Ref.TH.2747-CERN

GENERAL FORM OF EFFECTIVE LAGRANGIAN IN PATH
INTEGRAL QUANTIZATION FORMALISM

TU Tung-Sheng^{*)}
CERN -- Geneva

and

YIN Hung-Chun and YUAN Tu-Nan
The University of Science and Technology
Hefey -- China

ABSTRACT

The general form of the effective Lagrangian in path integral formalism is obtained. The effective Lagrangian includes the classical Lagrangian, the term proportional to $\delta(0)$ given by Lee and Yang and the additional term which is presented explicitly in series expansion. The formalism is explicitly covariant.

^{*)} After November 1979, the Institute of High Energy Physics, Peking, China.

Ref.TH.2747-CERN

21 September 1979

The concept of path integral quantization was proposed by Dirac¹⁾ in 1933. The method was perfected by Feynman²⁾ in 1948. The key step in this method is to calculate the generating functional $W(J)$ and to express it as an integral of the effective Lagrangian in co-ordinate space. For instance, in one dimensional Quantum Mechanics,

$$W[J] \sim \int [dq] \exp \left\{ i \int_{-\infty}^{\infty} dt [\mathcal{L}_{\text{eff}}(q, \dot{q}) + J(t)q(t)] \right\} \quad (1)$$

This is equivalent to calculating the transition amplitude³⁾

$$F(q, t; q_0, t_0) \equiv \langle q | e^{-iH(t-t_0)} | q_0 \rangle \quad (2)$$

and expressing it in terms of $\mathcal{L}_{\text{eff}}$. Where $|q_0\rangle$ and $|q\rangle$ are state vectors in Schrödinger representation, H is the Hamiltonian of the system and does not depend on time t explicitly. In what follows we shall concentrate on discussing only Quantum Mechanics. The extension to quantum field theory is straightforward. If the correspondence between the classical and quantum Hamiltonian is given by the Weyl ordering⁴⁾, we have

$$F(q, t; q_0, t_0) = \int \left[\frac{dq dp}{2\pi} \right] \exp \left\{ i \int_{t_0}^t dt [p\dot{q} - H(q, p)] \right\} \quad (3)$$

Note that different order procedures correspond to different forms of $H(q, p)$ ⁵⁾. Thus the correspondence between the classical and quantum Hamiltonian is not unique. As long as we can integrate the right-hand side of Eq. (3) with respect to p , we will find $\mathcal{L}_{\text{eff}}(q, \dot{q})$. Then we can calculate transition matrix elements (in Quantum Mechanics) or Green functions and Feynman rules (in field theory) by using Eq. (1). But to integrate Eq. (3) with respect to p is not easy.. Up to now this problem has been solved only in a few cases, for example, the case²⁾ where $\mathcal{L} = \frac{1}{2}\dot{q}^2 - V(q)$, and the case⁶⁾ where $\mathcal{L} = \frac{1}{2}\dot{q} f(q)$. In the following we shall give a general expression of $\mathcal{L}_{\text{eff}}$ in the general case.

In Eq. (3), p is only the integration variable, not the canonical momentum. Let us assume that the Lagrangian of a system is

$$\mathcal{L} = \mathcal{L}(q, \dot{q}) \quad (4)$$

where

$$\dot{q} = \frac{dq}{dt} \quad (5)$$

and introduce the canonical momentum

$$\tilde{p} = \frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}} \quad (6)$$

The Hamiltonian is then

$$H(q, \tilde{p}) = \tilde{p} \dot{q} - \mathcal{L}(q, \dot{q}) \quad (7)$$

where $\dot{q}$ is considered as a function of $q, \tilde{p}$. The quantum equations of motion of the system are

$$\begin{aligned} \dot{q} &= \frac{\partial}{\partial \tilde{p}} H(q, \tilde{p}) \\ \dot{\tilde{p}} &= - \frac{\partial}{\partial q} H(q, \tilde{p}) \end{aligned} \quad (8)$$

Expanding $H(q, p)$ in (3) at $\tilde{p}$, e.g.,

$$H(q, p) = H(q, \tilde{p}) + \left. \frac{\partial H(q, p)}{\partial p} \right|_{p=\tilde{p}} (p - \tilde{p}) + \sum_{l=2}^{\infty} \frac{1}{l!} \left. \frac{\partial^l H(q, p)}{\partial p^l} \right|_{p=\tilde{p}} (p - \tilde{p})^l \quad (10)$$

Then,

$$\begin{aligned} p \dot{q} - H(q, p) &= p \dot{q} - H(q, \tilde{p}) - \left. \frac{\partial H}{\partial p} \right|_{p=\tilde{p}} (p - \tilde{p}) - \sum_{l=2}^{\infty} \frac{1}{l!} \left. \frac{\partial^l H}{\partial p^l} \right|_{p=\tilde{p}} (p - \tilde{p})^l \\ &= \mathcal{L}(q, \dot{q}) + (p - \tilde{p}) \left[\dot{q} - \left. \frac{\partial H}{\partial p} \right|_{p=\tilde{p}} \right] - \sum_{l=2}^{\infty} \frac{1}{l!} \left. \frac{\partial^l H}{\partial p^l} \right|_{p=\tilde{p}} (p - \tilde{p})^l \end{aligned} \quad (11)$$

But

$$\left. \frac{\partial H(q, p)}{\partial p} \right|_{p=\tilde{p}} = \frac{\partial H(q, \tilde{p})}{\partial \tilde{p}} = \dot{q}$$

so the second term in Eq. (11) vanishes. Define the mass function as

$$m(q, \dot{q}) = \frac{\partial^2}{\partial \dot{q}^2} \mathcal{L}(q, \dot{q}) = \frac{\partial \tilde{p}}{\partial \dot{q}} \quad (12)$$

We have

$$\frac{\partial}{\partial \tilde{p}} = \frac{\partial \dot{q}}{\partial \tilde{p}} \frac{\partial}{\partial \dot{q}} = \frac{1}{m(q, \dot{q})} \frac{\partial}{\partial \dot{q}} \quad (13)$$

$$\frac{\partial^2 H(q, \tilde{p})}{\partial \tilde{p}^2} = \frac{1}{m(q, \dot{q})} \quad (14)$$

$$\frac{\partial^{l+2} H(q, \tilde{p})}{\partial \tilde{p}^{l+2}} = \left(\frac{1}{m(q, \dot{q})} \frac{\partial}{\partial \dot{q}} \right)^l \frac{1}{m(q, \dot{q})}, \quad l = 0, 1, 2, \dots$$

Equation (11) will become

$$p\dot{q} - H(q, p) = \mathcal{L}(q, \dot{q}) - \frac{1}{2m} (p - \tilde{p})^2 - \sum_{l=3}^{\infty} a_l(q, \dot{q}) (p - \tilde{p})^l \quad (15)$$

where

$$a_l(q, \dot{q}) = \frac{1}{l!} \left(\frac{1}{m} \frac{\partial}{\partial \dot{q}} \right)^{l-2} \frac{1}{m} m \equiv m(q, \dot{q}) \quad (16)$$

substituting (15) into Eq. (3) and changing the variable $p - \tilde{p} \rightarrow p$ we have

$$F(q, t; q_0, t_0) = \int [dq] e^{iS_0(t, t_0)} \int \left[\frac{dp}{2\pi} \right] \exp \left\{ -i \int_{t_0}^t d\tau \left[\frac{p^2}{2m} + \sum_{l=3}^{\infty} a_l(q, \dot{q}) p^l \right] \right\} \quad (17)$$

where

$$S_0(t, t_0) = \int_{t_0}^t d\tau \mathcal{L}(q, \dot{q}) \quad (18)$$

is the classical action.

Now, we introduce the external source $\eta(\tau)$ with the condition

$$\eta(\tau) \neq 0 \quad \text{when} \quad t_0 \leq \tau \leq t \quad (19)$$

Define the functional $F[\eta]$ as

$$F[\eta] = \int [dq] e^{iS_0(t, t_0)} \int \left[\frac{dp}{2\pi} \right] \exp \left\{ -i \int_{t_0}^t d\tau \left[\frac{p^2}{2m} - \eta p + \sum_{l=3}^{\infty} a_l(q, \dot{q}) p^l \right] \right\} \quad (20)$$

Eq. (20) can be rewritten as

$$F[\eta] = \int [dq] e^{iS_0(t, t_0)} \int \left[\frac{dp}{2\pi} \right] \exp \left\{ -i \int_{t_0}^t d\tau \sum_{l=3}^{\infty} a_l(q, \dot{q}) \left(\frac{\delta}{i\delta\eta} \right)^l \right\} \cdot \int \left[\frac{dp}{2\pi} \right] \exp \left\{ -i \int_{t_0}^t d\tau \left(\frac{p^2}{2m} - \eta p \right) \right\} \quad (21)$$

Obviously,

$$F[\eta] \Big|_{\eta=0} = F(q, t; q_0, t_0) \quad (22)$$

By using

$$\int_{-\infty}^{\infty} dx e^{i(ax^2 + bx)} = \left(\frac{\pi}{-ia}\right)^{1/2} e^{b^2/4ia} \quad (23)$$

We have

$$\begin{aligned} \int \left[\frac{dp}{2\pi}\right] \exp \left\{ -i \int_{t_0}^t d\tau \left(\frac{p^2}{2m} - \eta p \right) \right\} &= \int \prod_{j=1}^{n+1} \frac{dp_j}{2\pi} \exp \left\{ -i\epsilon \sum_{k=1}^{n+1} \left[\frac{p_k^2}{2m_k} - \eta_k p_k \right] \right\} \\ &= \prod_{k=1}^{n+1} \left(\frac{m_k}{2\pi i\epsilon} \right)^{1/2} e^{i\epsilon \frac{m_k}{2} \eta_k^2} \end{aligned} \quad (24)$$

where $\eta_k = \eta(t_k)$, $m_k = m[q(t_k), \dot{q}(t_k)]$ and n is the number of intervals of $(t-t_0)$. Note that

$$\delta(0) = 1/\epsilon, \quad \sqrt{m_k} = e^{\frac{\epsilon}{2} \delta(0) \ln m_k}$$

Then Eq. (24) becomes

$$\int \left[\frac{dp}{2\pi}\right] \exp \left\{ -i \int_{t_0}^t d\tau \left(\frac{p^2}{2m} - \eta p \right) \right\} = \left(\frac{1}{2\pi i\epsilon} \right)^{\frac{n+1}{2}} \exp \left\{ i \int_{t_0}^t d\tau \left[\frac{1}{2} m \eta^2 - \frac{i}{2} \delta(0) \ln m \right] \right\} \quad (25)$$

Substituting (25) into (2) we have

$$\begin{aligned} F[\eta] &= (2\pi i\epsilon)^{-1/2} \int \left[\frac{dq}{2\pi i\epsilon}\right] e^{i[\xi_0(t, t_0) + i\xi_1(t, t_0)]} \\ &\quad \cdot \exp \left\{ -i \int_{t_0}^t d\tau \sum_{l=3}^{\infty} q_l(q, i) \left(\frac{\delta}{i\delta\eta} \right)^l \right\} e^{i \int_{t_0}^t d\tau \frac{1}{2} m \eta^2(x)} \end{aligned} \quad (26)$$

where

$$\begin{aligned} \xi_1(t, t_0) &= \int_{t_0}^t \mathcal{L}_1(\tau) d\tau \\ \mathcal{L}_1(\tau) &= -\frac{i}{2} \delta(0) \ln m(q, i) \end{aligned} \quad (27)$$

This term was first obtained by Lee and Yang with mass $m(q, \dot{q}) = f(q)$ ⁶⁾. In our case m can be a function of q and $\dot{q}$. Taking $\eta \rightarrow 0$, we obtain from (26)

$$F(q, t; q_0, t_0) = (2\pi i \epsilon)^{-1/2} \int \left[\frac{dq}{2\pi i \epsilon} \right] \exp \left\{ i \left[S_0(t, t_0) + S_1(t, t_0) + S_r(t, t_0) \right] \right\} \quad (28)$$

where

$$e^{iS_r(t, t_0)} = \left\{ \exp \left[-i \int_{t_0}^t d\tau \sum_{\ell=3}^{\infty} a_{\ell}(q, \dot{q}) \left(\frac{\delta}{i\delta\eta} \right)^{\ell} \right] e^{i \int_{t_0}^t d\tau \frac{1}{2} m(q, \dot{q}) \eta^2(\tau)} \right\}_{\eta=0}$$

Now we simplify the expression $S_r(t, t_0)$. Define

$$D(\tau) = -i \sum_{\ell=3}^{\infty} a_{\ell}(q, \dot{q}) \left(\frac{\delta}{i\delta\eta(\tau)} \right)^{\ell} \quad (29)$$

$$\tilde{D} = \int_{t_0}^t d\tau D(\tau) \quad (30)$$

$$[\eta] = \int_{t_0}^t d\tau \frac{1}{2} m(q, \dot{q}) \eta^2(\tau) \quad (31)$$

Then we can write

$$\begin{aligned} e^{iS_r(t, t_0)} &= \left\{ e^{\tilde{D}} e^{i[\eta]} \right\}_{\eta=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{D}^n e^{i[\eta]} \Big|_{\eta=0} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{t_0}^t dt_1 dt_2 \dots dt_n D(t_1) D(t_2) \dots D(t_n) e^{i[\eta]} \Big|_{\eta=0} \end{aligned} \quad (32)$$

Now we calculate the "single time differentiation"

$$\left(\frac{\delta}{i\delta\eta} \right)^{\ell} e^{i[\eta]} = e^{i[\eta]} P_{\ell}(\eta) \quad (33)$$

By successive differentiation it is easy to verify that

$$P_{\ell+1}(\eta) = m(q, \dot{q}) \eta P_{\ell}(\eta) + \frac{1}{i\epsilon} \frac{dP_{\ell}(\eta)}{d\eta} \quad (34)$$

so we find

$$\begin{aligned} P_0(\eta) &= 1, \\ P_1(\eta) &= m(q, i) \eta, \\ P_2(\eta) &= [m(q, i) \eta]^2 + 2 \frac{m(q, i)}{2i\epsilon}, \\ &\vdots \end{aligned} \quad (35)$$

$$P_\ell(-\eta) = (-1)^\ell P_\ell(\eta) \quad (36)$$

We can have

$$P_\ell(\eta) = \sum_r \frac{\ell!}{r! \frac{\ell-r}{2}!} [m(q, i) \eta]^r \left[\frac{m(q, i)}{2i\epsilon} \right]^{\frac{\ell-r}{2}} \quad (37)$$

where

$$\begin{aligned} r &= \ell, \ell-2, \dots, 0, \quad \text{when } \ell \text{ is even} \\ r &= \ell, \ell-2, \dots, 1, \quad \text{when } \ell \text{ is odd} \end{aligned} \quad (38)$$

also

$$P_{2\ell+1}(0) = 0, \quad P_{2\ell}(0) = \frac{(2\ell)!}{\ell!} \left[\frac{m(q, i)}{2i\epsilon} \right]^\ell, \quad \ell = 0, 1, 2, \dots \quad (39)$$

For double time differentiation, put $\eta_1 \equiv n(t_1)$, $\eta_2 \equiv n(t_2)$, then

$$\left(\frac{\delta}{i\delta\eta_1} \right)^{\ell_1} \left(\frac{\delta}{i\delta\eta_2} \right)^{\ell_2} e^{i\tau\eta} = e^{i\tau\eta} \left\{ P_{\ell_1}(\eta_1) P_{\ell_2}(\eta_2) + \delta(t_1 - t_2) f_{\ell_1, \ell_2}(\eta_1) \right\} \quad (40)$$

put $t_1 = t_2 = \tau$ according to (40), the left-hand side is equal to

$$\left(\frac{\delta}{i\delta\eta(\alpha)} \right)^{\ell_1 + \ell_2} e^{i\tau\eta} = e^{i\tau\eta} P_{\ell_1 + \ell_2}(\eta(\alpha)) \quad (41)$$

The right-hand side of (40) is equal to

$$e^{i\tau\eta} [P_{\ell_1}(\eta) P_{\ell_2}(\eta) + \delta(0) f_{\ell_1, \ell_2}(\eta)] \quad (42)$$

so,

$$\delta(\omega) f_{\ell_1 \ell_2}(\eta) = p_{\ell_1 + \ell_2}(\eta) - p_{\ell_1}(\eta) p_{\ell_2}(\eta) \quad (43)$$

Also, for triple time differentiation,

$$\begin{aligned} \left(\frac{\delta}{i\delta\eta_1}\right)^{\ell_1} \left(\frac{\delta}{i\delta\eta_2}\right)^{\ell_2} \left(\frac{\delta}{i\delta\eta_3}\right)^{\ell_3} e^{i\tau\eta} &= e^{i\tau\eta} \left\{ p_{\ell_1}(\eta_1) p_{\ell_2}(\eta_2) p_{\ell_3}(\eta_3) \right. \\ &+ \delta(t_1 - t_3) p_{\ell_1}(\eta_1) f_{\ell_2 \ell_3}(\eta_3) + \delta(t_1 - t_2) f_{\ell_1 \ell_2}(\eta_2) p_{\ell_3}(\eta_3) \\ &\left. + \delta(t_1 - t_3) f_{\ell_1 \ell_3}(\eta_3) p_{\ell_2}(\eta_2) + \delta(t_2 - t_3) \delta(t_1 - t_2) f_{\ell_1 \ell_2 \ell_3}(\eta_3) \right\} \end{aligned} \quad (44)$$

Put $t_1 = t_2 = t_3 = \tau$ in Eq. (44) we can express $f_{\ell_1 \ell_2 \ell_3}(\eta)$ by $p_{\ell}(\eta)$ and $f_{\ell_1 \ell_2}(\eta)$, namely

$$\begin{aligned} [\delta(\omega)]^2 f_{\ell_1 \ell_2 \ell_3}(\eta) &= p_{\ell_1 + \ell_2 + \ell_3}(\eta) - p_{\ell_1}(\eta) p_{\ell_2}(\eta) p_{\ell_3}(\eta) - \delta(\omega) p_{\ell_1}(\eta) f_{\ell_2 \ell_3}(\eta) \\ &- \delta(\omega) f_{\ell_1 \ell_2}(\eta) p_{\ell_3}(\eta) - \delta(\omega) f_{\ell_1 \ell_3}(\eta) p_{\ell_2}(\eta) \end{aligned} \quad (45)$$

We can continue this kind of calculations to obtain multi-time differentiation.

It is easy to show that

$$D(\alpha) e^{i\tau\eta} = e^{i\tau\eta} f_1(\tau, \eta) \quad (46)$$

where

$$f_1(\tau, \eta) = -i \sum_{\ell=3}^{\infty} a_{\ell}(\tau, \dot{\eta}) p_{\ell}(\eta) \quad (47)$$

$$D(t_1) D(t_2) e^{i\tau\eta} = e^{i\tau\eta} \left\{ f_1(t_1, \eta_1) f_1(t_2, \eta_2) + \epsilon \delta(t_1 - t_2) f_2(t_1, \eta_1) \right\} \quad (48)$$

where

$$f_2(t, \eta) = \frac{(-i)^2}{\epsilon} \sum_{\ell_1, \ell_2=3}^{\infty} a_{\ell_1}(\tau) a_{\ell_2}(\tau) f_{\ell_1 \ell_2}(\eta) \quad (49)$$

and

$$a_{\ell}(\tau) = a_{\ell}(\eta(\tau), \dot{\eta}(\tau))$$

$$\begin{aligned} D(t_1)D(t_2)D(t_3) e^{i[\eta]} &= e^{i[\eta]} \left\{ f_1(t_1, \eta_1) f_1(t_2, \eta_2) f_1(t_3, \eta_3) \right. \\ &+ \epsilon \delta(t_1 - t_2) f_2(t_1, \eta_1) f_1(t_3, \eta_3) + \epsilon \delta(t_1 - t_3) f_2(t_1, \eta_1) f_1(t_2, \eta_2) \\ &\left. + \epsilon \delta(t_2 - t_3) f_2(t_2, \eta_2) f_1(t_1, \eta_1) + \epsilon^2 \delta(t_1 - t_2) \delta(t_2 - t_3) f_3(t_1, \eta_1) \right\} \end{aligned} \quad (50)$$

where

$$f_3(\tau, \eta) = \frac{(-i)^3}{\epsilon^2} \sum_{l_1, l_2, l_3=3}^{\infty} a_{l_1}(\tau) a_{l_2}(\tau) a_{l_3}(\tau) f_{l_1, l_2, l_3}(\eta) \quad (51)$$

Putting $\eta = 0$ in (46), (48) and (50), we have

$$\begin{aligned} D(\tau) e^{i[\eta]} \Big|_{\eta=0} &= f_1(\tau) \\ D(t_1)D(t_2) e^{i[\eta]} \Big|_{\eta=0} &= f_1(t_1) f_1(t_2) + \epsilon \delta(t_1 - t_2) f_2(t_1) \\ D(t_1)D(t_2)D(t_3) e^{i[\eta]} \Big|_{\eta=0} &= f_1(t_1) f_1(t_2) f_1(t_3) + \epsilon \delta(t_1 - t_2) f_2(t_1) f_1(t_3) \\ &+ \epsilon \delta(t_1 - t_3) f_2(t_1) f_1(t_2) + \epsilon \delta(t_2 - t_3) f_2(t_2) f_1(t_1) \\ &+ \epsilon^2 \delta(t_1 - t_2) \delta(t_2 - t_3) f_3(t_1) \\ &\vdots \end{aligned} \quad (52)$$

where

$$f_1(\tau) = f_1(\tau, 0), \quad f_2(\tau) = f_2(\tau, 0), \quad f_3(\tau) = f_3(\tau, 0) \quad (53)$$

The general term in (52) can be written as

$$\begin{aligned} &D(t_1)D(t_2) \cdots D(t_n) e^{i[\eta]} \Big|_{\eta=0} \\ &= \sum_{k_1, k_2, \dots, k_n=0} \sum_P P \prod_{r=1}^n \prod_{s=1}^{k_r} \frac{\delta(t_1^{rs} - t_2^{rs}) \delta(t_2^{rs} - t_3^{rs}) \cdots \delta(t_{r-1}^{rs} - t_r^{rs}) \epsilon^{r-1} f_r(t_1^{rs})}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \cdots (n!)^{k_n} k_n!} \end{aligned} \quad (54)$$

where the following convention must be understood, namely

the number of f_1 is equal to k_1

the number of f_2 is equal to k_2

the number of f_n is equal to k_n

the number of δ functions is equal to the number of ϵ . So, for $r = 1$, there is no δ function. $t_1^{rs}, t_2^{rs}, \dots, t_r^{rs}$ ($r = 1, 2, \dots, k_r$) is a permutation of $t, t, \dots, t_n$ and P denotes a permutation operation on $t_1, t_2, \dots, t_n$. $\sum_P$ denotes the summation for all permutations, also,

$$1 \cdot k_1 + 2 \cdot k_2 + \dots + n \cdot k_n = n \quad (55)$$

When we substitute (54) in to (32), different permutations in (54) will give the same result after integration with respect to $t_1 \dots t_n$. Thus, $\sum_P P \rightarrow n!$

$$e^{iS_r(t, t_0)} = \sum_{h=0}^{\infty} \frac{n!}{n!} \int_{t_0}^t dt_1 \dots dt_n \sum_{k_1, \dots, k_n} \frac{\prod_{r=1}^n \prod_{s=1}^{k_r} \delta(t_1^{rs} - t_2^{rs}) \dots \delta(t_{r-1}^{rs} - t_r^{rs}) \epsilon^{r-1} f_r(t_1^{rs})}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \dots (n!)^{k_n} k_n!}$$

$$= \sum_{h=0}^{\infty} \sum_{k_1, \dots, k_n} \frac{1}{k_1!} \left[\frac{1}{1!} \int_{t_0}^t dt_1 f_1(t_1) \right]^{k_1} \frac{1}{k_2!} \left[\frac{\epsilon}{2!} \int_{t_0}^t dt_2 f_2(t_2) \right]^{k_2} \dots \frac{1}{k_n!} \left[\frac{\epsilon^{n-1}}{n!} \int_{t_0}^t dt_n f_n(t_n) \right]^{k_n} \quad (56)$$

$$= \sum_{k_1=0}^{\infty} \frac{1}{k_1!} \left[\frac{1}{1!} \int_{t_0}^t dt_1 f_1(t_1) \right]^{k_1} \sum_{k_2=0}^{\infty} \frac{1}{k_2!} \left[\frac{\epsilon}{2!} \int_{t_0}^t dt_2 f_2(t_2) \right]^{k_2} \dots$$

$$= e^{\frac{1}{1!} \int_{t_0}^t dt_1 f_1(t_1)} \cdot e^{\frac{\epsilon}{2!} \int_{t_0}^t dt_2 f_2(t_2)} \dots$$

$$= \exp \left\{ \int_{t_0}^t d\tau \sum_{h=1}^{\infty} \frac{\epsilon^h}{h!} f_h(\tau) \right\} = \exp \left\{ \int_{t_0}^t d\tau \delta(\omega) \sum_{h=1}^{\infty} \frac{\epsilon^h}{h!} f_h(\tau) \right\}$$

Now, put $t_1 = t_2 \dots t_n = \tau$ in (54)

$$D^n(x) e^{i[t\eta]} \Big|_{\eta=0} = n! \sum_{k_1, \dots, k_n} \frac{f_1(\tau)^{k_1} f_2(\tau)^{k_2} \dots f_n(\tau)^{k_n}}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \dots (n!)^{k_n} k_n!} \quad (57)$$

$$\begin{aligned} \sum_{h=0}^{\infty} \frac{\epsilon^h}{h!} D^h(x) e^{i[t\eta]} \Big|_{\eta=0} &= \sum_{h=0}^{\infty} \sum_{k_1, \dots, k_n} \frac{\left[\frac{\epsilon}{1!} f_1(\tau) \right]^{k_1} \left[\frac{\epsilon^2}{2!} f_2(\tau) \right]^{k_2} \dots \left[\frac{\epsilon^n}{n!} f_n(\tau) \right]^{k_n}}{k_1! \dots k_n!} \\ &= \sum_{k_1=0}^{\infty} \frac{1}{k_1!} \left[\frac{\epsilon}{1!} f_1(\tau) \right]^{k_1} \dots \sum_{k_n=0}^{\infty} \frac{1}{k_n!} \left[\frac{\epsilon^n}{n!} f_n(\tau) \right]^{k_n} \end{aligned} \quad (58)$$

$$= \exp \left\{ \sum_{n=1}^{\infty} \frac{\epsilon^n f_n(\tau)}{n!} \right\}$$

According to (29)

$$\begin{aligned} G(x) &\equiv \sum_{h=0}^{\infty} \frac{\epsilon^h}{h!} D^h(x) e^{i[t\eta]} \Big|_{\eta=0} = \sum_{h=0}^{\infty} \frac{(-i\epsilon)^h}{h!} \sum_{l_1, \dots, l_n} q_{l_1}(x) q_{l_2}(x) \dots q_{l_n}(x) \left(\frac{\delta}{i\delta\eta} \right)^{l_1 + \dots + l_n} \\ &\quad \cdot e^{i[t\eta]} \Big|_{\eta=0} = \sum_{h=0}^{\infty} \frac{(-i\epsilon)^h}{h!} \sum_{l_1, \dots, l_n} q_{l_1}(x) \dots q_{l_n}(x) P_{l_1 + l_2 + \dots + l_n}(0) \end{aligned} \quad (59)$$

$$\sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} f_n(x) = \ln G(x) \quad (60)$$

$$e^{iS_r(t, t_0)} = \exp \left\{ \delta(\omega) \int_{t_0}^t d\tau \ln G(x) \right\} \quad (61)$$

$$S_r(t, t_0) \equiv \int_{t_0}^t d\tau \mathcal{L}_r(q, \dot{q}) = \int_{t_0}^t d\tau \{-i\delta(\omega) \ln G(\alpha)\} \quad (62)$$

$$\begin{aligned} \mathcal{L}_r(q, \dot{q}) &= -i\delta(\omega) \ln G(\alpha) \\ &= -i\delta(\omega) \ln \sum_{n=0}^{\infty} \frac{(i\epsilon)^n}{n!} \sum_{l_1, \dots, l_n=1}^{\infty} q_{l_1}(\alpha) \dots q_{l_n}(\alpha) P_{l_1+l_2+\dots+l_n}^{(0)} \end{aligned} \quad (63)$$

Finally, we get the general form of the effective Lagrangian on path integral quantization formalism

$$\mathcal{L}_{\text{eff}}(q, \dot{q}) = \mathcal{L}(q, \dot{q}) + \mathcal{L}_1(q, \dot{q}) + \mathcal{L}_r(q, \dot{q}) \quad (64)$$

where $\mathcal{L}(q, \dot{q})$ is the classical Lagrangian of the system, $\mathcal{L}_1(q, \dot{q})$, $\mathcal{L}_r(q, \dot{q})$ are given by (27) and (63). In Eqs (27), (63) and (64) we only give the $\mathcal{L}_{\text{eff}}$ in one dimensional quantum mechanics. The extension to the multi-dimensional case and to quantum field theory is straightforward. We shall not discuss it here.

One of us (TU) would like to thank W. Nahm for useful discussions.

REFERENCES

- 1) P.A.M. Dirac, Physik. Z. Sow-Jetunion, 3 (1933) 64.
- 2) R.P. Feynman, Rev. Mod. Phys. 20 (1948) 267.
- 3) See Ref. 2) and E.S. Abers and B.W. Lee, "Gauge Theories", Phys. Reports 9C (1973).
- 4) H. Weyl, Z. Physik 46 (1927) 1;
M. Mizrahi, J. Math. Phys. 16 (1975) 2201.
- 5) J.S. Dowker, J. Math. Phys. 17 (1976) 1873.
- 6) T.D. Lee and C.N. Yang, Phys. Rev. 128 (1962) 885.