σ model with its topological charge first introduced by Belavin and Polyakov²⁾. This particular example has already been discussed by Berg and Lüscher³⁾. Our definition, however, allows us to exorcize the anomalous behaviour of small instantons on the lattices encountered in Ref. 3).

In the next section we define the topological charge on the lattice and introduce the relevant notations. We also define, for the $O(3)$ non-linear σ model on the lattice, two different actions that have the same continuum limit. In Section 3 we describe our Monte Carlo experiments for the two actions and finally we discuss the numerical results.

The fact that we introduced a whole family of operators that measure the instantons larger than a certain scale L , allows us a direct comparison of the results given by the dilute gas approximation with the numerical results obtained by Monte Carlo simulations. The agreement is remarkable.

2. THE TOPOLOGICAL CHARGE OF THE $O(3)$ NON-LINEAR σ MODEL

In order to achieve the programme outlined in the Introduction it is practical to use a formalism where the topological charge can be expressed as a surface integral.

From now on we will discuss explicitly the $O(3)$ non-linear σ model; the generalization of our definition to the general CP^n models is straightforward. We start from the CP^1 version of the $O(3)$ theory. The spin field $\vec{\phi}$ of $O(3)$ can be expressed in terms of the fields z of the CP^1 model with the relation:

$$\vec{\phi}(\vec{x}) = \bar{z}(\vec{x}) \vec{\sigma} z(\vec{x}) \quad \bar{z}(\vec{x}) z(\vec{x}) = 1 \quad (2.1)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. If we choose on the lattice the following action

$$S_{CP^1} = \sum_{\vec{x}, \hat{\mu}} \left[\bar{z}(\vec{x} + \hat{\mu}) z(\vec{x}) U(\vec{x}, \hat{\mu}) + h.c. \right] \quad (2.2)$$

We can define a topological charge on the lattice by analogy with the discussion of the topological charge in the continuum. Let us start from the classical solution for U :

$$\frac{\delta S_{CP^1}}{\delta U_c} = 0 \Rightarrow U_c(\vec{x}, \hat{\mu}) = \frac{\bar{z}(\vec{x} + \hat{\mu}) z(\vec{x})}{|\bar{z}(\vec{x} + \hat{\mu}) z(\vec{x})|} \quad (2.3)$$

Then we define the topological charge Q as a surface integral

$$\exp [i 2\pi Q(S)] = \prod_{\text{Loop } S} U_c(\vec{x}_1, \hat{\mu}_1) U_c(\vec{x}_2, \hat{\mu}_2) \dots U_c(\vec{x}_f, \hat{\mu}_f) \quad (2.4)$$

Notice that while in the continuum one can introduce Q directly as a contour integral, here the only quantity we can determine from the values of the fields at the boundary surface is the fractional part of Q . This illustrates an argument originally developed by Peskin⁴⁾. In gauge theories he imagined performing a gauge transformation on a configuration so that it would correspond to a singular transformation in the continuum, but with the singularity located at a hole on the lattice. This reflects the compact nature of the group.

In general, such a transformation changes the contour integral by an integer and therefore the latter is not gauge invariant. From our point of view this argument reflects ambiguities involved in interpolating a field inside a region. It is obvious that we can affect the number of topological defects by a judicious choice of the interpolation, and therefore the integral part of the topological charge cannot be determined. On the other hand, there is no contradiction in assuming that the fractional part can be so derived. Therefore we assume that the operator [Eq. (2.4)] will have a continuum limit if, as usual, the size of the curve S is kept fixed in physical units when $a \rightarrow 0$. Let us call $Q_F(S)$ the so-defined fractional charge ($-1/2 \leq Q_F \leq 1/2$). Rather surprisingly, this operator can be expanded directly in terms of the $\vec{\phi}$ fields. In fact a little algebra shows that:

$$\exp[i 2\pi Q_F(S)] = K \text{Tr} \left[(1 + \vec{\phi}(m_1) \cdot \vec{\sigma}) (1 + \vec{\phi}(m_2) \cdot \vec{\sigma}) \cdots (1 + \vec{\phi}(\eta_p) \cdot \vec{\sigma}) \right]$$

with $-1/2 \leq Q_F \leq 1/2$ where $\vec{\phi}(\eta_i)$ are the spins belonging to the contour S and K is a real normalization constant. If we restrict S to three vertices belonging to the same plaquette and sum this charge over the whole lattice this definition coincides with the one given by Berg and Lüscher³⁾. As these authors have stressed, it corresponds to interpolating the $\vec{\phi}$ fields along the geodesics. However, their numerical experiments, corroborated by ours (see Section 3), show that the operator so defined does not have the correct renormalization group behaviour in the limit $a \rightarrow 0$. We interpret this result by saying that this is a wrong definition. As preannounced in Section 1, we introduce a whole family of operators Q_L , depending on a scale L . Q_L is interpreted as a regularized version of the topological charge, i.e., $Q = \lim_{L \rightarrow 0} Q_L$. We imagine superimposing on the total volume V a mesh of spacing L (in the lattice we will take $L_K = 2^K a$) as shown in Fig. 1. This mesh defines a partition of the original volume in V/L^2 subregions. The sum of the previously defined fractional topological charge Q_F of these subregions is, by definition, equal to the topological charge Q_L . It is very easy to show that this operator Q_L discriminates against

instantons whose dimension $\Lambda < L$. In fact, in the dilute gas approximation, a small instanton will contribute to Q_L only if it is centred near a vertex of the mesh (see Fig. 2). Therefore the volume available for it will be:

$$V_{\text{eff.}} = \frac{V}{L^2} \lambda^2 \pi \quad (2.6)$$

instead of V . The factor λ^2/L^2 damps the contribution of small instantons. In the limit where the instanton gas is sufficiently dense, we cannot use the discretization of the topological charge. The probability distribution for the charge inside a finite volume will be rather a smooth one. But even in this last case, and provided the instanton correlation length $\ell \ll L$, we can still prove that its contribution to Q_L will scale as ℓ^2/L^2 . In fact, in the limit of a very dense instanton gas, the probability distribution of the topological charge inside the L square will be:

$$P_1(Q)dQ = \left(\frac{2\pi\sigma L^2}{\ell^2} \right)^{-1/2} \exp\left(-\frac{Q^2 \ell^2}{2\sigma L^2}\right) dQ \quad -\infty < Q < +\infty \quad (2.7)$$

σ is a constant (possibly depending on the temperature). The same formula holds for the charge distribution in the total volume:

$$P_2(Q)dQ = \left(\frac{2\pi\sigma V}{\ell^2} \right)^{-1/2} \exp\left(-\frac{Q^2 \ell^2}{2\sigma V}\right) dQ \quad (2.8)$$

We can derive the distribution for Q_L from Eqs. (2.7) and (2.8).

The distribution of the fractional topological charge will clearly be:

$$P_F(Q)dQ = \left[\sum_{m=-\infty}^{+\infty} P_1(Q+m) \right] dQ \quad -\frac{1}{2} \leq Q \leq +\frac{1}{2} \quad (2.9)$$

We can calculate $P(Q_L) dQ_L$ by standard methods to derive:

$$P(Q_L)dQ_L = \left[\frac{2\pi f\left(\frac{\sigma L^2}{\ell^2}\right) V}{L^2} \right]^{-1/2} \exp\left[\frac{-Q_L^2 L^2}{2 f\left(\frac{\sigma L^2}{\ell^2}\right) V} \right] dQ_L \quad (2.10)$$

where $f(x)$, obtained by the saddle point method (for V/L^2 large), is equal to

$$f(x) = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \left\{ \exp \left[- (2\pi)^2 k^2 x \right] - 1 \right\} \quad (2.11)$$

By numerical computation it is possible to show that (with 10% accuracy) $f(x) = x$ for $x < 0.01$ and it is equal to $1/12$ for $x > 0.23$. Therefore, as soon as $L > 0.5\ell/V\sigma$ the contribution to $\langle Q_L^2 \rangle$ given by this source will be:

$$\frac{V}{12 L^2} \quad (2.12)$$

instead of $V\sigma/\ell^2$ [see Eq. (2.8)]. Therefore, we again found that in Q_L we cut off the small scales through a damping factor $12\ell^2/\sigma L^2$. Experimentally, we always find the situation described by the dilute gas approximation [Eq. (2.6)]. The fundamental advantage of the family of operators Q_L is that they can be measured in the lattice. In particular the lattice may be, and in a lot of cases is, a source of topological noise related to instantons of small size. This noise will depend on the details of the regularized action in such a way that even theories which belong to the same universality class may have different amounts of topological noise. Let us consider for instance two possible actions (with the same continuum limit) for the $O(3)$ non-linear σ model:

$$\begin{aligned} S_C &= -\beta \sum_{\vec{x}, \hat{\mu}} \left[\vec{\phi}(\vec{x}) \cdot \vec{\phi}(\vec{x} + \hat{\mu}) - 1 \right] \\ S_T &= \frac{\beta}{2} \sum_{\vec{x}, \hat{\mu}} \left[\arccos \left(\vec{\phi}(\vec{x}) \cdot \vec{\phi}(\vec{x} + \hat{\mu}) \right) \right]^2 \end{aligned} \quad (2.13)$$

The action of the instanton shown in Fig. 3 will be, for the two cases:

$$\begin{aligned} S_1 &= 8\beta \\ S_2 &= 2\pi^2 \beta \end{aligned} \quad (2.14)$$

to be compared with the continuum result $S = 4\pi\beta$. We see that the first action greatly over emphasizes that particular spin configuration while the second one under-estimates it^{*)}. However, it would be preposterous to pretend that the

^{*)} We thank G. Parisi for drawing our attention to this point and for suggesting the use of the action S_T for numerical experiments.

regularized action gives the correct continuum weight to instantons whose scale is of order of the cut-off a . The correct definition of the topological charge should be simply independent of these defects. This is precisely what we can do by using Q_L . In fact, in the limit $a \rightarrow 0$ at L fixed (in physical units), only instantons whose size is many times the lattice spacing will contribute to Q_L and their action will tend correctly to the continuum one. The contribution of instantons, whose scale is of order a , will go to zero at least as a^2 in the continuum limit.

The final limit

$$Q = \lim_{L \rightarrow 0} Q_L$$

is to be checked directly in the continuum theory. For asymptotically free theories, one can use the dilute gas of the instantons' approximation. It is known to exist for CP_{n-1} theories for $n \geq 3$ but does not exist for $O(3)$ ⁵⁾. The Q_L with $L \neq 0$ do exist, however, and have the correct continuum limit and normalization group properties. In the next section we will show that the behaviour of Q_L with L matches very nicely the logarithmic divergence of the continuum operator even with the correct absolute normalization.

When this paper was being written we received a paper by Lüscher where the peculiarity of the action S_{CV} of overestimating the action of small instantons is also noticed and discussed. Lüscher's conclusions, however, are orthogonal to ours because he believes that it is the action rather than the definition of the topological charge which is to blame. Furthermore, our numerical simulations indicate that the spurious topological noise produced by the lattice cannot be ascribed to one particular configuration.

3. DESCRIPTION OF THE MONTE CARLO SIMULATIONS AND DISCUSSION OF THE NUMERICAL RESULTS

We briefly describe our Monte Carlo experiment. The upgrading of the spins was made as in Ref. 7). The topological charge and correlation was measured at three different values of the temperature for each of the two actions considered (S_C and S_T).

For S_C we choose for β the values $\beta = 1.45, 1.55$ and 1.65 and make Monte Carlo simulations with 64×64 , 64×64 and 128×128 lattices respectively. For each action and values of β the two point correlation length was much smaller than the lattice size in order to avoid saturation problems⁸⁾. At $\beta = 1.65$ for the action S_C , because of the limitation of the computer memory, we packed each spin variable $[\vec{s} = (s_x, s_y, s_z)]$ into a single computer word. This procedure can introduce a systematic error in the determination of

of the thermodynamical quantities by making the system more disordered at a certain value of β because we lose precision in packing the spins. For example, for the magnetic susceptibility we get a value lower than that which was expected on the basis of previous measurements^{3),7),8)} although in agreement within the experimental errors. The same method of packing the spins was always used for the action S_T . We measured the topological charge every five sweeps.

In Table 1 we give the number of sweeps used to measure the relevant thermodynamical quantities for each of the two actions S_C and S_T at different values of β . A certain number of sweeps (about the first one third of the total) was used to reach the thermal equilibrium and we did not take measurements during these sweeps. We divided the remaining ones into bunches of 150 - 200 sweeps to estimate statistical errors.

Following the discussion of Section 2, we define and measure for each spin configuration the topological charge Q_L . For a lattice $N \times N$ with $N = KL$, K being an integer, we have L^2 independent (but correlated) possible definitions of the topological charge Q_L^i $i = 1, \dots, L^2$. We define the average squared topological charge as:

$$\frac{\langle Q_L^2 \rangle}{V} = \frac{1}{N^2} \left\langle \frac{1}{L^2} \sum_{i=1}^{L^2} (Q_L^i)^2 \right\rangle \quad (3.1)$$

$\langle \dots \rangle$ stands for the average over the field configurations. We expect that the contribution due to instantons of size λ much less than L goes to zero like λ^2/L^2 . Remember that for a lattice $N \times N$ Q_N is always zero. In order to have reasonable results (independent of the ultra-violet as well as of the infra-red cut-off) the following inequality must be satisfied:

$$1 \ll L \ll N \quad (3.2)$$

In Tables 2 and 3 we give the values of $\langle Q_L^2 \rangle/V$ for $L = 1, 2, 4, 8, 16$ at different values of β for the actions S_C and S_T respectively.

We parametrize $\langle Q_L^2 \rangle/V$ as a function of β as:

$$\frac{\langle Q_L^2 \rangle}{V} = \text{const. } \beta^2 \exp(-4\pi\beta\epsilon) \quad (3.3)$$

If the smallest instanton action were $4\pi\beta$ we would expect $\epsilon = 1$ for any L . Indeed, for the action S_C (from Table 2), we estimate for ϵ a value between $0.50 \div 0.60$ in agreement with the results of Ref. 3). For the action S_T we

we find much higher values for ϵ ($\epsilon \sim 0.70 \div 0.80$) because for any L we heavily cut the small instanton contribution. If we now compute, according to our philosophy, $\langle Q_L^2 \rangle / V$ against β at L/ξ fixed (approximatively) we get a real improvement for the value of ϵ . We find $\epsilon \sim 0.80$ for S_C and $\epsilon \sim 0.90$ for S_T^{**} . In Fig. 4 and Fig. 5 we plot the results of our experiment against the theoretical expectations.

The results we find for ϵ at L/ξ fixed are no worse than what was found for the magnetic susceptibility^{3),7)}. In that case the difference between the experimental results and the expected renormalization group behaviour was explained as being due to subleading corrections.

In the dilute gas approximation it is possible to give a quantitative estimate of the quantity:

$$\Delta(L, L')_{C, T} = \frac{\langle Q_L^2 \rangle - \langle Q_{L'}^2 \rangle}{V} = C_{PV} \left(\frac{\Lambda_{PV}}{\Lambda_{C, T}} \right)^2 \beta^2 \exp(-4\pi\beta) \quad (3.4)$$

C_{PV} is evaluated in the dilute gas approximation in the continuum limit and is given by $C_{PV} = 32\pi/e^2$. The value of C_{PV} reported here does not completely agree with published results^{1),9)}. The discrepancies are discussed in Appendix A. $\Lambda_{C, T, PV}$ are the renormalization scales of the running coupling constant for the action S_C, S_T and for the continuum theory using the Pauli-Villars renormalization prescription

$$\beta(Q^2)^{-1} = 1 / [4\pi \ln(Q^2 / \Lambda_{C, T, PV})]$$

we found $(\Lambda_{PV}/\Lambda_C)^2 = 32 \exp^{10})$ and $(\Lambda_{PV}/\Lambda_T)^2 = 32 \exp^{-\pi/3}$ (see Appendix B). The experimental results for $\Delta(L, L')_{C, T}$ are reported in Tables 4 and 5 at different values of β for $L' = 2L$ and $L = 1, 2, 4, 8$. We see that, as expected at fixed β , $\Delta(L, 2L)$ is practically independent of L ^{**)}. The most striking result is that, for the action S_T , the numbers we obtained with Monte Carlo simulations are in almost perfect agreement with the prediction of the dilute gas approximation [Eq. (3.4)]. On the other hand, for S_C , the small instanton contribution becomes stronger and stronger as β increases so that the ratio between the experimental value and the expected one goes from three to six between $\beta = 1.45$ and $\beta = 1.65$. In Table 6 we compare the average of $\Delta(L, 2L)$ with the numbers predicted by Eq. (3.4).

^{*}) Naturally, these numbers are affected by large errors. The accuracy could be improved by increasing the range of β , the lattice size and the statistics.

^{**}) This is really only true for $L \geq 2$ because $\Delta(1, 2)$ strongly feels cut-off effects.

In conclusion, all experimental data show a consistent picture and indicate that our definition of the topological charge is able to give the correct continuum limit. However, we think that another experiment with a higher, statistic larger lattice is required in order to improve our numerical results. Probably it will be necessary to use a finite subgroup of $O(3)$ in order to limit the computer time.

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APPENDIX A

In this Appendix we present a partial calculation of the determinant around the one instanton solution only in that part where we found a discrepancy with the published result¹⁾. In general we follow that reference^{*)}. The starting formula is:

$$Z = \frac{\int \mathcal{D}[\vec{\phi}] \prod_x \delta(\vec{\phi}_{(1)} \cdot \vec{\phi}(x)) \exp -\frac{1}{2} \langle \vec{\phi}, M^2 \vec{\phi} \rangle}{\int \mathcal{D}[\vec{\phi}] \prod_x \delta(\vec{v} \cdot \vec{\phi}(x)) \exp -\frac{1}{2} \langle \vec{\phi}, M_0^2 \vec{\phi} \rangle} \quad (A.1)$$

$\vec{\phi}_{(1)}$ is the classical one instanton solution, $\vec{v}$ is the classical vacuum solution and M^2 and M_0^2 are the operators appearing when one computes small oscillations around classical solutions.

$$\langle \vec{\phi}_1, \vec{\phi}_2 \rangle = \int d\mu(x) \vec{\phi}_1(x) \cdot \vec{\phi}_2(x) = \int \frac{4 d^2x}{[1+x^2+y^2]^2} \vec{\phi}_1(x) \cdot \vec{\phi}_2(x) \quad (A.2)$$

Expression (A.1) can be exactly computed; however, we need to separate the contribution in the functional integral due to the zero frequency modes of the operators M^2 and M_0^2 .

In the case of the one instanton solution these zero frequency modes are associated with the invariance of the action under the general transformation

$$\vec{\phi}_{[\vec{a}]}(z) = \vec{\phi} \left[\frac{a_1 z + a_2}{a_3 z + 1} \right] = T(\vec{a}) \vec{\phi}(z) \quad (A.3)$$

$$z = x + iy$$

$a_{1,2,3}$ being complex numbers.

Using standard collective co-ordinate methods, the expression (A.1) can be rewritten as:

$$Z = \frac{\prod_{i=1}^3 d^2 \vec{a}_i \left| \frac{1}{(a_1 - a_2 a_3)} \right|^2 \det \left| \left\langle \frac{\partial \vec{\phi}_{(1)}}{\partial \vec{a}_i}, \frac{\partial \vec{\phi}_{(1)}}{\partial \vec{a}_j} \right\rangle \right|^{1/2} (\det M^2)^{-1/2}}{g^2(\mu_0^2) (2\pi)^2 16 \pi^2 (\det M_0^2)^{-1/2}} \quad (A.4)$$

*) We thank A. Jevicki for communications concerning this calculation.

It is easy to show that

$$\det \left\langle \frac{\partial \vec{\phi}_{(1)}}{\partial a_i}, \frac{\partial \vec{\phi}_{(1)}}{\partial a_j} \right\rangle = \left\{ \det \left\langle \frac{\partial \vec{\phi}_{(1)}}{\partial a_i}, \frac{\partial \vec{\phi}_{(1)}}{\partial a_j^*} \right\rangle \right\}^2 = \left[\frac{1}{2} \left(\frac{8\pi}{3} \right)^3 \right]^2$$

we then have

$$Z = \frac{4}{27\pi g^2(\mu_0^2)} \int \prod_{i=1}^3 d^2 \vec{a}_i \left| \frac{1}{(a_1 - a_2 a_3)} \right|^2 \left[\frac{(\det M^2)^{-1/2}}{(\det M_0^2)^{-1/2}} \right] \quad (A.5)$$

After one simple variable transformation

$$Z = \frac{64\pi}{27 g^2(\mu_0^2)} \int d^2 x \frac{d\lambda}{\lambda^3} \left[\frac{(\det M^2)^{-1/2}}{(\det M_0^2)^{-1/2}} \right] \quad (A.6)$$

while the result reported in Ref. 1) was

$$Z = \frac{256\pi}{9 g^2(\mu_0^2)} \int d^2 x \frac{d\lambda}{\lambda^3} \left[\frac{(\det M^2)^{-1/2}}{(\det M_0^2)^{-1/2}} \right] \quad (A.7)$$

Moreover, we find that the ratio of the determinants will give

$$\exp \left(-\frac{4\pi}{g(\mu_0^2)} + \ln(\mu_0^2 \lambda^2) \right) * e^{-2} \frac{27}{2} \quad (A.8)$$

Instead of what was reported in Ref. 1)

$$\exp \left(-\frac{4\pi}{g(\mu_0^2)} + \ln(\mu_0^2 \lambda^2) \right) * e^{-6} 216 \quad (A.9)$$

the final result is then:

$$Z = \frac{32\pi}{e^2 g^2(\mu_0^2)} \exp \left(-\frac{4\pi}{g(\mu_0^2)} \right) \int d^2 x \frac{d\lambda}{\lambda} \quad (A.10)$$

APPENDIX B

In this Appendix we compute the relation between Λ_{PV} in the continuum and Λ_C for the action S_C and Λ_T for the action S_T on the lattice.

The action on the lattice in the limit $\beta \rightarrow \infty$ can be written as:

$$S = \frac{\beta}{2a^2(2\pi)^2} \int d^2p \vec{\pi}(p) V(p) \vec{\pi}(-p) + \frac{\beta}{8a^2(2\pi)^6} \int d^2p_1 d^2p_2 d^2p_3 d^2p_4 \delta^2(p_1 + p_2 + p_3 + p_4) \cdot V_4(p_1, p_2, p_3, p_4) \{ \vec{\pi}(p_1) \cdot \vec{\pi}(p_2) \} \{ \vec{\pi}(p_3) \cdot \vec{\pi}(p_4) \} \quad (B.1)$$

The integrals over the momenta are in the first Brillouin zone

$$V(p) = 4 \sum_{\vec{\mu} = \vec{\mu}_x, \vec{\mu}_y} \sin^2 \frac{\vec{p} \cdot \vec{\mu}}{2} + Ha^2 - \frac{a^2}{\beta} \quad (B.2)$$

$$V_4(p_1, \dots, p_4) = 4 \sum_{\vec{\mu} = \vec{\mu}_x, \vec{\mu}_y} \sin^2 \frac{(\vec{p}_1 + \vec{p}_2) \cdot \vec{\mu}}{2} + Ha^2 + \alpha \frac{16}{3} \left[\sum_{\vec{\mu} = \vec{\mu}_x, \vec{\mu}_y} \sin \frac{\vec{p}_1 \cdot \vec{\mu}}{2} \sin \frac{\vec{p}_2 \cdot \vec{\mu}}{2} \sin \frac{\vec{p}_3 \cdot \vec{\mu}}{2} \sin \frac{\vec{p}_4 \cdot \vec{\mu}}{2} \right]$$

$\vec{\mu}$ is the unit vector in the direction x and y $\begin{cases} \vec{\mu}_x = (a, 0) \\ \vec{\mu}_y = (0, a) \end{cases}$ $\alpha = 0$ for S_C ; $\alpha = 1$ for S_T .

We introduced as usual a magnetic field $\vec{H} = (H, 0, 0)$ to avoid infra-red divergences and we defined $\vec{\phi}(x) = [\sqrt{1 - \vec{\pi}^2(x)}, \vec{\pi}(x)] \simeq [1 - 1/2 \vec{\pi}^2(x), \dots, \vec{\pi}(x)]$ in the limit $\beta \rightarrow \infty$. At first order in perturbation theory the propagator will be given by

$$G(k_1)_{\alpha\beta} = \langle \pi(k_1)_\alpha \pi(k_2)_\beta \rangle = \delta_{\alpha\beta} \delta^2(k_1 - k_2) \left[G_0(k_1) + \frac{G_0(k_1)^2}{(2\pi)^2} - G_0(k_1)^2 \int d^2p \left\{ 4(n-1) \tilde{V}_4(-k_1, k_1, p, -p) + 8 \tilde{V}_4(-k_1, p, -p, k_1) \right\} G_0(p) \right] \quad (B.3)$$

$$G_0(k) = \frac{a^2 (2\pi)^2}{\beta \left[4 \sum_{\vec{\mu}} \sin^2 \frac{\vec{k} \cdot \vec{\mu}}{2} + Ha^2 \right]} \quad \tilde{V}_4 = \frac{\beta}{8a^2 (2\pi)^6} V_4 \quad (B.3) \text{ cont.}$$

Neglecting terms proportional to $k_1 a^2 Ha^2$ and writing $G(K_1)_{\alpha\beta} = \delta_{\alpha\beta} \delta^2(K_1 - K_2)$ $[G_0(k_1) - \Delta\Gamma(k_1)G_0(k_1)^2/(2\pi)^2]$ we find:

$$\Delta\Gamma(K_1) = \frac{1}{(2\pi)^2} \left\{ -\frac{(N-1)a^2}{2} I_0 H + \sum_{\vec{\mu}} \sin^2 \frac{\vec{k}_1 \cdot \vec{\mu}}{2} \cdot \left[\left(\frac{2\pi}{a} \right)^2 - 4 I_0 - \frac{(N+1)}{3} \left(\frac{2\pi}{a} \right)^2 \alpha \right] \right\} \quad (B.4)$$

$$I_0 = \int \frac{d^2 p}{4 \sum_{\vec{\mu}} \sin^2 \frac{\vec{p} \cdot \vec{\mu}}{2} + Ha^2} \quad (B.5)$$

In the limit $H \rightarrow 0$, $I_0 \approx -\pi/a^2 \ln(Ha^2/32)$

From Eqs. (B.4) and (B.5) we have

$$Z_1 = 1 + \frac{(N-1)}{4\pi\beta} \ln \frac{a^2}{32} \quad (B.6)$$

$$Z_2 = 1 + \frac{1}{\beta} \left[\frac{(N-2)}{4\pi} \ln \frac{a^2}{32} + \left(\frac{N+1}{12} \alpha - \frac{1}{4} \right) \right]$$

Z_1 and Z_2 are the field and coupling constant renormalization constants.

Using PV regulators for the continuum theory we had found:

$$Z_2 = 1 + \frac{1}{\beta} \frac{(N-2)}{4\pi} \ln \frac{1}{\mu^2} \quad (B.7)$$

μ^2 is the PV renormalization scale.

From (B.6) and (B.7) we get:

$$\Lambda_{PV}^2 = \Lambda_{LATTICE}^2 32 \exp - \frac{[(N+1)\alpha - 3]}{3(N-2)} \pi$$

This gives $\Lambda_{PV}^2 = 32 \exp \pi/(N-2) \Lambda_c^2$ and $\Lambda_{PV}^2 = 32 \exp -\pi/3 \Lambda_T^2$.

Note that with the Manton action the relation between $\Lambda_{lattice}$ and Λ_{PV} does not depend on N .

Table 1

β	Number of Sweeps
0.97	5200
1.05	5200
1.15	4000
1.45	4000
1.55	4000
1.65	8600

Table 2

L	1	2	4	8	16	β
$10^4 \cdot \frac{\langle Q_L^2 \rangle}{V}$	27.2±1.7	17.6±0.9	12.8±0.6	7.95±0.20	3.10±0.03	1.45
	15.9±0.7	10.4±0.6	7.10±0.40	5.05±0.15	2.64±0.03	1.55
	7.85±0.80	5.58±0.43	3.95±0.29	3.13±0.14	2.13±0.05	1.65

Table 3

L	1	2	4	8	16	β
$10^4 \frac{\langle Q_L^2 \rangle}{V}$	26.2 ± 0.9	19.5 ± 0.8	14.1 ± 0.4	8.65 ± 0.15	3.15 ± 0.02	0.97
	17.4 ± 1.1	13.5 ± 1.0	9.45 ± 0.70	6.50 ± 0.25	2.95 ± 0.04	1.05
	6.69 ± 0.43	4.74 ± 0.73	3.79 ± 0.51	3.44 ± 0.27	2.34 ± 0.04	1.15

Table 4

L, L'	1-2	2-4	4-8	8-16	β
$10^4 \Delta(L, L')$	9.6 ± 2.6	4.8 ± 1.5	4.85 ± 0.80	4.85 ± 0.23	1.45
	5.6 ± 1.3	3.30 ± 1.0	2.05 ± 0.55	2.41 ± 0.18	1.55
	2.3 ± 1.2	1.63 ± 0.72	0.82 ± 0.43	1.00 ± 0.19	1.65

Table 5

L, L'	1-2	2-4	4-8	8-16	β
$10^4 \Delta(L, L')$	6.7 ± 1.7	5.4 ± 1.2	5.45 ± 0.55	5.50 ± 0.17	0.97
	3.9 ± 2.1	4.1 ± 1.7	2.95 ± 0.95	3.55 ± 0.29	1.05
	2.0 ± 1.1	0.95 ± 1.20	0.35 ± 0.78	1.10 ± 0.30	1.15

TABLE 6

$\Delta(L,2L) \cdot 10^4$	$\Delta(L,2L)_{DGA} \cdot 10^4$	β
5.45	5.07	0.97
3.53	2.17	1.05
0.80	0.74	1.15
4.83	1.79	1.45
2.59	0.58	1.55
1.15	0.19	1.65

TABLE CAPTIONS

- Table 1 : Number of useful sweeps used to obtain our experimental data.
- Tables 2 and 3 : The values of $\langle Q_L^2 \rangle / V$ for various L and β are given: table 2 refers to the Monte Carlo results using the action S_C ; table 3 refers to S_T .
- Tables 4 and 5 : The values of $\Delta(L,L')$ defined in Eq. (3.4) against β .
- Table 6 : The average value of $\Delta(L,2L)$ and the values obtained using the dilute gas approximation $\Delta(L,2L)_{DGA}$ are compared at various values of β .

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FIGURE CAPTIONS

- Figure 1 : ● represent the lattice sites. At each site there is a spin $\vec{\phi}(\vec{x})$. Q_F is the fractional charge contained in a square $L \times L$. $L = 2$ in the figure. The total topological charge Q_L is the sum over all the $L \times L$ squares of Q_F . This quantity is always an integer for periodic boundary conditions.
- Figure 2a-b : In Figure 2a a topological configuration is shown which does not contribute to the total topological charge with a mesh of $L \times L$ squares. The instanton of Figure 2b instead will give a non-zero contribution.
- Figure 3 : Smallest possible instanton on the lattice.
- Figures 4 and 5 : We compare our data with the renormalization group expected behaviour. The dashed line is the expected behaviour. The dotted dashed line is obtained by taking L fixed as in Ref. 3). The full line is obtained by taking L/ξ fixed. Note that for S_T (Fig. 5) the predicted behaviour agrees very well with the experimental results.

Fig. 1

Fig. 2a

Fig. 2b

Fig. 3

Fig. 4

Fig. 5