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Erratum to "Behaviour at $x=0,1$, Sum Rules and Parametrizations for Structure Functions Beyond the Leading Order".^(*)

C. López and F.J. Ynduráin
Departamento de Física Teórica
Universidad Autónoma de Madrid
Canto Blanco, Madrid -34 Spain

Eq. (2.17): the 2 on the r.h.s. should be 6.

Eq. (2.18): a factor B_{NS} is missing on the r.h.s.

Page 8: In the expression for R , β_0^2 should be replaced by $2\beta_0^2$.

The expression for $\gamma^{(0)}(n)$ should read $-2\beta_0 D(n)$. The expression for $\underline{M}^{(r)}(n, r')$ should read $\exp(-3r' \underline{\gamma}^{(0)}/32)$.

Page 9: In the expression for $\underline{\Delta}$, replace $d\alpha'$ by $d\alpha'/4\pi$.

Eq. (2.21): the r.h.s. should read

$$\frac{1}{2\beta_0} \left\{ \left(\frac{\bar{\gamma}_{11}}{\bar{\gamma}_{21}(\alpha/\alpha_0)^{d_+ - d_-}} \frac{\bar{\gamma}_{12}(\alpha/\alpha_0)^{d_+ - d_-}}{d_+ - d_- + 1} \right) \frac{\alpha}{4\pi} - (\alpha \leftrightarrow \alpha_0) \right\}.$$

Eq. (2.23): the r.h.s. should read

$$\frac{-1}{2\beta_0} \left(\frac{\bar{\gamma}_{11} + 2\beta_1 d_+}{\bar{\gamma}_{21}} \frac{\bar{\gamma}_{12}}{d_+ - d_- + 1} \right).$$

Eq. (2.25.b) the r.h.s. should read

$$\begin{aligned} & -\frac{\beta_1 d_+(n)}{\beta_0} - \frac{\bar{\gamma}_{11}(n)}{2\beta_0} - \frac{\bar{\gamma}_{21}(n)}{2\beta_0(d_-(n) - d_+(n) + 1)} - \frac{D_{12}(n)}{d_-(n) - d_+(n)} \\ & - C_{11}(n) - [d_+(n) - D_{11}(n)] \frac{C_{12}(n)}{D_{12}(n)} \Big|_{n=1+\lambda_r} \end{aligned}$$

Eq. (2.25.c) the r.h.s. should read

$$\left[-\frac{\beta_1 d_+(n)}{\beta_0} - \frac{\bar{\gamma}_{11}(n)}{2\beta_0} \right] \frac{d_+(n) - D_{11}(n)}{D_{12}(n)} - \frac{\bar{\gamma}_{21}(n)}{2\beta_0 (d_-(n) - d_+(n) + 1)} \cdot \frac{d_-(n) - D_{11}(n)}{d_-(n) - d_+(n)}$$

$$- C_{12}(n) \frac{D_{21}(n)}{D_{12}(n)} - \left[C_{11}(n) + \frac{D_{22}(n) - D_{11}(n)}{D_{12}(n)} C_{12}(n) \right] \frac{d_+(n) - D_{11}(n)}{D_{12}(n)} \Big|_{n=1+\lambda_s}$$

Eq. (3.6.b) The expression for v_{INS} should read

$$v_{INS}(\alpha_s) = v_{NS}(\alpha_s) - \left[\frac{4}{3\pi} \psi(v_{NS}(\alpha_s) + 1) + a_1 \right] \alpha_s(Q^2).$$

Eq. (3.7) likewise

$$v_{IS}(\alpha_s) = v_S(\alpha_s) - \left[\frac{4}{3\pi} \psi(v_S(\alpha_s) + 1) + a_1 \right] \alpha_s(Q^2).$$

Page 15: The expression for $b_{01}(2)$ should read

$$b_{01}(2) = 3n_f \delta / (3n_f + 16), \delta = \langle Q_f^2 \rangle,$$

where Q_f is the charge of flavour f . In the four flavour model, $\delta = 5/18$.

Eq. (4.6): the r.h.s. should be multiplied by δ .

Eq. (4.7): the r.h.s. should be δ , not 1.

Eq. (4.8): the signs of $\bar{\gamma}_{21}$ should be (+) not (-). The term before last should read

$$k(2) \frac{\alpha_s^{1-d_-(2)}}{4\pi} \begin{pmatrix} -\Gamma_{12} - S_{12} \Gamma_{22} + C_{11} S_{12} + C_{12} S_{22} \\ -S_{21} - S_{22} \Gamma_{22} + C_{21} S_{12} + C_{22} S_{22} \end{pmatrix}$$

Eq. (4.9): the r.h.s. should be multiplied by δ .

Page 17: The first expression should be multiplied by δ .

Page 21: The first equation should read

$$\int_0^1 dx F_S(x, Q^2) = J_S.$$

In the like before last, P' should be ρ' .

Sec. 6: The δ here should not be identified with the quantity denoted by the same symbol in Sec. 4.

Page 27: The expression for $d(n)$ should read

$$d(n) = \frac{-1}{2\beta_0} \gamma_{NS}^{(0)}(n). \quad \text{The expression for } \gamma_{NS}^{(0)}(n) \text{ should be multiplied by } -32/3.$$

Page 28: The expression for $\underline{D}$ should read

$$\underline{D}(n) = \frac{-1}{2\beta_0} \gamma_{\underline{D}}^{(0)}(n), \quad \text{and that for } \gamma^{(0)} \text{ should be multiplied by } -32/3.$$

The expressions for $\gamma_{12}^{(1)}$, $\gamma_{21}^{(1)}$ should get an overall sign change.

Page 30: The expression for $C_{11}^{(1)}$ should start with $C_F\{\dots\}$, instead of $2C_F\{\dots\}$. The expression for $C_{12}^{(1)}$ should read

$$C_{12}^{(1)}(n) = 4T_R \left\{ -\frac{1}{n} + \frac{1}{n^2} + \frac{6}{n+1} - \frac{6}{n+2} - S_1(n) \frac{n^2+n+2}{n(n+1)(n+2)} \right\}.$$

Page 32: The values of a_0 , a_1 should

$$a_0 = -1.18, \quad a_1 = 0.66.$$

The authors apologize for this unusually large number of mistakes, most of them due to an unfortunate misunderstanding between them, caused by the physical separation during the writing of this paper.

BEHAVIOUR AT $x = 0, 1$, SUM RULES AND PARAMETRIZATIONS
FOR STRUCTURE FUNCTIONS BEYOND THE LEADING ORDER

C. Lopez

Departamento de Fisica Teorica
Universidad Autonoma de Madrid
Canto Blanco, Madrid - 34

and

F.J. Yndurain ^{*)}

CERN - Geneva

ABSTRACT

We write the simplest possible parametrizations of deep inelastic structure functions which satisfy the following requirements: i) exact compatibility with QCD at the end points $x = 0, 1$, to second order; ii) fulfilment of sum rules to second order; iii) leading and sub-leading Regge behaviour. In all we get that, including the QCD scale Λ , such parametrizations describe in a simple manner the three standard functions W_1, W_2, W_3 for all x, Q^2 in terms of four to six parameters only (two more if allowing for higher twists).

^{*)} Permanent address: Universidad Autonoma de Madrid.

1. INTRODUCTION

In a previous paper¹⁾ we considered the behaviour of the structure functions for deep inelastic scattering near the physical region end points, $x = 0, 1$. Using it, we were able to write very simple parametrizations of these structure functions, compatible with QCD to leading order in α_s at the end points, and satisfying approximately the relevant sum rules; these parametrizations turned out to represent well the experimental data.

In Ref. 1), only leading effects were considered: leading order in α_s , leading Regge singularities, verification of sum rules at the parton model level, twist two. The purpose of the present paper is to extend the analysis beyond the leading order. This will allow us not only to get explicit, accurate representations of the structure functions valid to very low Q^2 ; but it will, we hope, be a useful starting point for analyzing processes related to deep inelastic scattering via analytical continuation. Of course, the most rigorous method of analysis, for a single process, remains the point-like reconstruction as devised by Gross²⁾ for the non-singlet case to leading order and extended to second order, and the singlet case in Refs. 3,4).

Before going into detail, a few general points have to be clarified. First of all, a matter of notation. The structure functions of (spin-averaged) deep inelastic scattering are normally denoted by F_1, F_2, F_3 . Here we shall alter the notation to F^1, F^2, F^3 . This is because we wish to use suffixes to distinguish non-singlet, quark-singlet and glue: F_{NS}, F_S, F_G . We will, when not specifying to which F^a , $a = 1, 2, 3$ we refer, take it that we mean F^2 or F^3 ; F^1 will be treated on its own (via $F^L \sim \sigma_L$) in Section 6. Next we have the question of the renormalization scheme. Here we will use the modified minimal subtraction scheme $\overline{MS}$. Formulae using the minimal subtraction scheme MS ('t Hooft's) may be found in, e.g., Refs. 3,4). All our equations can be translated into the $\overline{MS}$ scheme by the substitution

$$\Lambda^2 \equiv \Lambda_{\overline{MS}}^2 = 4\pi e^{-\gamma_E} \Lambda_{MS}^2. \quad (1.1)$$

Phenomenologically, if using (as we do) the $\overline{MS}$ scheme, we expect $\Lambda \sim 0.45$ GeV; for the MS scheme, we have^{3,4)} $\Lambda_{MS} \sim 0.17$ GeV. This is of course consistent with (1.1).

We also have the point of the accuracy of our parametrizations. In Ref. 1), we said that, to leading order, and with approximate fulfilment of sum rules, we had a maximum discrepancy of $\sim 4\%$ between the exact moments and those computed with our parametrizations. Here, for the worst case (which is the non-singlet), we get accuracy at the per cent level. Finally we present here only the theoretical formulae. Phenomenological applications will be presented elsewhere.

2. BEHAVIOUR AS $x \rightarrow 0$ OF STRUCTURE FUNCTIONS

2.1 Behaviour to leading order

We start by assuming a Regge-like behaviour of the structure functions⁵⁾:

$$F_{NS}(x, Q^2) \underset{x \rightarrow 0}{\simeq} B_{0NS}(Q^2) x^{\lambda(Q^2)}, \quad (2.1)$$

$$F_i(x, Q^2) \underset{x \rightarrow 0}{\simeq} B_{0i}(Q^2) x^{-\lambda_i(Q^2)}, \quad i = S, G. \quad (2.2)$$

The notation $i = 1, 2$ (instead of S, G) will also be used when convenient. Note the minus sign in the definition of λ_{is} . There are indications⁶⁾ that one can get a Regge behaviour as a consequence of QCD; but we could also consider other, non-Regge behaviours. This has been done for the singlet case in Refs. 6, 7); we refer the reader to these for such behaviours, as well as to Refs. 1, 8) for a discussion of their merits as compared to (2.1) and (2.2), and will henceforth stick to a Regge-type behaviour.

What we have to do is to get the restrictions on the functions B, λ implied by the QCD evolution equations, which to leading order are

$$\mu_{NS}(n, Q^2) = \left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{d(n)} \mu_{NS}(n, Q_0^2), \quad (2.3)$$

$$\vec{\mu}(n, Q^2) = \left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{D(n)} \vec{\mu}(n, Q_0^2). \quad (2.4)$$

Here $\vec{\mu}$ is the vector with components μ_s, μ_G and the μ are the moments:

$$\mu(n, Q^2) = \int_0^1 dx x^{n-2} F(x, Q^2). \quad (2.5)$$

The values of the anomalous dimension $d(n)$ and anomalous dimension matrix have been collected in Appendix A. All we need now is that $d(n), D(n)$ can be uniquely defined as analytic functions of n so that (2.3) to (2.5) can be continued to complex n , except for the poles of $d(n)$ at $n = 0, -1, -2, \dots$ and of $D(n)$ at $n = 1, 0, -1, -2, \dots$. We will summarize here the analysis of the leading order case^{1,7)} as this will save us some work later on. First of all, it is easy to see that, because of (2.3), $\mu_{NS}(n, Q^2)$ and $\mu_{NS}(n, Q_0^2)$ must diverge (as functions of n) at the same value of n . Since, because of (2.5), this divergence will occur when $n = 1 - \lambda(Q^2)$ and when $n = 1 - \lambda(Q_0^2)$ respectively, we get at one stroke that λ is actually independent of Q^2 . Moreover, from (2.3), $\mu_{NS}(n, Q^2)$ must certainly diverge when $d(n)$ is singular, i.e., at $n = 0$: so necessarily $\lambda \leq 1$. To exclude $\lambda = 1$ is easy: from (2.5) both $\mu_{NS}(n, Q^2)$ and $\mu_{NS}(n, Q_0^2)$ would diverge when $n = 1 - \lambda + \epsilon, \epsilon \rightarrow 0$ as $(\text{constant})/\epsilon$; this is incompatible with (2.3) for it gives

$$\mu_{NS}(n, Q^2) = e^{(\text{const})\epsilon^{-1}} \mu_{NS}(n, Q_0^2).$$

So we conclude that λ is strictly less than unity. This is easily verified by the expected corresponding Regge trajectory since $\alpha_p(0) = 1 - \lambda \approx \frac{1}{2}$

To finish the analysis we take $n = 1 - \lambda + \epsilon$. From (2.5) and (2.1),

$$\mu_{NS}(1 - \lambda + \epsilon, Q^2) \underset{\epsilon \rightarrow 0}{\simeq} \frac{B_{oNS}(Q^2)}{\epsilon};$$

because $d(1 - \lambda)$ is non-singular, (2.3) then gives

$$B_{oNS}(Q^2) = \left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{d(1-\lambda)} B_{oNS}(Q_0^2),$$

i.e., the combination $B_{\text{ONS}}(Q^2) [\alpha_s(Q^2)]^{d(1-\lambda)}$ is independent of Q^2 . Defining it to be the constant B_{NS} we find, after substitution into (2.1),

$$F_{\text{NS}}(x, Q^2) \underset{x \rightarrow 0}{\simeq} B_{\text{NS}} [\alpha_s(Q^2)]^{-d(1-\lambda)} x^\lambda. \quad (2.6)$$

For the singlet case we can repeat the argument with a few simple modifications¹⁾: one gets that $\lambda_S = \lambda_G$ independent of Q^2 and, because the first singularity of $\underline{D}(n)$ is at $n = 1$, we obtain that λ_S is strictly larger than zero. This means that when Q^2 is large enough for perturbation theory to be valid, the leading vacuum trajectory is not the Pomeron for which $\alpha_P(0) = 1 + \lambda_P = 1$ but another one, $\alpha_L(0) = 1 + \lambda_S > 0$. This fact has been discussed previously^{1,8,9,10)} and we shall not comment further upon it.

Once we know that $\lambda_S > 0$, we can choose $n = 1 + \lambda_S + \varepsilon$, and get from (2.4)

$$\vec{B}_0(Q^2) = \left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{\underline{D}(1+\lambda_S)} \vec{B}_0(Q_0^2). \quad (2.7)$$

Let us define the matrix $\underline{S}(n)$ that diagonalizes $\underline{D}(n)$:

$$\underline{S}^{-1}(n) \underline{D}(n) \underline{S}(n) = \hat{\underline{D}}(n) = \begin{pmatrix} d_+(n) & 0 \\ 0 & d_-(n) \end{pmatrix}. \quad (2.8)$$

We can standardize $\underline{S}$ by requiring $\det S = S_{11} = 1$:

$$\underline{S}(n) = \begin{pmatrix} 1 & \frac{D_{12}(n)}{d_-(n) - d_+(n)} \\ \frac{d_+(n) - D_{11}(n)}{D_{12}(n)} & \frac{d_-(n) - D_{11}(n)}{d_-(n) - d_+(n)} \end{pmatrix}. \quad (2.9)$$

We have, from (2.7),

$$[\alpha_s(Q^2)]^{\hat{\underline{D}}(1+\lambda_S)} \underline{S}^{-1}(1+\lambda_S) \vec{B}(Q^2) = \text{independent of } Q^2 \equiv \vec{b};$$

therefore,

$$\vec{B}(Q^2) = \sum (1+\lambda_s) \begin{pmatrix} [\alpha_s(Q^2)]^{-d_+(1+\lambda_s)} & 0 \\ 0 & [\alpha_s(Q^2)]^{-d_-(1+\lambda_s)} \end{pmatrix} \vec{B} \quad (2.10)$$

Now, $d_{\pm}$ is chosen so that $d_+ > d_-$; hence, to leading order, we can neglect $\alpha_s^{-d_-}$ compared to $\alpha_s^{-d_+}$ and thus finally obtain the result

$$F_i(x, Q^2) = B_i [\alpha_s(Q^2)]^{-d_+(1+\lambda_s)} x^{-\lambda_s}, \quad (2.11a)$$

with

$$B_G / B_S = \frac{d_+(1+\lambda_s) - D_{11}(1+\lambda_s)}{D_{12}(1+\lambda_s)}. \quad (2.11b)$$

2.2 Sub-leading Regge trajectories

We will only discuss explicitly the non-singlet case, just indicating the corresponding results for the singlet case. If we introduce extra Regge trajectories in (2.1),

$$F_{NS}(x, Q^2) \underset{x \rightarrow 0}{\simeq} B_{0NS}(Q^2) x^{\lambda} + B_{1NS}(Q^2) x^{\lambda_1} + \dots, \quad \lambda < \lambda_1 < \dots, \quad (2.12)$$

then we can subtract the leading behaviour and repeat the analysis for the sub-leading one, and so on. We find

$$\begin{aligned} F_{NS}(x, Q^2) \underset{x \rightarrow 1}{\simeq} B_{NS} [\alpha_s(Q^2)]^{-d(1-\lambda)} x^{\lambda} + B_{NS}^{(1)} [\alpha_s(Q^2)]^{-d(1-\lambda_1)} x^{\lambda_1} \\ + \dots + B_{NS}^{(N)} [\alpha_s(Q^2)]^{-d(1-\lambda_N)} x^{\lambda_N} + \dots \end{aligned} \quad (2.13)$$

where $\lambda < \lambda_1 < \dots < \lambda_N < \dots < 1$, i.e., an infinity of Regge trajectories with accumulation point at $\alpha_P(0) = 0$. For the singlet, we get the same except that the accumulation point is now the Pomeron $\alpha_P(0) = 1$. This is satisfactory in as much as we obtain that the P', P (to which no known particle seems to correspond) are not ordinary Regge poles. However, we do not expect that a picture as simple as that of (2.13) should hold; very likely, Regge cuts will appear which will shield all but the leading term of (2.13), pushing the sub-leading poles to other Riemann sheets. For practical purposes, it is better not to impose exact QCD as $x \rightarrow 0$ except at the level of the leading Regge trajectory, and adopt a phenomenological representation of the rest with an effective, Q^2 dependent Regge trajectory. We will thus write

$$F_{NS}(x, Q^2) \simeq B_{NS} [\alpha_s(Q^2)]^{-d(1-\lambda)} x^\lambda + C_{NS}(Q^2) x^{\mu_{NS}(Q^2)}, \quad \mu_{NS}(Q^2) > \lambda; \quad (2.14)$$

$$F_i(x, Q^2) \simeq B_i [\alpha_s(Q^2)]^{-d_i(1+\lambda_s)} x^{-\lambda_s} + C_i(Q^2) x^{-\mu_i(Q^2)}, \quad \mu_i(Q^2) < \lambda_s, \quad (2.15)$$

and determine the Q^2 dependence of the C, μ by invoking the various sum rules available (cf. Section 5 below).

2.3 Behaviour to second order in α_s

The anomalous dimensions and coefficient functions to second order possess a very important property: they are analytic where the first order ones are. Thus, $\gamma_{NS}^{(1)}(n)$ has poles only at $n = 0, -1, -2, \dots$ and $\gamma^{(1)}(n)$ at $n = 1, 0, -1, -2, \dots$ etc. This means that the conclusions of the analysis made with leading order remain, and the only effect of second order corrections is to modify the $B(Q^2)$ of, say, (2.1), (2.2) and (2.6), (2.11), with $O(\alpha_s)$ corrections.

A.-Non-singlet

The QCD equations for the non-singlet moments to second order are^{3,11,12)}

$$\begin{aligned} \mu_{NS}(n, Q^2) &= \left[\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right]^{d(n)} \\ &\times \frac{1 + C_{NS}^{(1)}(n) \alpha_s(Q^2)/4\pi}{1 + C_{NS}^{(1)}(n) \alpha_s(Q_0^2)/4\pi} \left\{ \frac{1 + \beta_1 \alpha_s(Q^2)/4\pi\beta_0}{1 + \beta_1 \alpha_s(Q_0^2)/4\pi\beta_0} \right\}^{p(n)} \mu_{NS}(n, Q_0^2); \\ p(n) &= \frac{1}{2} \left\{ \frac{\gamma_{NS}^{(1)}(n)}{\beta_1} - \frac{\gamma_{NS}^{(0)}(n)}{\beta_0} \right\}, \end{aligned} \quad (2.16)$$

and α_s is to be calculated to two loops¹³⁾:

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2n_f) \log Q^2/\Lambda^2} \left\{ 1 - 2 \frac{153-19n_f}{(33-2n_f)^2} \cdot \frac{\log \log Q^2/\Lambda^2}{\log Q^2/\Lambda^2} \right\}. \quad (2.17)$$

The expressions for the C_{NS} , γ_{NS} , β may be found in Appendix A. As stated, the $d(n)$, $C_{NS}^{(1)}(n)$, $p(n)$ are analytic for $n = 1 - \lambda$, $\lambda < 1$; therefore the analysis of sub-section 2.1 may be repeated unchanged. We find

$$\begin{aligned} F_{NS}(x, Q^2) \\ \underset{x \rightarrow 1}{\simeq} \left[1 - \frac{C_{NS}^{(1)}(1-\lambda) \alpha_s(Q^2)}{4\pi} \right] \left\{ 1 + \frac{\beta_1 \alpha_s(Q^2)}{4\pi\beta_0} \right\}^{p(1-\lambda)} [\alpha_s(Q^2)]^{-d(1-\lambda)} x^\lambda. \end{aligned} \quad (2.18)$$

B.-Singlet

The coefficient functions $C_{11}^{(1)}(n)$, $C_{12}^{(1)}(n)$ and the anomalous dimension matrix $\gamma^{(1)}(n)$ of the second order QCD equations have been calculated in Refs. 4, 12, 14). They are only singular for $n = 1, 0, -1, \dots$, hence they are finite for $n = 1 + \lambda_S$, $\lambda_S > 0$. The analysis of sub-section 2.1, however, has to be extended to cope with some extra complications due to the matrix character of the equations. Our starting point will be the QCD equations in the form given in Ref. 4):

$$\vec{\mu}(n, Q^2) = \underline{\underline{C}}(n, \alpha) \underline{\underline{C}}^{-1}(n, \alpha_0) \underline{\underline{M}}(n; \alpha, \alpha_0) \vec{\mu}(n, Q_0^2). \quad (2.19)$$

For the sake of simplicity of notation, we have temporarily replaced $\alpha_s(Q^2)$ by α , and $\alpha_s(Q_0^2)$ by α_0 . Here

$$\underline{\underline{C}}(n, \alpha) = 1 + \frac{\alpha}{4\pi} \underline{\underline{C}}^{(1)}(n),$$

$$C_{21}^{(1)}(n) = \frac{D_{21}(n)}{D_{12}(n)} C_{12}^{(1)}(n); C_{22}^{(1)}(n) = C_{11}^{(1)}(n) + \frac{D_{22}(n) - D_{11}(n)}{D_{12}(n)} C_{12}^{(1)}(n);$$

with this, one can check that $\underline{\underline{C}}$ and $\underline{\underline{M}}$, and $\underline{\underline{C}}^{(1)}$ and $\underline{\underline{D}}$ commute. $\underline{\underline{M}}$ is

$$\underline{\underline{M}}(n; \alpha, \alpha_0) = \left(\frac{\alpha_0}{\alpha}\right)^{\underline{\underline{D}}(n)} \underline{\underline{R}}(n; \alpha, \alpha_0);$$

$$\underline{\underline{R}}(n; \alpha, \alpha_0) = 1 - \frac{\alpha - \alpha_0}{4\pi} \cdot \frac{\beta_1}{\beta_0^2} \underline{\underline{\gamma}}^{(0)}(n) + \underline{\underline{\Delta}}(n; \alpha, \alpha_0);$$

$$\underline{\underline{\Delta}}(n; \alpha, \alpha_0)$$

$$= -\frac{3}{32} \cdot \frac{\alpha_0}{4\pi} \int_0^r dr' e^{-3\beta_0 r'/16} [\underline{\underline{M}}^{(0)}(n, r')]^{-1} \underline{\underline{\gamma}}^{(1)}(n) \underline{\underline{M}}^{(0)}(n, r'),$$

with, finally, $\underline{\underline{\gamma}}^{(0)}(n) = (3\beta_0/16)\underline{\underline{D}}(n)$ and

$$r = \frac{16}{3\beta_0} \log \frac{\alpha_0}{\alpha}, \quad \underline{\underline{M}}^{(0)}(n, r') = e^{r' \underline{\underline{\gamma}}^{(0)}(n)}.$$

The expressions for the $\underline{\underline{\gamma}}^{(1)}$, $C_{11}^{(1)}$, $C_{12}^{(1)}$ may be found in Appendix A.

These equations are less complicated than they would appear at first sight. We start by defining the transform of $\underline{\underline{\gamma}}^{(1)}$ under $\underline{\underline{S}}$:

$$\underline{\underline{S}}^{-1}(n) \underline{\underline{\gamma}}^{(1)}(n) \underline{\underline{S}}(n) = \underline{\underline{\gamma}}(n). \quad (2.20)$$

By a change of variables, we may rewrite $\underline{\Delta}$ as

$$\underline{\Delta}(n; \alpha, \alpha_0) = \frac{1}{2\beta_0} \int_{\alpha_0}^{\alpha} d\alpha' \left(\frac{\alpha'}{\alpha_0} \right)^{D(n)} \underline{\gamma}^{(1)}(n) \left(\frac{\alpha_0}{\alpha'} \right)^{D(n)}$$

The integral may be performed by diagonalizing $\underline{D}$:

$$\begin{aligned} & \underline{S}^{-1}(n) \underline{\Delta}(n; \alpha, \alpha_0) \underline{S}(n) \\ &= \frac{1}{2\beta_0} \left\{ \begin{pmatrix} \bar{\gamma}_{11} & \bar{\gamma}_{12} \alpha^{d_+ - d_-} \\ \bar{\gamma}_{21} \alpha^{d_- - d_+ + 1} & \bar{\gamma}_{22} \end{pmatrix} \alpha - (\alpha \leftrightarrow \alpha_0) \right\} \end{aligned} \quad (2.21)$$

Now (2.19) may be rearranged to read^{*)}

$$\begin{aligned} & \alpha^{\hat{D}(n)} \underline{S}^{-1}(n) \underline{C}^{-1}(n, \alpha) \underline{\mu}(n, Q^2) \\ &= \alpha_0^{\hat{D}(n)} \underline{S}^{-1}(n) \underline{R}(n; \alpha, \alpha_0) \underline{S}(n) \underline{S}^{-1}(n) \underline{C}^{-1}(n, \alpha) \underline{\mu}(n, Q_0^2), \end{aligned}$$

and, using (2.21), we finally find that

$$\begin{aligned} & [\alpha_s(Q^2)]^{\hat{D}(n)} \left\{ 1 + \frac{\alpha_s(Q^2)}{4\pi} \underline{\Gamma}(n) \right\} \underline{S}^{-1}(n) \underline{C}^{-1}(n, \alpha_s(Q^2)) \underline{\mu}(n, Q^2) \\ &= \underline{b}(n) \text{ independent of } Q^2. \end{aligned} \quad (2.22)$$

Here we have defined

$$\underline{\Gamma}(n) = \begin{pmatrix} -\frac{\bar{\gamma}_{11}}{2\beta_0} + \frac{3\beta_1 d_+}{16\beta_0} & -\frac{1}{2\beta_0} \cdot \frac{\bar{\gamma}_{12}}{d_+ - d_- + 1} \\ -\frac{\bar{\gamma}_{21}}{d_- - d_+ + 1} \cdot \frac{1}{2\beta_0} & -\frac{\bar{\gamma}_{22}}{2\beta_0} + \frac{3\beta_1 d_-}{16\beta_0} \end{pmatrix}. \quad (2.23)$$

These equations are valid for all n ; they will be used later with $n \rightarrow \infty$ and with $n = 2$. For the $x \rightarrow 0$ behaviour, we require them for $n = 1 + \lambda_S$. Now, lowest order fits^{1,8)} suggest that $\lambda_S \approx 0.37 \pm 0.07$; for scattering on photon targets, Witten¹⁰⁾ has proved that $\lambda \approx 0.59$. We assume that $\lambda_S \leq 0.7$.

^{*)} Of course all the manipulations are valid only to leading and next-to-leading order in α_s .

Within this range,

$$d_+(1+\lambda_s) > 1 + d_-(1+\lambda_s). \quad (2.24)$$

This will greatly simplify the analysis, as it implies that to leading and next-to-leading order we can neglect $\alpha_s^{-d_-(1+\lambda_s)}$ against $\alpha_s^{-d_+(1+\lambda_s)}$. We get, with α_s given by (2.17),

$$\begin{aligned} \vec{F}(x, Q^2) &\underset{x \rightarrow 0}{\simeq} B_s [\alpha_s(Q^2)]^{-d_+(1+\lambda_s)} \\ &\times \left\{ \left(\frac{d_+(1+\lambda_s) - D_{11}(1+\lambda_s)}{D_{12}(1+\lambda_s)} \right) - \frac{\alpha_s(Q^2)}{4\pi} \vec{v}(1+\lambda_s) \right\} x^{-\lambda_s} \end{aligned} \quad (2.25a)$$

$$v_1(1+\lambda_s) = \frac{3\beta_1 d_+(n)}{16\beta_0} - \frac{\bar{\gamma}_{11}(n)}{2\beta_0} - \frac{\bar{\gamma}_{21}(n)}{2\beta_0 (d_-(n) - d_+(n) + 1)} \cdot \frac{D_{12}(n)}{d_-(n) - d_+(n)} \Big|_{n=1+\lambda_s} \quad (2.25b)$$

$$\begin{aligned} &v_2(1+\lambda_s) \\ &= \left[\frac{3\beta_1 d_+(n)}{16\beta_0} - \frac{\bar{\gamma}_{11}(n)}{2\beta_0} \right] \frac{d_+(n) - D_{11}(n)}{D_{12}(n)} - \frac{\bar{\gamma}_{21}(n)}{2\beta_0 (d_-(n) - d_+(n) + 1)} \cdot \frac{d_-(n) - D_{11}(n)}{d_-(n) - d_+(n)} \Big|_{n=1+\lambda_s} \end{aligned} \quad (2.25c)$$

We will also find it convenient to rewrite (2.25) as

$$F_i(x, Q^2) = B_i [\alpha_s(Q^2)]^{-d_+(1+\lambda_s)} x^{-\lambda_s} \left\{ 1 - \frac{\alpha_s(Q^2)}{4\pi} w_i(1+\lambda_s) \right\}, \quad (2.26a)$$

with

$$B_G/B_S = \frac{d_+(1+\lambda_S) - D_{11}(1+\lambda_S)}{D_{12}(1+\lambda_S)}, \quad (2.26b)$$

$$w_S(1+\lambda_S) = v_S(1+\lambda_S), \quad w_G(1+\lambda_S) = \frac{D_{12}(1+\lambda_S)}{d_+(1+\lambda_S) - D_{11}(1+\lambda_S)} v_G(1+\lambda_S). \quad (2.26c)$$

3. BEHAVIOUR AS $x \rightarrow 1$ OF STRUCTURE FUNCTIONS

3.1 Leading order

As before, we start with a summary of the leading order estimates. We assume now

$$F(x, Q^2) \underset{x \rightarrow 1}{\simeq} A_0(Q^2) (1-x)^{\nu(\alpha_S)} \quad (3.1)$$

and will find the Q^2 dependence of $A, \nu(\alpha_S)$ from the QCD evolution equations. Consider the non-singlet. As $n \rightarrow \infty$, only the $x \rightarrow 1$ part of F contributes to the moments, Eq. (2.5). Using (3.1), we see that

$$\mu_{NS}(n, Q^2) \underset{n \rightarrow \infty}{\simeq} A_{0NS}(Q^2) \frac{\Gamma(n-1) \Gamma(1+\nu_{NS}(\alpha_S))}{\Gamma(n+\nu_{NS}(\alpha_S))}. \quad (3.2)$$

From (2.3) on the other hand, and because

$$d(n) \underset{n \rightarrow \infty}{\simeq} \frac{-16}{33-2\alpha_f} \log n,$$

$$\frac{\mu_{NS}(n, Q^2)}{\mu_{NS}(n, Q_0^2)} \underset{n \rightarrow \infty}{\simeq} n^{[16/(33-2\alpha_f)] \log \alpha_S(Q^2)/\alpha_S(Q_0^2)}.$$

Equating to (3.2), we find^{*)} A_{NS} , v_{NS} and hence that³⁾

$$F_{NS}(x, Q^2) \underset{x \rightarrow 1}{\simeq} A_{NS} [\alpha_s(Q^2)]^{-d_0} \frac{(1-x)^{v_{NS}(\alpha_s)}}{\Gamma(1+v_{NS}(\alpha_s))}, \quad (3.3a)$$

$$\left. \begin{aligned} v_{NS}(\alpha_s) &= v_{ONS} - \frac{16}{33-2n_f} \log \alpha_s(Q^2); \\ d_0 &= \frac{16}{33-2n_f} \left(\frac{3}{4} - \gamma_E \right), \gamma_E = 0.57722\dots \end{aligned} \right\} \quad (3.3b)$$

We see that the behaviour of F as $x \rightarrow 1$ is controlled by the behaviour of $d(n)$ as $n \rightarrow \infty$. For the singlet case we will likewise require $\tilde{D}(n)$ for $n \rightarrow \infty$. The analysis is slightly complicated due to the coupled character of the equations. We have^{1,7)}

$$F_S(x, Q^2) \underset{x \rightarrow 1}{\simeq} A_S [\alpha_s(Q^2)]^{-d_0} \frac{(1-x)^{v_S(\alpha_s)}}{\Gamma(1+v_S(\alpha_s))}, \quad (3.4)$$

$$F_G(x, Q^2) \underset{x \rightarrow 1}{\simeq} \frac{2A_S}{5} [\alpha_s(Q^2)]^{-d_0} \frac{(1-x)^{v_S(\alpha_s)+1}}{\Gamma(2+v_S(\alpha_s)) \log(1-x)^{-1}}, \quad (3.5)$$

where

$$v_S(\alpha_s) = v_{OS} - \frac{16}{33-2n_f} \log \alpha_s(Q^2).$$

We expect¹⁵⁾ $v_{ONS} \simeq v_{OS}$, $A_{NS} \simeq A_S$.

*) Finding the exponent $-d_0$ of α_s requires use of another term in the asymptotic expansion of $d(n)$ for large n :

$$d(n) \underset{n \rightarrow \infty}{\simeq} \frac{-16}{33-2n_f} \left\{ \log n - \frac{3}{4} + \gamma_E + O\left(\frac{1}{n^2}\right) \right\}.$$

3.2 Next-to-leading order

A.-Non-singlet

As proved in Ref. 3), the $\gamma_{NS}^{(1)}(n)$ behaves for large n also as $\log n$. Therefore, the only substantial modifications we expect must be due to the coefficient functions which grow like $\log^2 n$. This is so. As shown in Ref. 3), a pure power behaviour is incompatible with QCD to second order; corrections $O(\alpha_s \log^2(1-x))$ must necessarily appear. We get

$$F_{NS}(x, Q^2) \underset{x \rightarrow 1}{\simeq} A_{NS} [\alpha_s(Q^2)]^{-d_0} \frac{e^{a(\alpha_s) \alpha_s(Q^2) (1-x)^{\gamma_{NS}(\alpha_s)}}}{\Gamma(1 + \gamma_{NS}(\alpha_s))} \times \left\{ 1 + \frac{2 \alpha_s(Q^2) \log^2(1-x)}{3\pi} \right\}, \quad (3.6a)$$

where

$$a(\alpha_s) = a_0 + a_1 \psi(\nu_{NS}(\alpha_s) + 1) + \frac{2}{3\pi} \left\{ [\psi(\nu_{NS}(\alpha_s) + 1)]^2 - \psi'(\nu_{NS}(\alpha_s) + 1) \right\},$$

$$\nu_{NS}(\alpha_s) = \nu_{NS}(\alpha_s) - \frac{4}{3\pi} \psi(\nu_{NS}(\alpha_s) + 1). \quad (3.6b)$$

ν_{NS} is still given by (3.3b), α_s is to be taken to two loops, Eq. (2.17) and the a_0, a_1 are given in Appendix B. Moreover, $\psi(z) = d \log \Gamma(z) / dz$, $\psi'(z) = d\psi(z) / dz$.

B.-Singlet

We have not calculated the second order correction to the behaviour of F_G , Eq. (3.5). Indeed, F_G is down by a factor $2(1-x)/5(1 + \nu_s(\alpha_s)) |\log(1-x)|$ with respect to F_s and, moreover, is not physically observable. Because we have chosen $\tilde{C}$ and $\tilde{M}$ commuting (recall Section 2B), the second order corrections to (3.4) and (3.5) essentially decouple. Therefore we can treat F_s to second order just as in the non-singlet case, and the result is totally analogous to Eqs. (3.6):

$$F_S(x, Q^2) \simeq A_S [\alpha_s(Q^2)]^{-d_0} \frac{e^{a_s(\alpha_s) \nu_s(Q^2)} (1-x)^{\nu_{1s}(\alpha_s)}}{\Gamma(1 + \nu_{1s}(\alpha_s))} \\ \times \left\{ 1 + \frac{2\alpha_s(Q^2) \log^2(1-x)}{3\pi} \right\},$$

where

$$a_s(\alpha_s) = a_0 + a_1 \psi(\nu_s(\alpha_s) + 1) + \frac{2}{3\pi} \left\{ [\psi(\nu_s(\alpha_s) + 1)]^2 - \psi'(\nu_s(\alpha_s) + 1) \right\}, \\ \nu_{1s}(\alpha_s) = \nu_s(\alpha_s) + \frac{4}{3\pi} \psi(\nu_s(\alpha_s) + 1), \quad (3.7)$$

with

$$\nu_s(\alpha_s) = \nu_{0s} - \frac{16}{33-2n_f} \log \alpha_s(Q^2).$$

4. SUM RULES TO SECOND ORDER

As is known, structure functions obey sum rules. This is because, for non-singlet ones, $d(1) = 0$, and, for singlet ones, $d_+(2) = 0$: therefore to leading order the corresponding moments are Q^2 independent, and hence we may calculate them at $Q^2 = \infty$ with the free parton model. Thus we may write

$$\int_0^1 dx \, x^{-1} F_{NS}(x, Q^2) \xrightarrow{Q^2 \rightarrow \infty} J_{0NS}, \quad (4.1)$$

$$\int_0^1 dx \, F_i(x, Q^2) \xrightarrow{Q^2 \rightarrow \infty} J_{0i}, \quad i = S, G. \quad (4.2)$$

The J_0 depend upon the specific process under consideration; they are compiled in Ref. 16). In ref. 1), such sum rules were treated approximately by requiring that they hold at a fixed Q_1^2 ; we shall now treat them exactly. Of course, since we are working beyond the leading order, we require the $O(\alpha_s)$ corrections to the sum rules:

$$\int_0^1 dx x^{-1} F_{NS}(x, Q^2) = J_{NS}(\alpha_s) \quad (4.3)$$

$$\begin{aligned} \int_0^1 dx F_i(x, Q^2) &= J_i(\alpha_s), \\ J(\alpha_s) &= J_0 + \frac{\alpha_s(Q^2)}{4\pi} J_1. \end{aligned} \quad (4.4)$$

The J_{1NS} may again be found in Ref. 16); but to our knowledge nobody has previously calculated the corrections for the singlet sum rules. This we will do now.

4.1 The singlet sum rules to leading order

From (2.4), and after manipulations similar to those of Section 2, we see that (cf. also the leading order of (2.22))

$$[\alpha_s(Q^2)]^{\hat{D}(n)} \underline{\underline{S}}^{-1}(n) \vec{\mu}(n, Q^2) = \vec{b}_0(n) \text{ independent of } Q^2. \quad (4.5)$$

For $n = 2$, $d_+(2) = 0$; since $d_-(2) < 0$, we can write to leading order

$$\vec{\mu}(2, Q^2) = \underline{\underline{S}}(2) \begin{pmatrix} b_0(2) \\ 0 \end{pmatrix} = b_{01}(2) \begin{pmatrix} S_{11}(2) \\ S_{21}(2) \end{pmatrix}.$$

The value of the $b_{01}(2)$ may be calculated with the free parton model. As stated, it depends upon the process in consideration. We will standardize it by eventually extracting a process-dependent coefficient and writing (cf. Ref. 16)):

$b_{01}(2) = 3n_f / (3n_f + 16)$. (This is the actual value for electroproduction in protons.)

Therefore we find, with the explicit values for $\underline{\underline{D}}(2)$, $\underline{\underline{S}}(2)$ of Appendix B,

$$\int_0^1 dx F_S(x, Q^2) \underset{Q^2 \rightarrow \infty}{\simeq} \frac{3n_f}{3n_f + 16}; \quad \int_0^1 dx F_G(x, Q^2) \underset{Q^2 \rightarrow \infty}{\simeq} \frac{16}{3n_f + 16}; \quad (4.6)$$

$$\int_0^1 dx \{F_S(x, Q^2) + F_G(x, Q^2)\} \underset{Q^2 \rightarrow \infty}{\sim} 1. \quad (4.7)$$

4.2 Next-to-leading corrections

We require the full Eq. (2.22) now. With the value of $b_{01}(2)$ selected, we can write

$$\begin{aligned} & \int_0^1 dx \vec{F}(x, Q^2) \\ &= b_{01}(2) \left\{ \begin{pmatrix} 1 \\ S_{21}(2) \end{pmatrix} + \frac{\alpha_s(Q^2)}{4\pi} \begin{pmatrix} \frac{-\bar{\gamma}_{11}(2)}{2\beta_0(d(2)+1)} S_{12}(2) + C_{11}^{(1)}(2) + C_{12}^{(1)}(2) S_{21}(2) \\ \frac{-\bar{\gamma}_{21}(2)}{2\beta_0(d(2)+1)} S_{22}(2) + C_{22}^{(1)}(2) S_{21}(2) + C_{21}^{(1)}(2) \end{pmatrix} \right. \\ & \quad \left. + k(2) [\alpha_s(Q^2)]^{-d_-(2)} \begin{pmatrix} S_{12}(2) \\ S_{22}(2) \end{pmatrix} \right. \\ & \quad \left. + k(2) \frac{[\alpha_s(Q^2)]^{1-d_-(2)}}{4\pi} \begin{pmatrix} C_{11}^{(1)}(2) S_{12}(2) + C_{12}^{(1)}(2) S_{22}(2) \\ C_{21}^{(1)}(2) S_{22}(2) + C_{22}^{(1)}(2) S_{22}(2) \end{pmatrix} \right\} + O(\alpha_s^2). \end{aligned} \quad (4.8)$$

We have used that $\bar{\gamma}_{11}(2) = 0$. The constant $k(2)$ is unfortunately unknown, so we cannot calculate the corrections to each individual sum rule (4.6): note that

$$-d_-(2) = \frac{2}{3} \cdot \frac{16+3n_f}{33-2n_f} < 1 \quad \text{for } n_f < 6,$$

hence the dominant correction is the one proportional to $k(2)\alpha_s^{-d_-(2)}$. There are, however, two combinations which involve only calculable corrections. One is the sum; for it, using the explicit values of the $\tilde{D}(2)$, $\tilde{S}(2)$, we find

$$\begin{aligned} & \int_0^1 dx \{F_S(x, Q^2) + F_G(x, Q^2)\} \\ &= 1 + \frac{\alpha_s(Q^2)}{4\pi} \left\{ C_{11}^{(1)}(2) + \frac{16}{3n_f} C_{12}^{(1)}(2) \right\} + O(\alpha_s^{1-d_-(2)}). \end{aligned} \quad (4.9)$$

For electroproduction on protons, the right-hand side is

$$1 - \frac{5}{9\pi} \alpha_s(Q^2).$$

The second contribution involves the integral of the singlet part of the longitudinal structure function, $\int_0^1 dx F_L^L(x, Q^2)$, and is therefore not of much practical use. Actually, because $-d_-(2)$ is less than, but near unity, a treatment with any hope of phenomenological success would be possible only when the second order corrections to the violation of the Callan-Gross relation have been performed.

5. PARAMETRIZATIONS OF STRUCTURE FUNCTIONS COMPATIBLE WITH QCD.

We will now write the simplest possible parametrizations of structure functions compatible with QCD. Of course, it is impossible to write compact functional forms which verify the QCD evolution equations at all values of x . If one wants compatibility everywhere, there is really no choice but to use point-like reconstruction methods, either approximate (with a degree of approximation increasing with the number of moments used, as in Ref. 17)), or exact, by inverting the Mellin transform as discussed in Refs. 2,3,4). However, given the very regular shape of the structure functions, it so happens that it is possible to get verification of the QCD equations to the per cent level provided we impose exact QCD at the end points $x = 0, 1$, and verification of the appropriate sum rules (which basically fix the area). This was shown in Ref. 1) to leading order; we will now extend the analysis beyond in successive steps, using the results just obtained.

5.1 QCD at the end points to leading order

We start by recalling the results of Ref. 1) to leading order:

$$F_{NS}(x, Q^2) = \{ B_{NS} [\alpha_s(Q^2)]^{-d(1-\lambda)} (x^\lambda - x^{\mu_{NS}}) + A_{NS} [\alpha_s(Q^2)]^{-d_0} \frac{\Gamma(1+\nu_{NS})}{\Gamma(1+\nu_{NS}(\alpha_s))} x^{\mu_{NS}} \} (1-x)^{\nu_{NS}(\alpha_s)}, \quad (5.1)$$

$$\mu_{NS} > \lambda.$$

$$F_S(x, Q^2) = \{ B_S [\alpha_S(Q^2)]^{-d_+(1+\lambda_S)} (x^{-\lambda_S} - x^{-\mu_S}) + A_S [\alpha_S(Q^2)]^{-d_0} \frac{\Gamma(1+\nu_{0S})}{\Gamma(1+\nu_S(\alpha_S))} x^{-\mu_S} \} (1-x)^{\nu_S(\alpha_S)} ; \quad (5.2)$$

$$F_G(x, Q^2) = \{ B_G [\alpha_S(Q^2)]^{-d_+(1+\lambda_S)} (x^{-\lambda_S} - x^{-\mu_G}) + \frac{2A_S}{5} [\alpha_S(Q^2)]^{-d_0} \frac{\Gamma(1+\nu_{0S})}{\Gamma(2+\nu_S(\alpha_S))} x^{-\mu_G} \} \frac{(1-x)^{\nu_S(\alpha_S)+1}}{1+|\log(1-x)|} ; \quad (5.3)$$

$$B_G/B_S = \frac{d_+(1+\lambda_S) - D_{11}(1+\lambda_S)}{D_{12}(1+\lambda_S)} , \quad \mu_S > \lambda_S .$$

To this order, we expect the exponents μ , μ_S , μ_G to be constant and $\mu_S = \mu_G$. From Regge theory, we may set

$$\lambda = 1 - \alpha_\rho(0) \simeq 0.5 , \mu_S = 1 - \alpha_{\rho'}(0) \simeq 1 , \mu_S = \mu_G = \alpha_2(0) - 1 \simeq 0 .$$

We also expect $\nu_{ONS} \simeq \nu_{OS}$, $A_{NS} \simeq A_S$. Finally, by imposing sum rules at a given, fixed Q^2 , we get two further relations linking A_{NS} and B_{NS} and B_S and A_S . Hence, and counting Λ , we are able to describe all three structure functions F_{NS} , F_S , F_G in terms of the four unknown parameters Λ , λ_S , ν_{ONS} , A_{NS} . The fits¹⁾ fix the first three very well, $\Lambda \simeq 0.45 \pm 0.5$ GeV, $\lambda_S \simeq 0.37 \pm 0.07$, $\nu_{ONS} = 2.1 \pm 0.2$. The other is less precisely determined (it may vary by a full order of magnitude); this is to be expected because of the approximations made. A precise determination requires, in particular, a less cavalier treatment of sum rules. This will be done in sub-section 5.3; we turn now to the extension of (5.1-3) to include second order corrections.

5.2 QCD at the end points to second order.

We shall only treat explicitly F_{NS} , F_S ; the extension to F_G is straightforward. Using the results of the previous sections we see that, in order to ensure QCD to second order at the end points, we have to modify (5.1) and (5.2) to

$$\begin{aligned}
 F_{NS}(x, Q^2) = & \left\{ B_{NS} [\alpha_s(Q^2)]^{-d(1-\lambda)} \left[1 - \frac{C_{NS}^{(1)}(1-\lambda) \alpha_s(Q^2)}{4\pi} \right] \right. \\
 & \times \left\{ 1 + \frac{\beta_1 \alpha_s(Q^2)}{4\pi\beta_0} \right\}^{-p(1-\lambda)} (x^\lambda - x^{\mu_{NS}(\alpha_s)}) \\
 & + A_{NS} \frac{\Gamma(1+\nu_{0NS})}{\Gamma(1+\nu_{NS}(\alpha_s))} [\alpha_s(Q^2)]^{-d_0} e^{a(\alpha_s)\alpha_s(Q^2)} x^{\mu_{NS}(\alpha_s)} \Big\} \\
 & \times (1-x)^{\nu_{NS}(\alpha_s)} \left\{ 1 + \frac{2\alpha_s(Q^2) \log^2(1-x)}{3\pi} \right\}; \quad (5.4)
 \end{aligned}$$

$$\begin{aligned}
 F_S(x, Q^2) = & \left\{ B_S [\alpha_s(Q^2)]^{-d_+(1+\lambda_s)} \left[1 - \frac{w_s(1+\lambda_s) \alpha_s(Q^2)}{4\pi} \right] \right. \\
 & \times (x^{-\lambda_s} - x^{-\mu_s(\alpha_s)}) \\
 & + A_S \frac{\Gamma(1+\nu_{0S})}{\Gamma(1+\nu_s(\alpha_s))} [\alpha_s(Q^2)]^{-d_0} e^{a_s(\alpha_s)\alpha_s(Q^2)} x^{-\mu_s(\alpha_s)} \Big\} \\
 & \times (1-x)^{\nu_s(\alpha_s)} \left\{ 1 + \frac{2\alpha_s(Q^2) \log^2(1-x)}{3\pi} \right\}. \quad (5.5)
 \end{aligned}$$

Of course, α_s is to be taken as given by Eq. (2.17). To this level, the $\mu_{NS}(\alpha_s)$, $\mu_s(\alpha_s)$ (which represent the effective sub-leading Regge singularities) are unknown. They will now be determined, together with the relations between the A's and B's by requiring fulfilment of the applicable sum rules.

5.3 Parametrizations compatible with the sum rules

We will now consider in detail the conditions implied by requiring sum rules in the limit $\alpha_s \rightarrow 0$. The extension to next-to-leading corrections is tedious but straightforward, and will be sketched later. We start with the non-singlet case, and thus require

$$\int_0^1 dx x^{-1} F_{NS}(x, Q^2) = J_{0NS}.$$

Using (5.4), and neglecting α_s corrections, this gives the condition

$$\frac{\Gamma(\mu_{NS}(\alpha_s))}{\Gamma(1+\mu_{NS}(\alpha_s)+\nu_{NS}(\alpha_s))} = \frac{\Gamma(\lambda)}{\Gamma(1+\lambda+\nu_{NS}(\alpha_s))}$$

$$\times \frac{1 - J_{NS} B_{NS}^{-1} \frac{\Gamma(1+\lambda+\nu_{NS}(\alpha_s))}{\Gamma(\lambda)\Gamma(1+\nu_{NS}(\alpha_s))}}{1 - A_{NS} B_{NS}^{-1} \frac{\Gamma(1+\nu_{NS}(\alpha_s))}{\Gamma(1+\nu_{NS}(\alpha_s))} [\alpha_s(Q^2)]^{d(1-\lambda)-d_0}} \quad (5.6)$$

This equation determines $\mu_{NS}(\alpha_s)$ as a function of J_{NS} , B_{NS} , A_{NS} . As $\alpha_s \rightarrow 0$, we find the explicit behaviour of μ_{NS} :

$$\mu_{NS}(\alpha_s) = \lambda$$

$$+ \frac{[\alpha_s(Q^2)]^{d(1-\lambda)}}{B_{NS} \Gamma(1+\nu_{NS}(\alpha_s))} \cdot \frac{J_{NS} \Gamma(1+\lambda+\nu_{NS}(\alpha_s))/\Gamma(\lambda) - A_{NS} \Gamma(1+\nu_{NS}(\alpha_s)) [\alpha_s(Q^2)]^{-d_0}}{\psi(1+\lambda+\nu_{NS}(\alpha_s)) - \psi(\lambda)} \quad (5.7)$$

In fact, Eq. (5.6) is the one that is easily generalized to second order; to first order we may use (5.7), since the error thereby committed is $\sim \alpha_s^{2d(1-\lambda)} \sim \alpha_s$.

This is not all, however. The condition $0 < \lambda < \mu_{NS} < 1$ implies that the left-hand side of (5.6) must be positive. This is only possible if both numerator and denominator of the right-hand side of (5.6) vanish at the same point^{*}). Let $\hat{\alpha}_{NS} \equiv \alpha_s(Q_{NS}^2)$, where Q_{NS}^2 is the value for which this vanishing occurs. We see that we can eliminate the two unknown constants, A_{NS} and B_{NS} in terms of the known J_{NS} , and $\hat{\alpha}_{NS}$:

$$B_{NS} = J_{NS} \frac{\Gamma(1+\lambda+\nu_{NS}(\hat{\alpha}_{NS}))}{\Gamma(\lambda) \Gamma(1+\nu_{NS}(\hat{\alpha}_{NS}))} \hat{\alpha}_{NS}^{d(1-\lambda)}$$

$$A_{NS} = J_{NS} \frac{\Gamma(1+\lambda+\nu_{NS}(\hat{\alpha}_{NS}))}{\Gamma(\lambda) \Gamma(1+\nu_{NS}(\hat{\alpha}_{NS}))} \hat{\alpha}_{NS}^{d_0} \quad (5.8)$$

^{*}) Actually this is only true if such a point, Q_{NS}^2 , lies in the region where QCD perturbation theory is applicable. However, this is the case in all practical applications as $Q_{NS}^2 \sim 10 \text{ GeV}^2$.

For the singlet the condition to be used is

$$\int_0^1 dx F(x, Q^2) = 1,$$

and we get

$$\begin{aligned} \frac{\Gamma(1-\mu_s(\alpha_s))}{\Gamma(2-\mu_s(\alpha_s)+\nu_s(\alpha_s))} &= \frac{\Gamma(1-\lambda_s)}{\Gamma(2-\lambda_s+\nu_s(\alpha_s))} \\ \times \frac{1 - J_s B_s^{-1} \frac{\Gamma(2-\lambda_s+\nu_s(\alpha_s))}{\Gamma(1-\lambda_s)\Gamma(1+\nu_s(\alpha_s))} [\alpha_s(Q^2)]^{d_+(1+\lambda_s)}}{1 - A_s B_s^{-1} \frac{\Gamma(1+\nu_{0s})}{\Gamma(1+\nu_s(\alpha_s))} [\alpha_s(Q^2)]^{d_+(1+\lambda_s)-d_0}} \end{aligned} \quad (5.9)$$

To leading order,

$$\begin{aligned} \mu_s(\alpha_s) &= \lambda_s \\ - \frac{[\alpha_s(Q^2)]^{d_+(1+\lambda_s)}}{B_s \Gamma(1+\nu_s(\alpha_s))} &\cdot \frac{J_s \Gamma(2-\lambda_s+\nu_s(\alpha_s))/\Gamma(1-\lambda_s) - A_s \Gamma(1+\nu_{0s}) \alpha_s^{-d_0}}{\psi(2-\lambda_s+\nu_s(\alpha_s)) - \psi(1-\lambda_s)} \end{aligned} \quad (5.10)$$

The relation between A_s , B_s , J_s , is again given via a parameter $\hat{\alpha}_s \equiv \alpha_s(Q_s^2)$:

$$\begin{aligned} B_s &= J_s \frac{\Gamma(2-\lambda_s+\nu_s(\hat{\alpha}_s))}{\Gamma(1-\lambda_s)\Gamma(1+\nu_s(\hat{\alpha}_s))} \hat{\alpha}_s^{d_+(1+\lambda_s)}, \\ A_s &= J_s \frac{\Gamma(2-\lambda_s+\nu_s(\hat{\alpha}_s))}{\Gamma(1-\lambda_s)\Gamma(1+\nu_{0s})} \hat{\alpha}_s^{d_0}. \end{aligned} \quad (5.11)$$

There is one important comment to be made. For both the singlet and non-singlet, the effective sub-leading Regge trajectory tends to coincide with the leading one, as $Q^2 \rightarrow \infty$, (cf. Eqs. (5.7) and (5.10)). However, for small Q^2 it separates itself from it and, as seen numerically, tends to fall on the P' or P , as the case may be. We find this very satisfactory.

5.4 Parametrizations to second order

We will treat both singlet and non-singlet at the same time; the extension to gluons is left to the reader. We may write (5.4) and (5.7) in a unified number as

$$F(x, Q^2) = \{ B_1(Q^2) [\alpha_s(Q^2)]^{d_+(1+\lambda_1)} (x^{\lambda_1} - x^{\mu_1(\alpha_s)}) \\ + A_1(Q^2) [\Gamma(1+\nu_0)/\Gamma(1+\nu_1(\alpha_s))] [\alpha_s(Q^2)]^{-d_0} x^{\mu_1(\alpha_s)} \\ \times (1-x)^{\nu_1(\alpha_s)} \{ 1 - \frac{2\alpha_s(Q^2)}{3\pi} \log^2(1-x) \}.$$

For example, for the singlet $B_1(Q^2) = B_s \{1 - W_s(1+\lambda_s)\alpha_s/4\pi\}$, $\lambda_1 = 1 - \lambda_s$, etc., and for the non-singlet $\lambda_1 = \lambda$, etc. With this notation the sum rules may be written also in a unified manner as

$$\int_0^1 dx x^{-1} F(x, Q^2) = J(\alpha_s).$$

We may introduce the notation

$$\beta_2(\lambda, \nu, \alpha_s) = \frac{\Gamma(\lambda)\Gamma(1+\nu)}{\Gamma(1+\lambda+\nu)} + \frac{2\alpha_s}{3\pi} \frac{\partial^2}{\partial \nu^2} \frac{\Gamma(\lambda)\Gamma(1+\nu)}{\Gamma(1+\lambda+\nu)}.$$

With this, the analogue of, say, (5.6) is

$$\beta_2(\mu_1(\alpha_s), \nu_1(\alpha_s); \alpha_s) \\ = \frac{J(\alpha_s) - B_1(\alpha_s) \alpha_s^{-d} \beta_2(\lambda, \nu_1(\alpha_s); \alpha_s)}{A_1(\alpha_s) \alpha_s^{-d_0} \Gamma(1+\nu_0)/\Gamma(1+\nu_1(\alpha_s)) - B_1(\alpha_s) \alpha_s^{-d}}, \quad (5.12)$$

where $d = d(1 - \lambda)$ or $d_+(1 + \lambda_s)$ as the case may be. Equation (5.12) gives $\mu_1(\alpha_s)$; the extra conditions linking A, B, J , are

$$B_1(\alpha_s) = \alpha_s^d \frac{J(\alpha_s)}{\beta_2(\lambda, \nu_1(\alpha_s); \alpha_s)}, \\ A_1(\alpha_s) = \alpha_s^{d_0} \frac{\Gamma(1+\nu_1(\alpha_s)) J(\alpha_s)}{\Gamma(1+\nu_0) \beta_2(\lambda, \nu_1(\alpha_s); \alpha_s)}. \quad (5.13)$$

This completes the analysis.

6. THE LONGITUDINAL STRUCTURE FUNCTION

The longitudinal structure function will be defined by

$$F^L(x, Q^2) = F^2(x, Q^2) - F^1(x, Q^2).$$

From QCD we have^{12,18,19)}

$$F_{NS}^L(x, Q^2) = \frac{4\alpha_s(Q^2)}{3\pi} \int_x^1 dy \frac{x^2}{y^3} F_{NS}^2(y, Q^2); \quad (6.1)$$

$$F_S^L(x, Q^2) = \frac{4\alpha_s(Q^2)}{3\pi} \int_x^1 dy \frac{x^2}{y^3} F_S^2(y, Q^2); \quad (6.2)$$

$$F_G^L(x, Q^2) = \frac{4\alpha_s(Q^2)}{3\pi} \delta \int_x^1 dy \frac{x^2}{y^3} \left(1 - \frac{x}{y}\right) F_G^2(y, Q^2); \quad (6.3)$$

here δ depends on the process. For electroproduction on protons, $\delta = 3n_f/2$. The F^L is the sum of all three contributions:

$$F^L = F_{NS}^L + F_S^L + F_G^L.$$

Since F^L is itself of $O(\alpha_s)$ we only have to compute it to leading order. This is easily done with the help of (6.1-3) and the previous results. We find

$$F_{NS}^L(x, Q^2) \underset{x \rightarrow 1}{\simeq} \frac{4\alpha_s(Q^2)}{3\pi} \cdot \frac{1-x}{1+\nu_{NS}(\alpha_s)} F_{NS}^2(x, Q^2),$$

$$F_S^L(x, Q^2) \underset{x \rightarrow 1}{\simeq} \frac{4\alpha_s(Q^2)}{3\pi} \cdot \frac{1-x}{1+\nu_S(\alpha_s)} F_S^2(x, Q^2),$$

$$F_G^L(x, Q^2) \underset{x \rightarrow 1}{\simeq} \frac{4\alpha_s(Q^2)}{3\pi} \cdot \frac{(1-x)^3 |\log(1-x)|^{-1}}{(2+\nu_S(\alpha_s))(3+\nu_S(\alpha_s))} \cdot \frac{2}{5} F_S^2(x, Q^2);$$

(6.4)

$$F_{NS}^L(x, Q^2) \underset{x \rightarrow 0}{\simeq} \frac{4\alpha_s(Q^2)}{3\pi(2-\lambda)} F_{NS}^2(x, Q^2),$$

$$F_S^L(x, Q^2) \underset{x \rightarrow 0}{\simeq} \frac{4\alpha_s(Q^2)}{3\pi(2+\lambda_S)} F_S^2(x, Q^2),$$

$$F_G^L(x, Q^2) \underset{x \rightarrow 0}{\simeq} \frac{4\delta\alpha_s(Q^2)}{3\pi(2+\lambda_S)}$$

$$\times \frac{d_+(1+\lambda_S) - D_{11}(1+\lambda_S)}{(3+\lambda_S)D_{12}(1+\lambda_S)} F_S^2(x, Q^2). \quad (6.5)$$

The F_{NS}^2, F_S^2 are to be taken to leading order, Eqs. (5.1,2), (5.7,8) and (5.10,11). When trying to write a parametrization for F^L we may choose to parametrize $R(x, Q^2) = F^L(x, Q^2)/F^2(x, Q^2)$, or F^L itself. The parametrizations for R may be found in Ref. 20); here we shall just write those for F^L . From (6.3-5), we see that the simplest parametrization compatible with the end point behaviour is

$$F_{NS}^L(x, Q^2) = \frac{4\alpha_s(Q^2)}{3\pi} \left\{ \frac{1-x}{2-\lambda} + \frac{x}{1+\lambda_{NS}(\alpha_s)} \right\} (1-x) F_{NS}^2(x, Q^2), \quad (6.6)$$

$$F_S^L(x, Q^2) = \frac{4\alpha_s(Q^2)}{3\pi} \left\{ \frac{1-x}{2+\lambda_S} + \frac{x}{1+\lambda_S(\alpha_s)} \right\} (1-x) F_S^2(x, Q^2), \quad (6.7)$$

$$F_G^L(x, Q^2) = \frac{4\alpha_s(Q^2)}{3\pi} \left\{ \frac{1-x}{2+\lambda_s} \cdot \delta \cdot \frac{d_+(1+\lambda_s) - D_{11}(1+\lambda_s)}{(3+\lambda_s)D_{12}(1+\lambda_s)} \right. \\ \left. + \frac{2}{5} \cdot \frac{x}{(2+\lambda_s(\alpha_s))(3+\lambda_s(\alpha_s))} \right\} \frac{(1-x)^3}{1+|\log(1-x)|} F_S^2(x, Q^2); \quad (6.8)$$

$$F^L = F_{NS}^L + F_S^L + F_G^L.$$

We should perhaps say a few words on the general features. At $x \rightarrow 1$, both F_{NS} and F_S contribute equally to F^L ; but the contribution of F_G is utterly negligible. At $x \rightarrow 0$, most of the contribution, however, comes from F_G with F_{NS} contributing a negligible amount. This very different shape is what makes it necessary to parametrize separately F_{NS}^L , F_S^L and F_G^L , and not the whole of F^L .

7. FINITE MASS EFFECTS

In QCD, one makes an expansion in two parameters: $\alpha_s(Q^2)$ and $\bar{m}^2/Q^2$, where $\bar{m}$ is a typical hadronic mass. Until now, we have been considering leading and second order in α_s , but have neglected $\bar{m}^2/Q^2$ corrections. These corrections enter because of a number of reasons: target mass effects²¹⁾, which are due to the fact that the calculations were made as if the target were massless; quark mass effects: the effective number of flavours depends upon the quarks excited at a given Q^2, x . This is particularly important²²⁾ for the c and b quarks. Higher twists^{18,23)} which give corrections involving the transverse momentum of the quarks p_t or the inverse nucleon radius R_N^{-1} . Finally, we have genuine non-perturbative effects. The theory of these corrections is far from being fully developed, so we feel justified in proposing a phenomenological treatment. Firstly, all estimates agree that, with the exception of quark mass effects, all the rest give large contributions of the form $\sim (\bar{m}^2/Q^2) x / (1-x)$, in the $x \sim 1$ region. We there take the effective number of flavours as given by

$$\eta_f = \eta_f(Q^2) = 3 + \sum_{q=c,b,\dots} \theta(Q^2 - 4m_q^2) \left(1 + \frac{2m_q^2}{Q^2}\right) \left(1 - \frac{4m_q^2}{Q^2}\right)^{1/2}. \quad (7.1)$$

This will give a good approximation to the effect except in the large x region where, however, we may lump quark mass effects with the other finite mass corrections. Then we take into account collectively all the remaining finite mass effects by multiplying our parametrizations by the correction

$$1 + \frac{\bar{m}^2}{Q^2} \left\{ k + \frac{x}{1-x} \right\}. \quad (7.2)$$

The constants $\bar{m}$, k are to be determined empirically. We would expect $\bar{m} \sim \Lambda \sim 0.5$ GeV, $k \sim 1$. Target mass corrections can be taken into account in the usual way¹⁸⁾.

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APPENDIX A

All our formulae are in the $\overline{\text{MS}}$ subtraction scheme¹²⁾

$$\beta_0 = 11 - \frac{2}{3} n_f ; \quad \beta_1 = 102 - \frac{38}{3} n_f .$$

A.1 Non-singlet

$$d(n) = \frac{16}{3\beta_0} \gamma_{NS}^{(0)}(n) , \quad \gamma_{NS}^{(0)}(n) = \frac{1}{2n(n+1)} + 3/4 - S_1(n) .$$

$$p(n) = \frac{1}{2} \left\{ \frac{\gamma_{NS}^{(1)}(n)}{\beta_1} - \frac{\gamma_{NS}^{(0)}(n)}{\beta_0} \right\} ,$$

$$\gamma_{NS}^{(1)}(n) = \frac{32}{9} S_1(n) \left\{ 67 + 8 \frac{2n+1}{n^2(n+1)^2} \right\} - 64 S_1(n) S_2(n)$$

$$- \frac{32}{9} [S_2(n) - S_2(n/2)] \left\{ 2 S_1(n) - \frac{1}{n(n+1)} \right\}$$

$$- \frac{128}{9} \tilde{S}(n) + \frac{32}{2} S_2(n) \left[\frac{3}{n(n+1)} - 7 \right] + \frac{16}{9} S_3(n/2)$$

$$- 28 - 16 \frac{151n^4 + 260n^3 + 96n^2 + 3n + 10}{9n^3(n+1)^3}$$

$$+ \frac{32}{9} \cdot \frac{2n^2 + 2n + 1}{n^3(n+1)^3}$$

$$+ \frac{32}{7} n_f \left\{ -10 S_1(n) + 6 S_2(n) + \frac{3}{4} + \frac{11n^2 + 5n - 3}{n^2(n+1)^2} \right\} .$$

For electroproduction on proton targets,

$$C_{NS}^{(1)}(n)$$

$$= \frac{4}{3} \left\{ 2 [S_1(n)]^2 + 3 S_1(n) - 2 S_2(n) \right.$$

$$\left. - \frac{2 S_1(n)}{n(n+1)} + \frac{3}{n} + \frac{4}{n+1} + \frac{2}{n^2} - 9 \right\}$$

For other processes, cf. Ref. 16).

A.2 Singlet

$$\underline{D}(n) = \frac{16}{3\beta_0} \underline{\gamma}^{(0)}(n);$$

$$\underline{\gamma}^{(0)}(n) = \begin{pmatrix} \frac{1}{2n(n+1)} + \frac{3}{4} - S_1(n) & \frac{3n_f}{8} \cdot \frac{n^2+n+2}{n(n+1)(n+2)} \\ \frac{n^2+n+2}{2n(n^2-1)} & \frac{33-2n_f}{16} + \frac{9}{4} \left[\frac{1}{n(n-1)} + \frac{1}{(n+1)(n+2)} - S_1(n) \right] \end{pmatrix}$$

$$\gamma_{11}^{(1)}(n) = C_F^2 \left\{ 16 S_1(n) \frac{2n+1}{n^2(n+1)^2} + 16 \left[2S_1(n) - \frac{1}{n(n+1)} \right] \left[S_2(n) - S_2\left(\frac{n}{2}\right) \right] \right.$$

$$+ 24 S_2(n) + 64 \tilde{S}(n) - 8 S_3(n/2) - 3 - 8 \frac{3n^2+n^2-1}{n^3(n+1)^3}$$

$$\left. - 16 \frac{2n^2+2n+1}{n^3(n+1)^3} \right\}$$

$$+ C_F C_A \left\{ \frac{536}{9} S_1(n) - 8 \left[2S_1(n) - \frac{1}{n(n+1)} \right] \left[2S_2(n) - S_2\left(\frac{n}{2}\right) \right] \right.$$

$$- \frac{88}{3} S_2(n) - 32 \tilde{S}(n) + 4 S_3\left(\frac{n}{2}\right) - \frac{17}{3}$$

$$\left. - \frac{4}{9} \cdot \frac{151n^4 + 236n^3 + 88n^2 + 3n + 18}{n^3(n+1)^3} + 8 \frac{2n^2+2n+1}{n^3(n+1)^3} \right\}$$

$$+ C_F T_R \left\{ - \frac{160}{9} S_1(n) + \frac{32}{3} S_2(n) + \frac{4}{3} \right.$$

$$\left. + \frac{16}{9} \cdot \frac{11n^7 + 49n^6 + 5n^5 - 329n^4 - 514n^3 - 350n^2 - 240n - 72}{(n-1)n^3(n+1)^3(n+2)^2} \right\}$$

$$\gamma_{12}^{(1)}(n) = 8 C_A T_R \left\{ \left[-2S_1^2(n) + 2S_2(n) - 2S_2\left(\frac{n}{2}\right) \right] \frac{n^2+n+2}{n(n+1)(n+2)} \right.$$

$$+ 8 S_1(n) \frac{2n+3}{(n+1)^2(n+2)^2} + \frac{3n^4 + 15n^3 + 29n^2 + 50n + 44}{n(n+1)^3(n+2)^3}$$

$$+ \frac{2n^9 + 12n^8 + 27n^7 + 38n^6 + 58n^5 + 149n^4 + 26n^3 + 252n^2 + 128n + 32}{(n-1)n^3(n+1)^3(n+2)^3} \left. \right\}$$

$$+ 8 C_F T_R \left\{ \left[2S_1^2(n) - 2S_2(n) + 5 \right] \frac{n^2+n+2}{n(n+1)(n+2)} - 4 \frac{S_1(n)}{n^2} \right.$$

$$\left. + \frac{11n^4 + 26n^3 + 15n^2 + 8n + 4}{n^3(n+1)^3(n+2)} \right\}.$$

$$\begin{aligned}
 \gamma_{21}^{(1)}(n) = & 4 C_F^2 \left\{ \left[-2 S_1^2(n) + 10 S_1(n) - 2 S_2(n) \right] \frac{n^2+n+2}{(n^2-1)n} \right. \\
 & - 4 \frac{S_1(n)}{(n+1)^2} - \frac{12n^6 + 30n^5 + 43n^4 + 28n^3 - n^2 - 12n - 4}{(n-1)(n+1)^3 n^3} \left. \right\} \\
 & + 8 C_F C_A \left\{ \left[S_1^2(n) + S_2(n) - S_2\left(\frac{n}{2}\right) \right] \frac{n^2+n+2}{n(n^2-1)} \right. \\
 & - S_1(n) \frac{17n^4 + 41n^2 - 22n - 12}{3(n-1)^2 n^2 (n+1)} + \frac{n^3 + n^2 + 4n + 2}{n^3 (n+1)^3} \\
 & + \frac{109n^8 + 512n^7 + 879n^6 + 772n^5 - 104n^4 - 954n^3 - 278n^2 + 288n + 72}{9(n-1)^2 n^3 (n+1)^2 (n+2)^2} \left. \right\} \\
 & + \frac{32}{3} C_F T_R \left\{ \left[S_1(n) - \frac{8}{3} \right] \frac{n^2+n+2}{n(n^2-1)} + \frac{1}{(n+1)^2} \right\}. \\
 \gamma_{22}^{(1)}(n) = & C_A T_R \left\{ -\frac{160}{9} S_1(n) + \frac{32}{3} + \frac{16}{9} \cdot \frac{38n^4 + 76n^3 + 94n^2 + 56n + 12}{(n-1)n^2(n+1)^2(n+2)} \right\} \\
 & + C_F T_R \left\{ 8 + 16 \frac{2n^6 + 4n^5 + n^4 - 10n^3 - 5n^2 - 4n - 4}{(n-1)n^3(n+1)^3(n+2)} \right\} \\
 & + C_A^2 \left\{ S_1^2(n) \frac{-4n^2 + 14n + 10}{(n^2-1)n} + \frac{536}{9} S_1(n) \right. \\
 & + S_1(n) \frac{10n^7 - 10n^6 - 122n^5 - 182n^4 + 432n^3 + 89n^2 + 160n - 32}{(n-1)^2 n^2 (n+1)^2 (n+2)^2} \\
 & + S_2(n) \frac{28n^3 + 86n^2 + 22n + 20}{(n-1)n(n+1)(n+2)} - 16 S_1(n) S_2(n) + 32 \tilde{J}(n) \left. \right\}
 \end{aligned}$$

(Equation continues on
next page).

$$\begin{aligned}
 & + 16 \left[S_2(n) - S_2\left(\frac{n}{2}\right) \right] \left[S_1(n) - \frac{1}{(n+1)(n+2)} - \frac{1}{n(n-1)} \right] - 4 S_3\left(\frac{n}{2}\right) - \frac{64}{3} \\
 & + \frac{-468n^{10} - 3970n^9 - 11256n^8 - 11785n^7 - 4205n^6 - 4387n^5 - 5123n^4}{9(n-1)^2 n^3 (n+1)^3 (n+2)^3} \\
 & + \frac{-2742n^3 - 7640n^2 - 7896n - 2736}{(n-1)n^3(n+1)^3(n+2)^3} \left. \vphantom{\frac{-468n^{10} - 3970n^9 - 11256n^8 - 11785n^7 - 4205n^6 - 4387n^5 - 5123n^4}{9(n-1)^2 n^3 (n+1)^3 (n+2)^3}} \right\}.
 \end{aligned}$$

$$C_F = 4/3, C_A = 3, T_R = \frac{1}{2} \eta_f.$$

For electroproduction on protons

$$\begin{aligned}
 C_{11}^{(1)}(n) = 2 C_F \left\{ 2 S_1^2(n) + 3 S_1(n) - 2 S_2(n) - \frac{2 S_1(n)}{n(n+1)} \right. \\
 \left. - 9 + \frac{3}{n} + \frac{4}{n+1} + \frac{2}{n^2} \right\}
 \end{aligned}$$

$$\begin{aligned}
 C_{12}^{(1)}(n) = 8 T_R \left\{ \frac{1}{n} - \frac{1}{n^2} - \frac{6}{n+1} + \frac{6}{n+2} \right. \\
 \left. - S_1(n) \frac{n^2+n+2}{n(n+1)(n+2)} \right\}
 \end{aligned}$$

For other processes, cf. Ref. 16).

A.3 The functions $S, \tilde{S}$

The unique definition of the functions $S_\ell(n), \tilde{S}(n)$ that makes them Carlsonian and of Carlsonian exponent in n (for n arbitrary, real or complex) is

$$S_\ell(n) = \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(k+1)}{\Gamma(k+\ell+1)} - \frac{\Gamma(k+n+1)}{\Gamma(k+\ell+n+1)} \right\},$$

$$\tilde{S}(n) = -\frac{5}{8} \zeta(3) - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+n)^2} S_\ell(k+n)$$

ζ is Riemann's function. The behaviour of the $S_\ell, \tilde{S}$ as $n \rightarrow \infty$ may be found in Refs. 3), 4). Generally, it is $\sim C \log n + C_0 + C_1 1/n + O(1/n^2)$.

*) The formulae given here correspond to those of Refs. 3), 4), 11), and 14) for $n = \text{even}$. The expression given in the present paper is the one to be used for analytical continuation to general n . All the theoretical expressions have been double checked (see Ref. 24)); only in the C_A^2 coefficient for $\gamma_{22}^{(1)}$ does the calculation of Ref. 24) disagree with previous results.

APPENDIX B

For $n = 2$,

$$\gamma^{(0)}(2) = \begin{pmatrix} 64/9 & -4n_f/3 \\ -64/9 & 4n_f/3 \end{pmatrix}$$

$$\underline{S}(2) = \begin{pmatrix} 1 & -\frac{2n_f}{16+3n_f} \\ \frac{16}{3n_f} & \frac{3n_f}{16+3n_f} \end{pmatrix},$$

$$\gamma^{(1)}(2) = \frac{1}{243} \begin{pmatrix} 64[367-39n_f] & -3666n_f \\ -64[367-39n_f] & 3666n_f \end{pmatrix}.$$

For electroproduction on protons,

$$C_{11}^{(1)}(2) = \frac{4}{9}, \quad C_{12}^{(1)}(2) = -\frac{n_f}{2}.$$

APPENDIX C

The coefficients a_0, a_1 of Eq. (3.6) are

$$a_0 = -1.32$$

$$a_1 = 1.48$$

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