Ref.TH.2986-CERN

SHIFMAN-VAINSHTEIN-ZAKHAROV MOMENTS AND
QUARK - ANTIQUARK POTENTIALS

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A B S T R A C T

From the SVZ moments, for heavy quark-antiquark systems, we extract an "equivalent" non-relativistic potential. It increases as the fourth power of the distance and linearly with the quark mass. Numerical integration of the "equivalent" Schrödinger equation leads to much higher estimates of the gluon condensate parameter ϕ_1 than the original inverse power moment method or even the Borel transformed exponential version.

*) Supported in part by "Fonds zur Förderung der wissenschaftlichen Forschung in Österreich", Project No. 3800.

1. INTRODUCTION

Shifman, Vainshtein and Zakharov ¹⁾ (SVZ hereafter) have developed a very original method for obtaining the masses of quark-antiquark bound states from QCD. It involves certain approximations for certain integrals (moments) over the cross-section for e^+e^- annihilation into quark-antiquark pairs. To gain some insight into the method, it has been applied also to potential models ^{2),3)}, where the approximate results can be compared with accurate values obtained by integration of the Schrödinger equation. We considered ³⁾ in particular potentials of the type conventionally used in potential models of charmonium. But these potentials were unrelated to the specific QCD calculations of SVZ. Our object here is to repair this deficiency.

In Section 2 we present formulae for the exponential (or Borel-transformed ¹⁾) moments. They are obtained by a limiting process from QCD formulae for inverse power moments, kindly supplied to us by Reinders, Rubinstein and Yazaki (RRY hereafter). These authors ⁴⁾ generalized in a suitable way the original calculations of SVZ. Although SVZ advocated ¹⁾ the exponential version as an improvement, and used it in their work with light quarks, it was the original inverse-power version which both SVZ ¹⁾ and RRY ⁴⁾ used in their work on charmonium. We find that the exponential moment method requires a considerably higher value for the gluon condensate parameter ϕ_1 , if the quark mass and coupling constant are held fixed.

In Section 3 we argue for the use of an "on-shell" quark mass $\bar{m}$, rather than the "off-shell" mass m of SVZ and RRY, in the exponential version of the method.

In Section 4 we compare the QCD moments, in non-relativistic approximation, with those appropriate to a superposition of power-law potentials. This permits identification of an "equivalent" potential - a Coulomb potential plus one increasing as the fourth power of the distance - the strength of the latter being proportional to the quark mass.

In Section 5 we compare with the original results of SVZ and RRY (obtained with inverse power moments) results for charmonium of the exponential moment method, of the non-relativistic approximation to the latter, and of the numerical integration of the "equivalent" Schrödinger equation. We find that to fit the given level position, the exponential moment method

requires a much higher value for the gluon condensate parameter ϕ_1 than the original inverse-power version (holding fixed $\bar{m}$ and α_s). The non-relativistic approximation to the exponential moment method differs little from the relativistic. But with numerical integration of the Schrödinger equation a still higher value of ϕ_1 is required.

2. EXPONENTIAL MOMENTS FROM QCD

We work with the exponential moments

$$\mathcal{M}(\sigma) = \int ds e^{-\sigma s} \text{Im} \Pi(s) \quad (1)$$

If s_1 is the position of the lowest resonance, clearly

$$s_1 \leq -\frac{d}{d\sigma} \log \mathcal{M}(\sigma) \quad (2)$$

We will use on the right the expression for free quarks augmented by corrections of first order in the strong coupling α_s and the SVZ confinement parameter ϕ ¹⁾. Minimizing this with respect to σ defines our approximation for s_1 . We found that the non-relativistic version of this procedure (with corrections of first order in a potential) worked surprisingly well³⁾.

The exponential moments can be obtained by a limiting process from the inverse-power moments

$$\mathcal{M}_n(Q^2) = \int ds \text{Im} \Pi(s) (s + Q^2)^{-1-n} \quad (3)$$

These have been calculated for arbitrary Q^2 for (in particular) systems like $c\bar{c}$ and $b\bar{b}$ in 1^- states by Reinders, Rubinstein and Yazaki⁴⁾. Using their results and

$$\lim_{n \rightarrow \infty, Q^2 = n/\sigma} (Q^2)^{n+1} \mathcal{M}_n(Q^2) = \mathcal{M}(\sigma) \quad (4)$$

we find, to first order in α_s and ϕ

$$\mathcal{M}(\sigma) = e^{-4m^2\sigma} \pi A [1 + \alpha_s a + \phi b] \quad (5)$$

where

$$\pi A = \frac{3}{16} \frac{1}{\sqrt{\pi}} \frac{1}{\sigma} G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (6)$$

$$a = \frac{4}{3} \frac{1}{\sqrt{\pi}} \left[\pi - c_1 G(1, 2, \omega) + \frac{1}{3} c_2 G(2, 3, \omega) \right] / G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \\ - c_2 - c_3 \omega G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) / G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (7)$$

$$b = -\frac{1}{2} \omega^2 G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) / G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (8)$$

with

$$\omega = 4m^2 \sigma \quad (9)$$

$$c_1 = \pi/3 + \pi/4 - 3/(8\pi) \quad (10)$$

$$c_2 = \pi/2 - 3/(4\pi) \quad (11)$$

$$c_3 = (4/\pi) \ln 2 \quad (12)$$

$$G(b, c, \omega) = (\Gamma(c))^{-1} \int_0^\infty dt e^{-t} t^{c-1} (\omega+t)^{-b} \quad (13)$$

Finally m is the off-shell quark mass of SVZ, and ϕ is related to their "gluon condensate" :

$$\phi = \phi_1 / (4m^2)^2 \quad (14)$$

$$\phi_1 = (4\pi\alpha_s/9) \langle 0 | G_{\mu\nu}^a G_{\mu\nu}^a | 0 \rangle \quad (15)$$

3. MASS RENORMALIZATION

The c_3 term in a is a mass renormalization term associated with the use of the off-shell mass m rather than the on-shell mass, say $\bar{m}$, where

$$\bar{m}^2 = m^2 (1 + c_3 \alpha_s) \quad (16)$$

SVZ found that in the inverse power moments, the c_3 term tends to cancel with others, reducing the net correction to the free quark terms. We do not find such a cancellation among first order corrections in

$$-\frac{d}{d\sigma} \log M(\sigma)$$

The c_3 term tends like the others to push s_1 upwards from $2m$. We prefer therefore to work in terms of $\bar{m}$, modifying the above formulae by the substitutions

$$m \rightarrow \bar{m} \quad c_3 \rightarrow 0 \quad (17)$$

One can of course verify that this makes no difference to first order in α_s and ϕ . The concept of "on-shell-perturbative" mass, i.e., $\bar{m}$, may well lose meaning when confinement is fully allowed for. But its use presents no difficulty at this stage.

From (5), with (17), we have now

$$-\frac{d}{d\sigma} \log M(\sigma) = 4\bar{m}^2 - \frac{d}{d\sigma} (\log A + \alpha_s a + \phi b) \quad (18)$$

retaining only first order corrections in α_s and ϕ .

4. "EQUIVALENT" POTENTIAL

For large σ the lower s values are weighted in (1). Non-relativistic approximations might then be relevant. Retaining in each term only the leading power of σ^{-1} , and using

$$G(b, c, \omega) \sim \omega^{-b} \quad (19)$$

for large ω , we have

$$\left. \begin{aligned} A &\sim \frac{3}{16} \frac{1}{\sqrt{\pi}} \frac{1}{2\bar{m}} \sigma^{-3/2} \\ a &\sim \frac{4}{3} \sqrt{\pi} \omega^{1/2} - c_2 \\ b &\sim -\omega^3/2 \end{aligned} \right\} \quad (20)$$

and from (18),

$$-\frac{d}{d\sigma} \log M(\sigma) \sim 4\bar{m}^2 + \frac{3}{2\sigma} + 4\bar{m}^2 \left[-\frac{2}{3} \sqrt{\frac{\pi}{\omega}} \alpha_s + \frac{3}{2} \omega^2 \phi \right] \quad (21)$$

Consider now the non-relativistic Hamiltonian

$$p^2/(2\mu) + V(r) \quad (22)$$

with

$$V(r) = \sum_s \lambda_s r^s \quad (23)$$

For moments

$$M(r) = \int dE e^{-rE} \text{Im} \Pi \quad (24)$$

we have from BB³⁾, to first order in the λ ,

$$-\frac{d}{dr} \log M(r) = \frac{3}{2r} + \sum_s \lambda_s \Gamma(2+s/2) (r/2\mu)^{s/2} \quad (25)$$

Let us define E by

$$(E + 2\bar{m})^2 = S \quad (26)$$

Note that this fixes an otherwise arbitrary constant in defining E and λ_0 . Then we have the non-relativistic correspondence between (1) and (24)

$$4\bar{m}\sigma \rightarrow \tau \quad (27)$$

$$e^{4\bar{m}^2\sigma} M(\sigma) \rightarrow 4\bar{m} M(\tau) \quad (28)$$

$$\begin{aligned} (4\bar{m})^{-1} \left\{ -\frac{d}{d\sigma} \log M(\sigma) - 4\bar{m}^2 \right\} \\ \rightarrow -\frac{d}{d\tau} \log M(\tau) \end{aligned} \quad (29)$$

Comparing (21) and (25),

$$\begin{aligned} -\alpha_s \frac{2}{3} \sqrt{\pi \bar{m}/\tau} + \phi \frac{3}{2} \bar{m}^3 \tau^2 \\ \rightarrow \sum_s \lambda_s \Gamma(2+s/2) (\tau/2\mu)^{s/2} \end{aligned} \quad (30)$$

Using

$$2\mu = \bar{m} \quad (31)$$

(which must be correct at least in zero order) we see then that the "equivalent" potential is a superposition of Coulomb and quartic :

$$V = -\frac{4}{3} \frac{\alpha_s}{\tau} + \frac{\phi_1}{64} \bar{m} \tau^4 \quad (32)$$

5. NUMERICAL RESULTS

The relativistic exponential moment method gives for the ground state energy

$$\sqrt{s_1} = \left(4\bar{m}^2 + \left(\min_{\sigma} \right) \left(\frac{-d}{d\sigma} \right) \left(\log A + \alpha_s a + \phi b \right) \right)^{1/2} \quad (33)$$

The corresponding non-relativistic exponential moment method gives, from (25) with (32),

$$\overline{S}_1 = 2\overline{m} + \left(\frac{m}{r}\right)\left(\frac{-d}{dr}\right)\left(\frac{3}{2r} - \frac{2}{3}\alpha_s\sqrt{\frac{\pi\overline{m}}{r}} + \frac{3}{32}\overline{m}\phi_1\left(\frac{r}{\overline{m}}\right)^2\right) \quad (34)$$

We will compare results obtained with (33) and (34) with those obtained by numerical integration of the Schrödinger equation for potential (32).

In applying the inverse power moments to the $c\bar{c}$ system, RRY⁴⁾ determined as suitable parameters

$$\alpha_s = .27 \quad \phi_1 = .0484 \text{ (GeV)}^4 \quad (35)$$

and

$$\overline{m} = 1.26 \text{ GeV} \quad (36)$$

or [from (16)]

$$\overline{m} = 1.402 \text{ GeV} \quad (37)$$

With these parameters they had, for the J/ψ mass

$$3.100 \text{ GeV} \quad (38)$$

With the same parameters, (35)-(37), we find from the relativistic exponential moment method (33)

$$3.010 \text{ GeV} \quad (39)$$

and from the "equivalent" non-relativistic exponential moment method (34)

$$3.014 \text{ GeV} \quad (40)$$

while integration of the "equivalent" Schrödinger equation gives instead (for $2\overline{m}+E$)

$$2.928 \text{ GeV} \quad (41)$$

That the non-relativistic exponential moment method gives a result, (40), of order 100 MeV above the Schrödinger equation is quite typical of the variety of potentials we have considered for charmonium³⁾, including logarithmic and low power potentials (Appendix).

Instead of determining the level position for fixed values of the parameters we can ask for the parameters which give the desired level position. Let us suppose for simplicity that α_s and $\bar{m}$ have been determined correctly from relatively low moments [small n in (3) or small σ in (1)]. Then to obtain the correct (38) from the relativistic exponential moment method requires that ϕ_1 be increased from the RRY value (35) by a factor of about 2.3 :

$$\phi_1 \rightarrow .11 \text{ (GeV)}^4$$

The non-relativistic exponential moment method is in close agreement with the relativistic, but with the non-relativistic Schrödinger equation we would have to increase ϕ_1 by a factor of about 5.2 :

$$\phi_1 \rightarrow .25 \text{ (GeV)}^4$$

Very similar results are obtained with the original parameters of Shifman, Vainshtein and Zakharov ¹⁾

$$\alpha_s = .20 \quad m = 1.26 \quad (\bar{m} = 1.367) \quad \phi_1 = .0544 \text{ (GeV)}^4$$

The relativistic (or non-relativistic) exponential moment method requires instead

$$\phi_1 = .13 \text{ (GeV)}^4$$

and the non-relativistic Schrödinger equation requires

$$\phi_1 = .25 \text{ (GeV)}^4$$

These results strengthen our conjecture that ϕ_1 was substantially underestimated in the original work. But a conjecture it remains, for potential theory is not field theory - as emphasized by Shifman, Vainshtein and Zakharov in a private communication.

ACKNOWLEDGEMENTS

We are much indebted to L.J. Reinders, H.R. Rubinstein and S. Yazaki, for allowing us the use of their results in advance of publication, and to Dr. Rubinstein for valuable discussions of the method. We also profited from discussions with H. Grosse - and from the use of his Schrödinger equation computer program, developed from that of P. Hertel ⁵⁾.

APPENDIX - LOGARITHMIC AND LOW POWER POTENTIALS

The logarithmic potential

$$V = K \log r/r_0 \quad (\text{A.1})$$

in the Hamiltonian

$$p^2/(2\mu) + V$$

gives a ground state energy

$$E_1 = K \left(c - \frac{1}{2} \log 2\mu K r_0^2 \right) \quad (\text{A.2})$$

$$c = 1.0443 \quad (\text{A.3})$$

- by scaling arguments and numerical integration of the Schrödinger equation⁶⁾.
Since

$$K \log r/r_0 = \lim_{s \rightarrow 0} \frac{K}{s} \left[\left(\frac{r}{r_0} \right)^s - 1 \right] \quad (\text{A.4})$$

we can infer the result of applying the non-relativistic exponential moment method from the formulae given before for pure power potentials. The result is

$$E_1(\text{NREM}) = K \left(c' - \frac{1}{2} \log 2\mu K r_0^2 \right) \quad (\text{A.5})$$

$$c' = 1 + \frac{1}{2} \Gamma'(1) - \frac{1}{2} \log \frac{1}{3} = 1.2607 \quad (\text{A.6})$$

So

$$E_1(\text{NREM}) - E_1(\text{exact}) = K (c' - c) \quad (\text{A.7})$$

For charmonium and bottomonium a value of about

$$750 \text{ MeV} \quad (\text{A.8})$$

for K is appropriate. Then

$$E_1(\text{NREM}) - E_1(\text{exact}) \approx 1.62 \text{ MeV} \quad (\text{A.9})$$

Martin ⁷⁾ has proposed a potential

$$V = (-7.17 + 6.584 r^{.104}) \text{ GeV} \quad (\text{A.10})$$

with r in GeV^{-1} . Using for charmed quark mass

$$\bar{m}_c = 1.45 \text{ GeV} \quad (\text{A.11})$$

this gives on numerical integration of the Schrödinger equation

$$E_1 = 167 \text{ GeV} \quad (\text{A.12})$$

The exponential moment method, as usual, gives a higher value

$$E_1(\text{NREM}) - E_1(\text{exact}) \approx 140 \text{ MeV} \quad (\text{A.13})$$

These errors (A.9) and (A.13) are not very different from those which we obtained with Coulomb plus linear and Coulomb plus cubic potential in Ref. 3), (107 MeV and 83 MeV).

R E F E R E N C E S

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