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BOUNDS ON THE RENORMALIZATION GROUP FUNCTIONS
IN QED AND QCD

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A B S T R A C T

We prove the inequalities $\psi(y, \alpha) \leq \alpha$ (for the Gell-Mann-Low function in QED) and $\gamma_0(\alpha_s) \leq 1$ [for the anomalous dimension of the gauge invariant operator $O(x)$]. The consequences of the inequalities are discussed: for modern energies, comparison of theoretical and experimental moments of deep inelastic structure functions has a meaning only for $N \leq 8$ (singlet case) and $N \leq 50$ (non-singlet case).

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1. INTRODUCTION

As is well known, the anomalous dimensions of the gauge invariant operators give us important information on the behaviour of physical processes with high momentum transfer. For example, using the Wilson expansion, we can relate the high energy behaviour of the moments of the deep inelastic structure functions with the anomalous dimensions of the gauge invariant composite operators.

The aim of this paper is to derive rigorous bounds on the behaviour of the GLM (Gell-Mann-Low) function in QED and the anomalous dimensions of the gauge invariant local operators in QCD. We will derive the inequalities $\psi(\alpha, y) \leq \alpha$ (for the GLM function in QED) and $\gamma_0(\alpha_s) \leq 1$ [for the anomalous dimension of the gauge invariant operator $O(x)$ in QCD]. The derivation of these inequalities is based on the use of rigorous properties of local quantum field theory: namely, our main tool is the KL (Källén-Lehmann) representation. Using the inequality for the GLM function, we will show that the QED with $N \gtrsim 25$ identical fermions does not fulfil this inequality ^{*)}.

We will also show that the comparison of the theoretical and experimental moments of the deep inelastic structure functions has sense ^{**)} only for $N \leq 8$ (singlet case) and $N \leq 50$ (non-singlet case).

2. A BOUND ON THE GLM FUNCTION IN QED

The KL representation for the transverse part of the photon propagator has the form ¹⁾

$$D^{tr}(k^2, \alpha_0, m^2) = \frac{1}{k^2 + i\varepsilon} + \int_0^\infty \frac{\rho(t, \alpha_0, m^2)}{k^2 - t + i\varepsilon} dt, \quad (1)$$

$$\rho(t, \alpha_0, m^2) \geq 0$$

Here, m is the mass of the electron and $\alpha_0 = 1/137$ is the fine structure constant. The invariant charge in QED is proportional to the transverse photon propagator ¹⁾⁻³⁾

*) I think that the non-fulfilment of this inequality in many-charged QED is evidence in favour of the unstability of the vacuum state in this model. Unfortunately, I cannot prove this statement.

**) We suppose that for modern energies ($Q^2 \leq 100 \text{ GeV}^2$) the strong coupling constant $\alpha_s = 0.2-0.3$.

$$\bar{\alpha}(x, y, \alpha) = \alpha_0 D^{tz}(K^2, \alpha_0, m^2) K^2 \quad (2)$$

Here, $x = -k^2/\mu^2$, $y = m^2/\mu^2$ and the renormalization condition is

$$\alpha = \alpha_0 K^2 D^{tz}(K^2, \alpha_0, m^2) \Big|_{K^2 = -\mu^2} \quad (3)$$

Using the representation (1), the definition (2) of the invariant charge and the renormalization condition (3), we obtain

$$\bar{\alpha}(x, y, \alpha) = \alpha_0 + \alpha_0 x \int_0^\infty \frac{\rho(t, \alpha_0, m^2)}{x + t/\mu^2} dt, \quad (4)$$

$$\alpha = \bar{\alpha}(1, y, \alpha) = \alpha_0 + \alpha_0 \int_0^\infty \frac{\rho(t, \alpha_0, m^2)}{1 + t/\mu^2} dt \quad (5)$$

The renormalization group equation for the invariant charge is ¹⁾⁻³⁾:

$$x \frac{\partial \bar{\alpha}(x, y, \alpha)}{\partial x} = \psi\left(\frac{y}{x}, \bar{\alpha}\right), \quad (6)$$

where

$$\psi(y, \alpha) = x \left. \frac{\partial \bar{\alpha}(x, y, \alpha)}{\partial x} \right|_{x=1} \quad (7)$$

Using the representations (4) and (5) and the definition (7) of the GLM function, we immediately obtain

$$\begin{aligned} \psi(y, \alpha) &= \alpha_0 \int_0^\infty \frac{\rho(t, \alpha_0, m^2)}{(1 + t/\mu^2)^2} \frac{t}{\mu^2} dt \leq \\ &\alpha_0 \int_0^\infty \frac{\rho(t, \alpha_0, m^2)}{1 + t/\mu^2} dt = \alpha - \alpha_0 < \alpha \end{aligned} \quad (8)$$

For QED with N identical fermions, the GLM function ^{*)} in the one-loop approximation is ¹⁾

$$\psi_1(\alpha) \equiv \psi_1(y=0, \alpha) = N \frac{\alpha^2}{3\pi} \quad (9)$$

Inequality (8) is not fulfilled for the approximation (9) for $\alpha \geq 3\pi/N$. The two-loop correction to the approximation (9) is equal to $\psi_2 = N(\alpha^3/4\pi^2)$. The relative contribution of the two-loop correction to the one-loop approximation (9) for $\alpha = 3\pi/N$ is equal to $9/4N$ and for $N \gtrsim 25$ is less than 10 %. Thus, we may think that the one-loop approximation is correct for $\alpha = 3\pi/N$.

So we obtained our second result : inequality (8) is not fulfilled in many-charged QED for $N \gtrsim 25$. I think that the non-fulfilment of inequality (8) in many-charged QED is evidence in favour of the instability of the vacuum state in this model.

3. A BOUND ON THE ANOMALOUS DIMENSIONS OF THE GAUGE INVARIANT OPERATORS IN QCD

For the two-point function of the gauge invariant local operators, the KL representation has the form (for simplicity, we consider the case of scalar operators, the proof is trivially generalized in the case of tensor operators)

$$\begin{aligned} F(-p^2) &= i \int e^{ipx} \langle 0 | T(O(x) O^\dagger(0)) | 0 \rangle d^4x = \\ &= \sum_{k=0}^{N-1} c_k (p^2)^k + \int_0^\infty \rho(t) \left[\frac{1}{t - p^2 - i\varepsilon} - \right. \\ &\quad \left. \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{1}{t - p^2 - i\varepsilon} \right)^{(k)} \right]_{p^2 = -\mu^2} dt, \end{aligned} \quad (10)$$

$$\rho(t) \geq 0$$

*) The derivation of inequality (8) is generalized trivially in the case of many-charged QED.

Due to the asymptotic freedom of QCD, the number of subtractions of the representation (10) coincides with the number of subtractions of the corresponding representation with free operators, and is equal to $d_0 - 1$, where d_0 is the dimension of the operator $O(x)$ *).

The N th derivative on $F(x)$

$$\phi(x) \equiv x \frac{(-1)^N}{N!} \frac{\partial^N F(x)}{\partial x^N} = x \int_0^\infty \frac{\rho(t)}{(t+x)^{N+1}} dt \quad (11)$$

transforms multiplicatively under the renormalization group transformations

$$\phi(x) \rightarrow Z_0^2 \phi(x) \quad (12)$$

Note that when we have a few operators $O_i(x)$ with the same twist, the renormalization transformation of the operators $O_i(x)$ in general has the form $O_i(x) \rightarrow Z_{ij} O_j(x)$. Using unitary transformations, we can always diagonalize the matrix Z_{ij} .

The renormalization group equation for the $\phi(x)$ has the form ^{1), 4)}

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + 2\gamma_0(\alpha_s) \right] \phi = 0 \quad (13)$$

Here, $\beta(\alpha_s)$ is the QCD β function and $\gamma_0(\alpha_s)$ is the anomalous dimension of the operator $O(x)$. It should be noted that the values of $\beta(\alpha_s)$ and $\gamma_0(\alpha_s)$ depend on the renormalization scheme ⁴⁾. In the one-loop approximation, $\gamma_0(\alpha_s)$ does not depend on the renormalization scheme ⁴⁾. Our renormalization is the following

$$\phi(x) \big|_{x=\mu^2} = \phi^{\text{free}}(\mu^2) \quad (14)$$

Using the renormalization group equation (13) and the renormalization condition (14), we immediately obtain

*) Since we are investigating the ultra-violet behaviour, the dependence on the quark masses is omitted.

$$f(\alpha_s) = \frac{x \frac{\partial}{\partial x} \phi(x) |_{x=\mu^2}}{\phi(x) |_{x=\mu^2}} = \frac{\mu^2 \int_0^\infty \frac{\rho(t) dt}{(\mu^2+t)^{N+1}} - \mu^4 (N+1) \int_0^\infty \frac{\rho(t) dt}{(\mu^2+t)^{N+2}}}{\mu^2 \int_0^\infty \frac{\rho(t) dt}{(\mu^2+t)^{N+1}}} \leq 1 \quad (15)$$

Note that when we use other renormalization schemes which are more convenient for the calculations ('t Hooft's scheme ⁵⁾), we have

$$\phi(x=\mu^2) = \phi^{free}(x=\mu^2) \bar{\phi}(\alpha_s),$$

$$\bar{\phi}(\alpha_s) = 1 + \sum_{n=1}^{\infty} \alpha_s^n C_n$$

Here, the coefficients C_n depend on the renormalization scheme. In this case, the inequality (15) will have the form

$$f(\alpha_s) + \beta(\alpha_s) \frac{\partial \phi(\alpha_s)}{\partial \alpha_s} \frac{1}{\phi(\alpha_s)} \leq 1$$

The difference between the various renormalization schemes arises ⁴⁾ only at the two-loop level, because

$$\beta(\alpha_s) \frac{\partial \phi(\alpha_s)}{\partial \alpha_s} \frac{1}{\phi(\alpha_s)} = \sum_{k=2}^{\infty} b_k (\alpha_s)^k$$

Let us now discuss the consequences of inequality (15) for the deep inelastic eN scattering. The high energy behaviour of the structure function moments ⁴⁾

$$M_n(\alpha^2) = \int_0^1 dx x^{n-2} F_i(x, \alpha^2)$$

is related to the anomalous dimensions of the operators

$$O_{NS,K}^{\mu_1 \dots \mu_n} = \bar{S} (\bar{\psi} \lambda^K f^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi - \text{traces})$$

(non-singlet case), and

$$O_{S,\psi}^{\mu_1 \dots \mu_n} = \bar{S} (\bar{\psi} f^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \psi - \text{traces}),$$

$$O_{S,G}^{\mu_1 \dots \mu_n} = \bar{S} (G^{\mu_1 \nu} D^{\mu_2} \dots D^{\mu_{n-1}} G^{\mu_n \nu} - \text{traces})$$

(singlet case).

The anomalous dimensions in the one-loop approximation are ⁴⁾

$$\delta_{NS}^n = \delta_{S,\psi}^n = \frac{\alpha_s}{4\pi} \frac{g}{3} \left[1 - \frac{2}{n(n+1)} + 4 \sum_{j=2}^n \frac{1}{j} \right],$$

$$\delta_{S,G}^n = 6 \frac{\alpha_s}{4\pi} \left[\frac{1}{3} - \frac{4}{n(n+1)} - \frac{4}{(n+1)(n+2)} + 4 \sum_{j=2}^N \frac{1}{j} \right] \quad (16)$$

For $\alpha_s = 0.3$, inequality (15) is not fulfilled for the approximation (16), for $N \geq 8$ (singlet case) and $N \geq 50$ (non-singlet case).

The non-fulfilment of inequality (15) for high N means that we cannot use approximation (16).

One way of improving the situation is to take into account all higher-loop corrections; however, I think it is an unreal task.

4. CONCLUSIONS

In conclusion, I would like to stress that the use of general properties of local quantum field theory enables us to obtain non-trivial information about the behaviour of the renormalization group functions.

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