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SPONTANEOUS COMPACTIFICATION IN HIGHER DIMENSIONAL GRAVITY

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A B S T R A C T

Examples for spontaneous compactification in pure higher dimensional gravity are presented. Invariants at least in second order of the curvature tensor have to be included in the action in order to stabilize the scalar potential.

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It is an old conjecture ¹⁾ that gravity in more than four dimensions could serve to unify ordinary four-dimensional gravity and gauge interactions. This requires a ground state consisting of a direct product (at least locally) of four-dimensional Minkowski space with a D dimensional compact manifold admitting a group G of co-ordinate transformations. The early ideas have been generalized ²⁾ to non-Abelian gauge groups G . A model based on $G = SO(10)$ has been proposed ³⁾ as a candidate for a unified gauge theory.

Here we explore some of the conditions for such a ground state to be realized. We only consider pure higher dimensional gravity without supplementary matter fields. We work within the classical field theory which we consider to be a good approximation for distances large compared with the characteristic length scale appearing in the action. In this context, the action for higher dimensional gravity should only be interpreted as an effective action for long distances. (One may hope that such an effective action can be derived from some consistent short-distance quantum theory of gravity, which unfortunately does not exist so far.) Especially, only quantum fluctuations involving length scales large compared with the characteristic length scale can be treated consistently within this approach. As well, unitarity is required only for large length scales, but may break down at or above the characteristic length scale.

In general, the action for such an effective infra-red theory can contain all possible invariants under general co-ordinate transformations and local Lorentz transformations in d dimensions. (Even if not present in the underlying short-distance action, such invariants are often generated by quantum fluctuations.) However, the couplings of invariants with higher mass dimensions are suppressed by inverse powers of the characteristic mass ^{*)} compared with the couplings of invariants with a low mass dimension. At large length scales only the invariants with a low mass dimension will give sensible contributions. Four-dimensional gravity may serve as an example for this kind of infra-red theory: the invariant with lowest mass dimension is the scalar curvature R and the Einstein equations are derived from an action only containing R . However, nothing excludes invariants like R^2 or $R_{\mu\nu} R^{\mu\nu}$ in the effective action. At a small mass scale M all effects of such supplementary terms are suppressed by M^2/M_p^2 (M_p is the Planck mass) and are therefore unobservable for present experiments.

We will see that without supplementary matter fields invariants with mass dimension greater than two are crucial for spontaneous compactification. Indeed, an effective action containing only R cannot lead to spontaneous compactification.

^{*)} Couplings of invariants with mass dimension greater than the total space dimension d may be forbidden or suppressed by inverse powers of an even higher mass scale existing in the underlying quantum theory.

On the other hand, we demonstrate that an action containing the invariants R , R^2 , $R_{\mu\nu} R^{\mu\nu}$ and $R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda}$ leads to spontaneous compactification.

How can one decide if an assumed ground state is acceptable? The most obvious strategy would be to require that it minimizes the action. However, this condition may well be too strong since it is not met by ordinary Einstein gravity which nevertheless is believed to be a good infra-red theory with Minkowski space as a ground state. We therefore require four weaker conditions for the ground state of higher dimensional gravity :

- 1) the ground state obeys the classical field equations ;
- 2) four-dimensional gravity is governed by an effective R term with a positive non-vanishing Planck constant ;
- 3) the effective four-dimensional cosmological constant is zero. One parameter of the action may be adjusted unnaturally to achieve this ;
- 4) in the Euclidean version of the reduced four-dimensional theory all non-gravitational (vector and scalar) excitations, characterized by length scales larger than the Planck length, have positive action.

A ground state obeying these four conditions is roughly on equal footing with Minkowski space as a ground state of ordinary Einstein gravitation. The well-known problems of four-dimensional gravity ^{*)} are still there, but at least no new problems are added.

It is easy to see that an effective action of the form (e is the determinant of the vielbein)

$$S = -\rho \int d^d x e (R + \lambda) \quad (1)$$

cannot lead to spontaneous compactification of a $d-4$ dimensional internal space. Condition 3) would require the curvature R_I of the internal space to be $-\lambda$ whereas the field equations lead to $R_I = -\lambda(d-4)/(d-6)$ which is only compatible for $R_I = \lambda = 0$. In order to circumvent this problem some authors ⁴⁾ have considered the combined action of gravity and Yang-Mills gauge fields in higher dimensions. We will not follow this line here, but rather concentrate on the possibility of invariants with higher mass dimension in the effective action.

^{*)} Here we include the possible danger of tachyons or ghosts in the gravitational part of the theory.

The most general action involving invariants with mass dimension ≤ 4 and vanishing torsion is a five-parameter action :

$$S = -\frac{1}{N} \int d^d x e (\alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda} + \delta R + \epsilon) \quad (2)$$

Here $R_{\mu\nu\sigma\lambda}$ is the d dimensional curvature tensor. N is a normalization constant chosen to be the volume of the $d-4$ dimensional compact manifold characterizing the ground state. Then α, β and γ are dimensionless parameters, δ has dimension mass squared and $\epsilon \sim (\text{mass})^4$. For $\epsilon \neq 0$ the d dimensional Minkowski space does not fulfil the field equations and the ground state must have non-vanishing curvature.

First we ask if a ground state consisting of a direct product of a D dimensional hypersphere ($D = d-4$) and four-dimensional Minkowski space can fulfil the above four conditions. For $S^D \times \mathcal{M}^4$ the curvature tensor is given by

$$R_{\mu\nu\sigma\lambda} = \begin{cases} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) y & \text{for } \mu, \nu, \sigma, \lambda \text{ } S^D \text{ indices} \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The variable y is related to the radius L of the hypersphere by $y = L^{-2}$ and has to be determined by minimizing the (Euclidean) action with respect to y . We may write S in the form

$$S = -\frac{1}{N} \int d^d x e (K + \epsilon) \quad (4)$$

where K is given in our case by

$$K = \{ \alpha D^2(D-1)^2 + \beta D(D-1)^2 + 2\gamma D(D-1) \} y^2 - \delta D(D-1) y \quad (5)$$

Condition (3) implies that we have to minimize K and then adjust ϵ in order to obtain $K_{\min} + \epsilon = 0$. Since e is positive definite this also minimizes the Euclidean action with respect to y and guarantees a zero four-dimensional cosmological constant. One easily finds for the minimum value $\bar{y}$:

$$\bar{y} = \frac{\delta}{2\alpha D(D-1) + 2\beta(D-1) + 4\gamma} \quad (6)$$

where $\partial^2 K / \partial y^2 > 0$ requires

$$\alpha + \frac{1}{D} \beta + \frac{2}{D(D-1)} \gamma > 0 \quad (7)$$

Furthermore, for a hypersphere γ has to be positive implying

$$\delta > 0 \quad (8)$$

Finally ϵ has to be adjusted to be

$$\epsilon = \frac{\delta^2}{4} \frac{D(D-1)}{\alpha D(D-1) + \beta(D-1) + 2\gamma} \quad (9)$$

It remains to check if the conditions (1), (2) and (4) are fulfilled.

Planck's constant is given by the term linear in the four-dimensional curvature when we expand the action (2) around the presumed ground state :

$$\frac{1}{4\kappa^2} = \delta - 2\alpha D(D-1)\bar{\gamma} = \delta \frac{\beta(D-1) + 2\gamma}{\alpha D(D-1) + \beta(D-1) + 2\gamma} \quad (10)$$

Thus condition (2) is met provided

$$\beta(D-1) + 2\gamma > 0 \quad (11)$$

At this point it is interesting to note that for α , β and γ all of order one, the Planck mass squared is suppressed by a factor $1/D$ compared with the characteristic mass scale given by δ . The inverse radius squared of the hypersphere is even suppressed by $1/D^2$ compared with δ . Thus the natural small parameter $1/D$ related to a high dimensional hypersphere could justify the use of the approximation of an effective infra-red theory. (Note that larger values of α even enhance this effect.)

The classical field equations derived from (2) are ⁵⁾ :

$$\begin{aligned} & 2\alpha R R^{\mu\nu} + 2\beta R^{\mu}_{\sigma} R^{\sigma\nu} + 2\gamma R^{\mu}_{\sigma\tau\rho} R^{\sigma\tau\rho\nu} + \delta R^{\mu\nu} \\ & - \frac{1}{2} g^{\mu\nu} (\alpha R^2 + \beta R_{\sigma\tau} R^{\sigma\tau} + \gamma R_{\sigma\tau\rho\lambda} R^{\sigma\tau\rho\lambda} + \delta R + \epsilon) \\ & + 2\alpha (g^{\mu\nu} R_{;\sigma}{}^{\sigma} - R_{;\mu}{}^{\mu}) + \beta (g^{\mu\nu} R^{\sigma\tau}_{;\sigma\tau} + R^{\mu\nu}_{;\sigma}{}^{\sigma}) \\ & - R^{\mu\sigma}_{;\sigma}{}^{\nu} - R^{\nu\sigma}_{;\sigma}{}^{\mu}) + 2\gamma (R^{\mu\sigma\tau\nu}_{;\sigma\tau} + R^{\mu\sigma\tau\nu}_{;\tau\sigma}) = 0 \end{aligned} \quad (12)$$

For a ground state $S^D \times M^4$ the covariant derivatives of the curvature tensor vanish and Eqs. (6) and (9) assure that the field equations are indeed fulfilled.

Let us finally discuss condition (4) which necessitates a dimensional reduction of the action (2). Consistent with $SO(D+1)$ and Poincaré invariance, the Lagrange density for the vector and scalar fields after dimensional reduction will contain many different invariants with up to four derivatives. [For example, there will be terms like $(F_{\mu\nu} F^{\mu\nu})^2$ or $R_{\mu\nu} F^{\mu\rho} F^{\nu\rho}$, etc., where $F_a^{\mu\nu}$ is the field strength of the gauge fields.] However, since we are only interested in excitations with mass scales below the Planck mass, only second derivatives are relevant for the kinetic terms of vectors and scalars. To leading approximation in $1/D$ we can neglect the terms $\beta R_{\mu\nu} R^{\mu\nu}$ and $\gamma R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda}$ in (2). Dimensional reduction for the kinetic terms of scalars and vectors is then completely parallel to the action (1), with $\rho = (1/N) \cdot (1/4K^2)$ and $\lambda = 0$. It has been carried out in Ref. 3) and indeed yields positive gauge invariant kinetic terms^{*)}. Especially the gauge coupling has been found to be

$$g^2 \approx \frac{D+1}{2} 4\kappa^2 \bar{y} = \frac{1}{4} \frac{D+1}{D-1} \frac{1}{\beta + 2\gamma/(D-1)} \quad (13)$$

For $\beta \approx 1$ this is in reasonable agreement with the value $g^2 \approx 1/4$ required for a realistic asymptotically free $SO(10)$ theory. The kinetic terms also have contributions from the terms $\beta R_{\mu\nu} R^{\mu\nu}$ and $\gamma R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda}$ where one of the factors is the curvature tensor for S^D . For α, β and γ of the same order of magnitude, however, these contributions are suppressed by $1/D$ and cannot change the sign of the kinetic terms.

Then condition (4) is met provided the classical scalar potential

$$V = \frac{1}{N} \int d^D x E (\alpha R_I^2 + \beta R_{I\mu\nu} R_I^{\mu\nu} + \gamma R_{I\mu\nu\sigma\lambda} R_I^{\mu\nu\sigma\lambda} + \delta R_I + \varepsilon) \quad (14)$$

has a minimum for our assumed ground state. Here $R_{I\mu\nu\sigma\lambda}$ is the curvature of the D dimensional space, E is the determinant of the D dimensional vielbein and the integration is over D internal co-ordinates. We note that (14) is just the Euclidean version of the action (2), except that all quantities are now related to the $d=4$ dimensional internal space. (In the further discussion, we drop the index I for the internal curvature.) Even if our four conditions do not require

*) Dimensional reduction for the massive spin one fields has not been carried out in Ref. 3). We expect, however, no problems to arise from these fields.

the total action of the ground state to be minimal, this nevertheless must be true for the internal manifold. If not, there would be unstable scalar excitations which are local in Minkowski space and this is certainly not acceptable for a viable ground state.

In order to check if $S^D \times \mathcal{M}^4$ corresponds to a minimum of V we split the curvature tensors into irreducible pieces :

$$\begin{aligned}\tilde{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{D} R g_{\mu\nu} \\ \tilde{R}_{\mu\nu\sigma\lambda} &= R_{\mu\nu\sigma\lambda} + \frac{1}{D(D-1)} R (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) \\ &\quad - \frac{1}{D-2} (\tilde{R}_{\mu\sigma} g_{\nu\lambda} + \tilde{R}_{\nu\lambda} g_{\mu\sigma} - \tilde{R}_{\mu\lambda} g_{\nu\sigma} - \tilde{R}_{\nu\sigma} g_{\mu\lambda})\end{aligned}\quad (15)$$

which obey the relations

$$\begin{aligned}\tilde{R}_{\mu\nu} g^{\mu\nu} &= 0 \\ \tilde{R}_{\mu\nu\sigma\lambda} g^{\mu\sigma} &= \tilde{R}_{\mu\nu\sigma\lambda} g^{\nu\lambda} = 0\end{aligned}\quad (16)$$

The quantity K [Eq. (4)] which has to be minimized can be expressed as a function of the independent tensors R , $\tilde{R}_{\mu\nu}$ and $\tilde{R}_{\mu\nu\sigma\lambda}$:

$$K = \tilde{\alpha} (R + \tilde{\delta})^2 + \tilde{\beta} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + \tilde{\gamma} \tilde{R}_{\mu\nu\sigma\lambda} \tilde{R}^{\mu\nu\sigma\lambda} - \tilde{E}\quad (17)$$

with

$$\begin{aligned}\tilde{\alpha} &= \alpha + \frac{1}{D} \beta + \frac{2}{D(D-1)} \gamma \\ \tilde{\beta} &= \beta + \frac{4}{D-2} \gamma \\ \tilde{\gamma} &= \gamma \\ \tilde{\delta} &= \delta / 2\tilde{\alpha} \\ \tilde{E} &= \delta^2 / 4\tilde{\alpha}\end{aligned}\quad (18)$$

For $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ all positive, V has its minimum at

$$\begin{aligned}\tilde{R}_{\mu\nu\sigma\lambda} &= 0 \\ \tilde{R}_{\mu\nu} &= 0 \\ R &= -\tilde{\delta}\end{aligned}\tag{19}$$

provided we adjust $\epsilon = \tilde{\epsilon}$. It is easy to check that the above ansatz for a ground state $S^D \times \mathcal{M}^4$ satisfies (19). In conclusion, for $\tilde{\alpha}$, $\beta+2\gamma/(D-1)$, γ and δ all positive and $\epsilon = \tilde{\epsilon}$, d dimensional gravity spontaneously compactifies with the ground state $S^D \times \mathcal{M}^4$.

What about more complex ground states ? [For example, a ground state $S^9 \times S^4 \times \mathcal{M}^4$ in 17 dimensional gravity has been proposed ³⁾ to combine the unification group $SO(10)$ with a generation group $SO(5)$.] Rather generally, we believe that the search for a ground state obeying our four conditions can be reduced to the search for a minimum of V , the adjustment of the d dimensional cosmological constant ϵ so that $V_{\min} = 0$ and the check that Planck's constant is positive. (The generalization of V for more general actions is obvious.) Indeed, the minimization of V and the adjustment of ϵ automatically guarantee that the field equations are fulfilled. For a ground state $\mathcal{M}^4 \times$ compact manifold the curvature tensors and all its covariant derivatives vanish if they carry at least one Minkowski index. Then the only contribution to the field equations where Minkowski indices appear is of the form $-\frac{1}{2}g^{\mu\nu}(K+\epsilon)$ which vanishes by the prescription for the adjustment of ϵ . The only equations which remain are equations only involving quantities corresponding to the compact manifold. These equations can be entirely derived from the variation of the "internal" action V and are fulfilled for the minimum of V . The same arguments as above apply for the positive sign of scalar and vector kinetic terms. [The dimensional reduction in Ref. 3) has been carried out for arbitrary compact internal manifolds admitting a symmetry group.]

Let us again investigate a concrete example : a ground state based on $S^E \times S^F \times \mathcal{M}^4$. This corresponds to

$$\begin{aligned}R &= -E(E-1)y_E - F(F-1)y_F \quad \text{for } \mu, \nu \text{ } S^E \text{ indices} \\ \tilde{R}_{\mu\nu} &= \begin{cases} \frac{F}{E+F}((F-1)y_F - (E-1)y_E) g_{\mu\nu} & \text{for } \mu, \nu \text{ } S^F \text{ indices} \\ \frac{E}{E+F}((E-1)y_E - (F-1)y_F) g_{\mu\nu} & \text{otherwise} \\ 0 \end{cases}\end{aligned}\tag{20}$$

$$\tilde{R}_{\mu\nu\sigma\lambda} = \begin{cases} \frac{F(F-1)}{(E+F-1)(E+F-2)} (y_E + y_F) (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) & \text{for } \mu, \nu, \sigma, \lambda \text{ } S^E \text{ indices} \\ \frac{E(E-1)}{(E+F-1)(E+F-2)} (y_E + y_F) (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}) & \text{for } \mu, \nu, \sigma, \lambda \text{ } S^F \text{ indices} \\ \frac{(E-1)(F-1)}{(E+F-1)(E+F-2)} (y_E + y_F) g_{\mu\sigma} g_{\nu\lambda} & \text{for } \mu, \sigma \text{ } S^E \text{ indices ; } \nu, \lambda \text{ } S^F \text{ indices} \\ + \text{ terms obtained from index permutations} \\ 0 & \text{otherwise} \end{cases}$$

(20)
Cont.

Clearly a value $\tilde{R}_{\mu\nu\sigma\lambda} \neq 0$ is needed, which is only possible for $\gamma < 0$. In this case, however, V is not bounded from below and we need to include invariants up to a mass dimension 8 [For example, terms like $(R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda})^2$] in the action in order to stabilize the scalar potential. Without going into detail, it becomes obvious that an action with many invariants of high mass dimension can lead to a variety of rather complex ground states.

At this point, some remarks on the predictive power of such models of higher dimensional gravity are in order. In general, we expect the effective infra-red theory to contain all possible invariants at least up to a mass dimension $\leq d$. For large d , the number of free parameters becomes very large. It may be possible to arrange the parameters to obtain nearly every phenomenologically required scalar potential. This enhances the chances of constructing a realistic model out of higher dimensional gravity, but also diminishes considerably the predictive power for the scalar part of the theory. However, we remind ourselves that all Yukawa couplings of fermions are in principle calculable, once the ground state of the theory has been fixed.

In conclusion, we have shown how pure higher dimensional gravity can lead to spontaneous compactification. In order to have a stable ground state, invariants with mass dimension greater than two have to be included in the action. In general, rather complex ground states can be realized by an appropriate choice of parameters.

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