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QCD PREDICTIONS FOR THE DISTRIBUTION FUNCTIONS OF
THE PHOTON USING ALTARELLI-PARISI TYPE EQUATIONS

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A B S T R A C T

In a leading approximation, the distribution functions of the photon are investigated using Altarelli-Parisi type equations including point-like and hadron-like components. Using a new method to make a Mellin transformation, the full inhomogeneous equation is solved numerically. The point-like and hadron-like parts of photon distribution function are discussed separately. The point-like part has a form $f(X,t)\ln Q^2$, where $t = \ln Q^2$. Analytic expressions for the photon distribution functions near $X = 1$ and $X = 0$ are also given. These results are compared with others which have been given using other methods.

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1. INTRODUCTION

We have known for some time that the photon possesses both point-like and hadron-like behaviours. In the point-like case, the photon is an elementary field and interacts with other charged particles through the electromagnetic interaction. On the other hand, when the photon acts with other particles like a hadron, we use, say, a vector dominance model, to describe it. Here, we hope to have a unified and quantitative description in the QCD language, i.e., to reveal the behaviour of the quarks, antiquarks and gluons inside the photon.

Recently, some authors have discussed the structure functions of the photon in three different ways: the operator product expansion method, a diagrammatic procedure and the Altarelli-Parisi equations. Witten¹⁾, Hill et al.²⁾ and Bardeen, Buras³⁾ use the operator product expansion. Witten points out that, in contrast to the structure function of the nucleon, there is a new term which comes from the point-like contribution in the structure functions of the photon. Llewellyn Smith⁴⁾ uses a diagrammatic procedure and also gives the contribution of this anomalous term. DeWitt et al.⁵⁾ use the A-P type equations of the photon and get a numerical solution using an iteration procedure. Brodsky et al.⁶⁾ and Frazer and Gunion⁷⁾ also use the structure function of the photon and emphasize the measurement of it in high energy $\gamma\text{-}\gamma$ scattering. In this paper, using A-P type equations⁸⁾ and a new method to make the Mellin transformation given by F. Martin⁹⁾, we discuss the behaviours of the distribution function of the photon, not only the point-like but also the hadron-like component. We also give analytic expressions for the photon structure function near $X = 1$ and $X = 0$. These results are compared with some others. Finally, the numerical results for the distribution functions are given for different Q^2 values.

2. ALTARELLI-PARISI TYPE EQUATIONS

In the lowest approximation, the structure functions of the photon obey the following A-P type equations:

$$\begin{aligned} \frac{d q_Y^i(x, t)}{dt} &= \epsilon_i^+ \frac{\alpha}{2\pi} a(x) + \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[P_{qg}\left(\frac{x}{y}\right) q_Y^i(y, t) + P_{qQ}\left(\frac{x}{y}\right) G_Y(y, t) \right] \\ \frac{d G_Y(x, t)}{dt} &= \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[P_{Gq}\left(\frac{x}{y}\right) \sum_{j=1}^{2f} q_Y^j(y, t) + P_{GG}\left(\frac{x}{y}\right) G_Y(y, t) \right] \end{aligned} \quad (1)$$

where $q_Y^i(X, t)$, $G_Y(X, t)$ are the distribution functions of the i^{th} quark and the gluon inside the photon respectively. ϵ_i is the fractional charge of the i^{th} quark.

$$\alpha_s(t) = \frac{d_s(t_0)}{1 + b \alpha_s(t_0) \ln Q^2/\Lambda^2} \approx 1/bt$$

$$t = \ln Q^2/\Lambda^2, \quad b = \frac{33 - 2f}{12\pi} \quad (2)$$

$$a(x) = x^2 + (1-x)^2$$

f is the flavour number, and the anomalous term $\epsilon_1^2 \alpha/2\pi a(X)$ comes from the point-like box diagram.

As is usually done, by taking the moment of each side of Eq. (1), we obtain an inhomogeneous equation:

$$\frac{d}{dt} \langle q_i^i(t) \rangle_n = \epsilon_i^2 \frac{d}{2\pi} a(n) + \frac{d_s(t)}{2\pi} \frac{g}{3} A_n^{q\bar{q}} \langle q_i^i(t) \rangle_n + \frac{d_s(t)}{2\pi} \frac{g}{3} A_n^{qG} \langle G_i(t) \rangle_n$$

$$\frac{d}{dt} \langle G_i(t) \rangle_n = \frac{d_s(t)}{2\pi} \frac{g}{3} A_n^{Gq} \sum_{i=1}^{2f} \langle q_i^i(t) \rangle_n + \frac{d_s(t)}{2\pi} \frac{g}{3} A_n^{GG} \langle G_i(t) \rangle_n \quad (3)$$

where the definitions of the moments are

$$\langle q_i^i(t) \rangle_n = \int_0^1 x^{n-1} dx q_i^i(x, t)$$

$$\langle G_i(t) \rangle_n = \int_0^1 x^{n-1} dx G_i(x, t) \quad (4)$$

and

$$a(n) = \int_0^1 x^{n-1} dx a(x)$$

$$A_n^{q\bar{q}} = \frac{3}{4} + \frac{1}{2n} - \frac{1}{2(n+1)} - \psi(n+1) - C$$

$$A_n^{qG} = \frac{3(2+n+n^2)}{16n(n+1)(n+2)}$$

$$A_n^{Gq} = \frac{2+n+n^2}{2n(n^2-1)} \quad (5)$$

$$A_n^{GG} = \frac{g}{4} \left[\frac{33-2f}{36} + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2} - \psi(n+1) - C \right],$$

$\psi(n+1) = \Gamma'(n+1)/\Gamma(n+1)$ is the digamma function, c is Euler's constant, $c = 0.5772\dots$. Up to a multiplicative factor, the A 's are the usual anomalous dimensions in QCD¹⁰⁾.

After diagonalization, we obtain the solution of this inhomogeneous equation as follows:

$$\begin{aligned}\langle q_Y^i(t) \rangle_n &= \left[\langle q_Y^i(t_0) \rangle_n - \frac{1}{2f} \langle q_Y(t_0) \rangle_n \right] F_{qq_1}(n, t) \\ &+ \frac{1}{2f} \langle q_Y(t_0) \rangle_n F_{qq_2}(n, t) \\ &+ \langle G_Y(t_0) \rangle_n F_{qG}(n, t) \\ &+ F_{qp}(n, t) \\ \langle G_Y(t) \rangle_n &= \langle q_Y(t_0) \rangle_n F_{Gq}(n, t) \\ &+ \langle G_Y(t_0) \rangle_n F_{GG}(n, t) \\ &+ F_{Gp}(n, t)\end{aligned}\tag{6}$$

The functions $\tilde{F}(n, t)$ are given in Appendix A. We have defined

$$\langle q_Y(t) \rangle_n = \sum_{i=1}^{2f} \langle q_Y^i(t) \rangle_n\tag{7}$$

and t_0 is the initial value of t .

From (6), we see that, besides the hadron-like contribution to the moment of quarks and gluons, there are point-like contributions $F_{qp}(n, t)$, $F_{Gp}(n, t)$. In order to obtain the contribution from the hadron-like part, we must have an initial input of the structure functions of the photon, but for the point-like, it is not necessary.

The next step is to calculate the inverse Mellin transformation to get back the X distribution as

$$q_Y^i(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn \, x^{-n} \langle q_Y^i(t) \rangle_n\tag{8}$$

$$G_Y(x, t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dn \ x^{-n} \langle G_Y(t) \rangle_n \quad (8)$$

cont.

where the contour $(\gamma-i\infty, \gamma+i\infty)$ is to the right of all singularities of $\langle q_Y(t) \rangle_n, \langle G_Y(t) \rangle_n$ in the complex n plane (Fig. 1). The formulae (8) have another form which we used in our calculation:

$$\begin{aligned} q_Y^i(x, t) = & \int_x^1 \frac{dz}{z} \left[q_Y^i(z, t_0) - \frac{1}{2f} q_Y(z, t_0) \right] \tilde{F}_{qY_1} \left(\frac{x}{z}, t \right) \\ & + \int_x^1 \frac{dz}{z} \frac{1}{2f} q_Y(z, t_0) \tilde{F}_{qY_2} \left(\frac{x}{z}, t \right) \\ & + \int_x^1 \frac{dz}{z} G_Y(z, t_0) \tilde{F}_{qG} \left(\frac{x}{z}, t \right) \\ & + \tilde{F}_{qY} (x, t) \end{aligned} \quad (9a)$$

$$\begin{aligned} G_Y(x, t) = & \int_x^1 \frac{dz}{z} q_Y(z, t_0) \tilde{F}_{Gq} \left(\frac{x}{z}, t \right) \\ & + \int_x^1 \frac{dz}{z} G_Y(z, t_0) \tilde{F}_{GG} \left(\frac{x}{z}, t \right) \\ & + \tilde{F}_{GY} (x, t) \end{aligned} \quad (9b)$$

where $\tilde{F}(X, t)$ is the inverse Mellin transformation of $F(n, t)$,

$$q_Y(z, t) = \sum_{i=1}^{2f} q_Y^i(z, t)$$

and the s variable used later on is defined as

$$s = \frac{4}{3\pi b} \ln t/t_0$$

In order to get the inverse Mellin transformation of $F(n, t)$, we use two different but mathematically equivalent forms of $F(n, t)$, (see Appendix B) which are very useful when studying the behaviours of the distribution functions near $X = 0$ and $X = 1$ respectively.

From (6) and Appendix A, we can also see that, if we only consider the point-like contribution and neglect the term $t_0 \exp A_n^+$ compared with t , these are just the results given by Llewellyn Smith⁴⁾ with a colour factor 3. Thus, we have a point-like piece of the form $F(X,t)t$, where $F(X,t)$ is not only a function of X but also of t (also see numerical results). In the limiting case where t is large enough, $F(X,t)$ is independent of t . The t dependence in $F(X,t)$ that we have is a non-leading logarithmic effect. For complete consistency, all such non-leading logarithms should be included, but this would be a very complicated calculation.

3. BEHAVIOUR NEAR THE KINEMATIC BOUNDARIES

In this section we shall discuss the behaviour of the quark and gluon densities near $X = 1$ and $X = 0$ assuming a general behaviour of the initial densities.

A) Behaviour near $X = 1$

To study the behaviour of a parton density near $X = 1$ is equivalent to studying the limit of its Mellin transform for n going to infinity. Therefore, we make a $1/n$ expansion of its Mellin transform and calculate the inverse Mellin transform. We assume that the behaviours of the quark and gluon densities inside the photon near $X = 1$ for $t = t_0$ are

$$x q_i^i(x, t_0) \sim a_i (1-x)^{b_i}$$

$$x G_g(x, t_0) \sim a_g (1-x)^{b_g}$$

For the quark distribution function $X q_Y^i(X, t)$, using formula (9.a) and (B.2) of Appendix B we get three dominant terms near $X = 1$, that is

$$\begin{aligned} x q_Y^i(x, t) \sim & a_i e^{s(3/4 - c)} \frac{\Gamma(b_i + 1)}{\Gamma(s + b_i + 1)} (1-x)^{s+b_i} \\ & + a_g \frac{3}{20} e^{s(3/4 - c)} \frac{\Gamma(b_g + 1)}{\Gamma(b_g + 2 + s)} \frac{1}{[\lambda + \ln \frac{1}{1-x} + \psi(b_g + 2 + s)]} (1-x)^{b_g + 1 + s} \\ & + \varepsilon_i^2 \frac{\alpha}{\alpha_s} \frac{3}{8} \left[\frac{1}{\ln \frac{1}{1-x} + \frac{3\pi b}{4} - \frac{3}{4}} - \frac{e^{s(3/4 - c) - 53\pi b/4}}{[\ln \frac{1}{1-x} + \frac{3\pi b}{4} - \frac{3}{4} + c + \psi(s+1)]} \frac{(1-x)^s}{\Gamma(s+1)} \right] \end{aligned} \quad (10)$$

The first one corresponds to the a quark which modifies its momentum by radiating gluons. The second term corresponds to a $q\bar{q}$ pair produced from the gluon. The third one comes from the point-like contribution $\lambda = c - (21 - 2f)/20$.

From (10), obviously, the main contribution near $X = 1$ is given by the point-like part^{*)}. In the point-like part, when X is close to 1, if s is large enough we can neglect the second term, and the point-like contribution then has the form

$$\varepsilon_i^+ \frac{d}{ds} f(x)$$

where

$$f(x) = \frac{1}{\frac{8}{3} \ln \frac{1}{1-x} + 2\pi b - 2}$$

is a scaling function. This behaviour is similar to the result given by the diagrammatic approach^{7),11)}.

For the gluon distribution, using formulae (9.b) and (B.2) of Appendix B, we get three dominant contributions near $X = 1$. That is:

$$\begin{aligned} x G_T(x, t) \sim & \frac{2}{5} \sum_i a_i e^{S(\frac{3}{4}-c)} \frac{\Gamma(b_i+1)}{\Gamma(b_i+2+s)} \frac{(1-x)^{b_i+1+s}}{[\lambda + \ln \frac{1}{1-x} + \psi(b_i+2+s)]} \\ & + a_g e^{S\frac{g}{4}(\frac{33-2f}{36}-c)} \frac{\Gamma(b_g+1)}{\Gamma(\frac{g}{4}s+b_g+1)} (1-x)^{\frac{g}{4}s+b_g} \quad (11) \\ & + \frac{3}{20} \sum_i \varepsilon_i^+ \frac{d}{ds} (1-x) \frac{1}{\ln \frac{1}{1-x}} \frac{1}{\ln \frac{1}{1-x} + \frac{3\pi b}{4} + \frac{1}{4}} \end{aligned}$$

The first one corresponds to the gluons produced by the quarks through bremsstrahlung. The second term corresponds to the gluons radiated by the gluons themselves. The last one comes from the point-like contribution.

B) Behaviour near $X = 0$

To study the behaviour of a parton distribution near $X = 0$ is equivalent to studying the behaviour of its Mellin transform near its rightmost singularity in the complex n plane. Therefore, after calculating the contribution

^{*)} Note that non-leading logarithms are expected to contribute non-uniformity near the kinematic boundary.

of the moment at the rightmost singularity and integrating it on the contour C of Fig. 1, we obtain an expansion of the parton density near $X = 0$.

Let us assume that the behaviours of the quark and gluon distributions near $X = 0$ for $t = t_0$ are

$$x q_Y^i(x, t_0) \sim C_i$$

$$x G_Y(x, t_0) \sim C_g$$

For the quark distribution function, using formula (9.a) and (A.1) of Appendix A we get three dominant contributions corresponding to the quark which modifies its momentum by radiating gluons, $q\bar{q}$ pair production and the point-like contribution respectively.

$$\begin{aligned} x q_Y^i(x, t) \sim & \frac{2}{81} \sum_i C_i e^{-s(\frac{33}{16} + \frac{1}{72}f)} \left(\frac{qs}{4 \ln 1/x}\right)^{\frac{1}{2}} I_1(3\sqrt{s \ln 1/x}) \\ & + \frac{1}{18} C_g e^{-s(\frac{33}{16} + \frac{1}{72}f)} \left(\frac{qs}{4 \ln 1/x}\right)^{\frac{1}{2}} I_1(3\sqrt{s \ln 1/x}) \\ & + \frac{41}{729} \epsilon_i^2 \alpha b t_0 e^{-s(\frac{33}{16} + \frac{1}{72}f)} \left(\frac{qs}{4 \ln 1/x}\right)^{\frac{3}{2}} I_3(3\sqrt{s \ln 1/x}). \end{aligned} \quad (12)$$

For the gluon distribution, $x G_Y(X, t)$, we also get three dominant terms corresponding to quark bremsstrahlung, gluons radiated by the gluons themselves and the point-like contribution.

$$\begin{aligned} x G_Y(x, t) \sim & \frac{4}{9} \sum_i C_i e^{-s(\frac{33}{16} + \frac{1}{72}f)} I_0(3\sqrt{s \ln 1/x}) \\ & + C_g e^{-s(\frac{33}{16} + \frac{1}{72}f)} I_0(3\sqrt{s \ln 1/x}) \\ & + \frac{4}{81} \sum_i \epsilon_i^2 \alpha b t_0 e^{-s(\frac{33}{16} + \frac{1}{72}f)} \left(\frac{qs}{4 \ln 1/x}\right) I_2(3\sqrt{s \ln 1/x}) \end{aligned} \quad (13)$$

Similarly, the non-leading logarithms are expected to give contributions to the point-like piece for the parton boundary behaviours.

In the leading logarithmic approximation, we obtain the boundary behaviours near $X = 1$ and $X = 0$ (10)-(13), but in this region, higher order contributions must be considered. This would be a very complicated calculation.

4. NUMERICAL RESULTS

In this section, we give the numerical results for the distribution functions of the photon.

In order to get the inverse Mellin transformation of the hadron-like part, we have to input an initial value of the distribution function of the photon. Unfortunately, up to now, we have no data on it. However, we can estimate it by using crude, but apparently reasonable, assumptions.

In the general case, both the quark and gluon distribution functions of the photon contain hadron-like and point-like components. For the hadron-like piece we use vector meson dominance and SU(3) symmetry to estimate it, e.g. ^{*)},

$$q_r^i(x, t_0) = \sum_j \frac{4\pi\alpha}{f_v^2} q_v^i(x, t_0) \quad (14)$$

(hadron-like)

and

$$\frac{f_\rho^2}{4\pi} : \frac{f_\omega^2}{4\pi} : \frac{f_\phi^2}{4\pi} = 1 : 9 : \frac{9}{2}$$

$$\frac{f_\rho^2}{4\pi} = 2.5$$

Next, we assume the valence quark of ρ_0, ω (u,d-quark) and ϕ (s-quark) have the same distribution as that of the valence quark of π^0 and the sea quark of ρ_0, ω and ϕ have the same distribution as that of the π^0 sea-quark if the spin has little effect on distribution functions. Assuming SU(2) symmetry, π^0 and $\pi^\pm$ have the same sea-quark and gluon distributions, but the valence quark distributions in π^0 for each flavour (u and d) are half those in the relevant charged pion.

Recently, a CERN-SPS group has given the distribution function of $\pi^\pm$ ¹²⁾. According to the principle mentioned above, we take

$$\begin{aligned} x q_{\gamma}^{u,d}(x, t_0) &= \left(\frac{4\pi\alpha}{f_\rho^2} + \frac{4\pi\alpha}{f_\omega^2} \right) \times q_{\pi^0(v)}(x, t_0) \\ &+ \left(\frac{4\pi\alpha}{f_\rho^2} + \frac{4\pi\alpha}{f_\omega^2} + \frac{4\pi\alpha}{f_\phi^2} \right) \times q_{\pi^0(s)}(x, t_0) \end{aligned}$$

^{*)} I thank John Ellis for this suggestion.

$$\begin{aligned} \times q_8^s(x, t_0) &= \frac{4\pi\alpha}{f_\phi^2} 2 \times q_{\pi^0(v)} + \left(\frac{4\pi\alpha}{f_\rho^2} + \frac{4\pi\alpha}{f_\omega^2} + \frac{4\pi\alpha}{f_\phi^2} \right) \times q_{\pi^0(s)} \\ \times G_8(x, t_0) &= \left(\frac{4\pi\alpha}{f_\rho^2} + \frac{4\pi\alpha}{f_\omega^2} + \frac{4\pi\alpha}{f_\phi^2} \right) \times G_{\pi^0}(x, t_0) \end{aligned}$$

where

$$\times q_{\pi^0(v)}(x, t_0) = 0.175 x^{0.4} (1-x)^{0.9} \text{ for one kind of the valence quark of } \pi^0$$

$$\times q_{\pi^0(s)}(x, t_0) = 0.09 (1-x)^{4.4} \text{ for one kind of the sea quark of } \pi^0$$

$$\times G_{\pi^0}(x, t_0) = a(1-x)^5$$

$$a = 3.36$$

a is determined by momentum conservation of π^0 .

$$Q_s^2 = 15 \text{ GeV}^2/c^2$$

For the point-like piece, compared with the hadron-like piece, we assume that at low Q^2 it can be neglected. Thus, using the expressions $F_{qp}(n, t)$, $F_{Gp}(n, t)$ and making a Mellin transformation we can get the point-like piece at larger Q^2 . For calculational convenience, we assume that at $Q^2 = 4.6 \text{ GeV}^2/c^2$, it can be neglected, which corresponds to $s = 0.2$ from $Q^2 = 4.6 \text{ GeV}^2/c^2$ to $Q^2 = 15 \text{ GeV}^2/c^2$. After fitting the numerical output, we get the approximate expressions at $Q^2 = 15 \text{ GeV}^2/c^2$:

$$\times q_{fp}(x) = \frac{\alpha}{\pi} (4.081) \left[(-0.12 + 0.5x)(1-x)^{0.1} + \frac{0.07}{x} - 0.07 \right] \text{ for charge} = 2/3$$

$$\times q_{fp}(x) = \frac{\alpha}{\pi} (4.081) \left[(-0.032 + 0.1x)(1-x)^{0.1} + \frac{0.01}{x} - 0.035 \right] \quad (16)$$

for charge = 1/3

$$\times G_{fp}(x) = \frac{\alpha}{\pi} (4.081) \left[\frac{0.0012}{x^3} + 0.0001x \right]$$

Then we take the sum of Eqs (15) and (16) as initial input^{*)}.

*) I thank R. Petronzio for this suggestion.

For small values of X , we use the formula (B.1) of Appendix B and compute the contributions of all the singularities of $F(n,t)$ in the complex n plane: $n = 1, 0, -1, -2, \dots$. Using the fact that the inverse Mellin transform of $1/(n+p)^r$ is $X^p 1/\Gamma(r) (\ln 1/X)^{r-1}$, we use the computer to sum up all the terms, and finally obtain the $Xq_Y^i(X,t)$, $XG_Y(X,t)$ for small X using Eqs (9.a) and (9.b).

For large values of X , we use formula (B.2) of Appendix B and expand the integrand in a series in $1/(n-1)$. Using the relation that the inverse Mellin transform of $1/(n-1)^r$ is $1/X \Gamma(r) (\ln 1/X)^{r-1}$, we obtain the inverse Mellin transform of each term of the series. After integration over v , $\tilde{F}(X,t)$ is obtained. Also, $Xq_Y^i(X,t)$, $XG_Y(X,t)$ are obtained from Eqs (9.a) and (9.b) for large values of X .

Finally we check that the two types of calculation give the same results for $X \sim 0.6 - 0.7$.

The numerical results are shown in Figs 2-9. There is a free parameter Λ in the theory which can be chosen. If we take $\Lambda^2 = 0.25 \text{ GeV}^2/c^2$ as in the proton case and take $Q_0^2 = 15 \text{ GeV}^2/c^2$, then $s = 0.2, 0.1$ and 0.05 correspond to $Q^2 = 77.6, 31.8$ and $21.5 (\text{GeV}/c)^2$ respectively.

We have discussed the point-like piece before. From Figs 2-3 we see again that we have a point-like piece of the form $F(X,t)t$. As a comparison, the contribution in the leading logarithm approximation is also shown⁴⁾.

About the hadron-like piece, we compare, as an example in Fig. 5, our up-quark distribution function with the result given by Jones et al.⁵⁾. Compared with their input, we have not only a hadron-like but also a point-like piece in our input. We also take not only the ρ meson but also ω and ϕ in vector dominance. Thus our results are larger than theirs.

From Figs 6-8, we can see that in the general case, the hadron-like piece of Xq_Y^u , Xq_Y^d and Xq_Y^s are not equal to one another. In their input, Xq_Y^d and Xq_Y^s have the same point-like piece but different hadron-like pieces, Xq_Y^u and Xq_Y^d have the same (approximate) hadron-like piece but different point-like pieces. Thus the outputs are not equal.

I would like to thank F. Martin for useful discussions and some relevant programs. I also wish to acknowledge useful discussions with S.D. Drell, J. Ellis and R. Petronzio and to thank the CERN theory group for kind hospitality while this work was in progress.

APPENDIX A

$$F_{qq_1}(u, t) = e^{A_n^{qq} s}$$

$$F_{qq_2}(u, t) = \frac{1}{2}(e^{A_n^{+} s} + e^{A_n^{-} s}) - \frac{1}{2}(A_n^{GG} - A_n^{qq}) \frac{e^{A_n^{+} s} - e^{A_n^{-} s}}{A_n^{+} - A_n^{-}}$$

$$F_{qG}(u, t) = A_n^{qq} \frac{e^{A_n^{+} s} - e^{A_n^{-} s}}{A_n^{+} - A_n^{-}} \quad (A.1)$$

$$F_{qp}(u, t) = \left(\varepsilon_i^{+} - \frac{1}{2f} \sum_i \varepsilon_i^{+} \right) \frac{\alpha}{2\pi} a(u) \frac{1}{1 - \frac{4}{3\pi b} A_n^{qq}} [t - t_0 e^{A_n^{qq} s}]$$

$$+ \frac{1}{2f} \sum_i \varepsilon_i^{+} \frac{\alpha}{2\pi} a(u) \frac{1}{2} \left\{ \frac{1}{1 - \frac{4}{3\pi b} A_n^{+}} (t - t_0 e^{A_n^{+} s}) + \frac{1}{1 - \frac{4}{3\pi b} A_n^{-}} (t - t_0 e^{A_n^{-} s}) \right.$$

$$\left. - \frac{A_n^{GG} - A_n^{qq}}{A_n^{+} - A_n^{-}} \left[\frac{1}{1 - \frac{4}{3\pi b} A_n^{+}} (t - t_0 e^{A_n^{+} s}) - \frac{1}{1 - \frac{4}{3\pi b} A_n^{-}} (t - t_0 e^{A_n^{-} s}) \right] \right\}$$

$$F_{Gq}(u, t) = A_n^{qq} \frac{e^{A_n^{+} s} - e^{A_n^{-} s}}{A_n^{+} - A_n^{-}}$$

$$F_{GG}(u, t) = \frac{1}{2}(e^{A_n^{+} s} + e^{A_n^{-} s}) + \frac{1}{2} \frac{A_n^{GG} - A_n^{qq}}{A_n^{+} - A_n^{-}} (e^{A_n^{+} s} - e^{A_n^{-} s}) \quad (A.2)$$

$$F_{GP}(u, t) = \sum_i \varepsilon_i^{+} \frac{\alpha}{2\pi} a(u) \frac{A_n^{qq}}{A_n^{+} - A_n^{-}} \left[\frac{1}{1 - \frac{4}{3\pi b} A_n^{+}} (t - t_0 e^{A_n^{+} s}) - \frac{1}{1 - \frac{4}{3\pi b} A_n^{-}} (t - t_0 e^{A_n^{-} s}) \right]$$

where

$$A_n^{\pm} = \frac{1}{2} \left[A_n^{GG} + A_n^{qq} \pm \sqrt{(A_n^{GG} - A_n^{qq})^2 + 8f A_n^{qq} A_n^{GG}} \right]$$

$$s = \frac{4}{3\pi b} \ln \frac{t}{t_0}$$

APPENDIX B

Two different forms of $F(n,t)$. (a) is useful near $X = 0$, and (b) is useful near $X = 1$.

(a)

$$F_{qq_1}(u,t) = e^{A_n^{qq} s}$$

$$F_{qq_2}(u,t) = \sum_{l=0}^{\infty} \frac{1}{l!} s^l (P_n)_l$$

$$F_{qG}(u,t) = A_n^{qG} \sum_{l=0}^{\infty} \frac{1}{l!} s^l (L_n)_l$$

$$F_{qb}(u,t) = \left(\varepsilon_i^+ - \frac{1}{2f} \sum_i \varepsilon_i^+ \right) \frac{d}{2\pi} t_0 a(u) \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{3\pi b}{4} s \right)^l (Q_n)_l \\ + \frac{1}{2f} \sum_i \varepsilon_i^+ \frac{d}{2\pi} t_0 a(u) \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{3\pi b}{4} s \right)^l (R_n)_l$$

$$F_{Gq}(u,t) = A_n^{Gq} \sum_{l=0}^{\infty} \frac{1}{l!} s^l (L_n)_l$$

$$F_{GG}(u,t) = \sum_{l=0}^{\infty} \frac{1}{l!} s^l (S_n)_l \quad (B.1)$$

$$F_{Gb}(u,t) = \sum_i \varepsilon_i^+ \frac{d}{2\pi} t_0 a(u) A_n^{Gq} \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{3\pi b}{4} s \right)^l (H_n)_l$$

where

$$(L_n)_l = \frac{(A_n^+)^l - (A_n^-)^l}{A_n^+ - A_n^-}$$

$$(P_n)_l = \frac{1}{2} \left[(A_n^+)^l + (A_n^-)^l - \frac{A_n^{GG} - A_n^{qq}}{A_n^+ - A_n^-} \left\{ (A_n^+)^l - (A_n^-)^l \right\} \right]$$

$$(S_n)_l = \frac{1}{2} \left[(A_n^+)^l + (A_n^-)^l + (A_n^{GG} - A_n^{qq}) \frac{(A_n^+)^l - (A_n^-)^l}{A_n^+ - A_n^-} \right]$$

$$(Q_n)_l = \frac{1 - \left(\frac{4}{3\pi b} A_n^{qq} \right)^l}{1 - \frac{4}{3\pi b} A_n^{qq}}$$

$$\begin{aligned}
 (R_n)_2 &= \frac{1}{2} \left[\frac{1 - \left(\frac{4}{3\pi b} A_n^+\right)^2}{1 - \frac{4}{3\pi b} A_n^+} + \frac{1 - \left(\frac{4}{3\pi b} A_n^-\right)^2}{1 - \frac{4}{3\pi b} A_n^-} \right. \\
 &\quad \left. - \frac{A_n^{GG} - A_n^{qq}}{A_n^+ - A_n^-} \left\{ \frac{1 - \left(\frac{4}{3\pi b} A_n^+\right)^2}{1 - \frac{4}{3\pi b} A_n^+} - \frac{1 - \left(\frac{4}{3\pi b} A_n^-\right)^2}{1 - \frac{4}{3\pi b} A_n^-} \right\} \right] \\
 (H_n)_2 &= \frac{1}{A_n^+ - A_n^-} \left[\frac{1 - \left(\frac{4}{3\pi b} A_n^+\right)^2}{1 - \frac{4}{3\pi b} A_n^+} - \frac{1 - \left(\frac{4}{3\pi b} A_n^-\right)^2}{1 - \frac{4}{3\pi b} A_n^-} \right]
 \end{aligned}$$

and they have the following recursive formulae:

$$\begin{aligned}
 (L_n)_2 &= (A_n^{GG} + A_n^{qq})(L_n)_{2-1} - (A_n^{GG} A_n^{qq} - 2f A_n^{Gq} A_n^{qG})(L_n)_{2-2} \\
 (P_n)_2 &= (A_n^{GG} + A_n^{qq})(P_n)_{2-1} - (A_n^{GG} A_n^{qq} - 2f A_n^{Gq} A_n^{qG})(P_n)_{2-2} \\
 (S_n)_2 &= (A_n^{GG} + A_n^{qq})(S_n)_{2-1} - (A_n^{GG} A_n^{qq} - 2f A_n^{Gq} A_n^{qG})(S_n)_{2-2} \\
 (H_n)_2 &= (A_n^{GG} + A_n^{qq})(H_n)_{2-1} - (A_n^{GG} A_n^{qq} - 2f A_n^{Gq} A_n^{qG})(H_n)_{2-2} + \frac{4}{3\pi b} \\
 (R_n)_2 &= (A_n^{GG} + A_n^{qq})(R_n)_{2-1} - (A_n^{GG} A_n^{qq} - 2f A_n^{Gq} A_n^{qG})(R_n)_{2-2} + 1 - \frac{4}{3\pi b} A_n^{GG} \\
 (Q_n)_2 &= 1 + \frac{4}{3\pi b} A_n^{qq} (Q_n)_{2-1}
 \end{aligned}$$

with

$$\begin{aligned}
 (L_n)_0 &= 0 & (L_n)_1 &= 1 \\
 (P_n)_0 &= 1 & (P_n)_1 &= A_n^{qq} \\
 (S_n)_0 &= 1 & (S_n)_1 &= A_n^{GG} \\
 (H_n)_0 &= 0 & (H_n)_1 &= 0 \\
 (R_n)_0 &= 0 & (R_n)_1 &= 1 \\
 (Q_n)_0 &= 0 & (Q_n)_1 &= 1.
 \end{aligned}$$

(b)

$$F_{qq_1}(n, t) = e^{A_n^{qq}}$$

$$F_{qq_2}(n, t) = e^{s A_n^{qq}} + \int_0^s dv \sum_{j=1}^{\infty} U^j \frac{v^j (s-v)^{j-1}}{j! (j-1)!} e^{(s-v) A_n^{GG} - v A_n^{qq}}$$

$$F_{qG}(n, t) = A_n^{qG} \int_0^s dv \sum_{j=0}^{\infty} U^j \frac{v^j (s-v)^j}{j! j!} e^{(s-v) A_n^{GG} + v A_n^{qq}}$$

$$F_{qp}(n, t) = \left(\varepsilon_i^2 - \frac{1}{2f} \sum_i \varepsilon_i^2 \right) \frac{d}{2\pi} a(n) \frac{3\pi b}{4} t \int_0^s ds' e^{(A_n^{qq} - \frac{3\pi b}{4}) s'} \\ + \frac{1}{2f} \sum_i \varepsilon_i^2 \frac{d}{2\pi} \frac{3\pi b}{4} t a(n) \int_0^s ds' e^{-\frac{3\pi b}{4} s'} \left[e^{s' A_n^{qq}} + \int_0^{s'} dv \sum_{j=1}^{\infty} U^j \frac{v^j (s'-v)^{j-1}}{j! (j-1)!} e^{(s'-v) A_n^{GG} + v A_n^{qq}} \right]$$

$$F_{Gq}(n, t) = A_n^{Gq} \int_0^s dv \sum_{j=0}^{\infty} U^j \frac{v^j (s-v)^j}{j! j!} e^{(s-v) A_n^{GG} + v A_n^{qq}}$$

$$F_{GG}(n, t) = e^{s A_n^{GG}} + \int_0^s dv \sum_{j=1}^{\infty} U^j \frac{v^j (s-v)^{j-1}}{j! (j-1)!} e^{v A_n^{GG} + (s-v) A_n^{qq}}$$

$$F_{Gp}(n, t) = \sum_i \varepsilon_i^2 \frac{d}{2\pi} \frac{3\pi b}{4} t \quad (B.2)$$

$$a(n) A_n^{Gq} \int_0^s ds' e^{-\frac{3\pi b}{4} s'} \int_0^{s'} dv \sum_{j=0}^{\infty} U^j \frac{v^j (s'-v)^j}{j! j!} e^{(s'-v) A_n^{GG} + v A_n^{qq}}$$

where

$$U = 2f A_n^{qG} A_n^{Gq}$$

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FIGURE CAPTIONS

- Fig. 1 : The contour C in the complex n plane.
- Fig. 2 : The contribution of the point-like part to the quark distribution function of photon $Xq_Y(X,t)$ for $s = 0.2$ (a), $s = 0.1$ (b) and $s = 0.05$ (c). The structure function in units of $(\alpha/\pi)t$ for $\epsilon_i = 2/3$. (d) is the result of Ref. 4) for comparison.
- Fig. 3 : The contribution of the point-like part to the gluon distribution of the photon $XG_Y(X,t)$ for $s = 0.2$ (a), $s = 0.1$ (b) and $s = 0.05$ (c) in units of $(\alpha/\pi)t$.
- Fig. 4 : Comparing the contribution of the point-like part to the quark distribution for $\epsilon_i = 2/3$ (a) and $\epsilon_i = 1/3$ (b) and to the gluon distribution function (c) for $s = 0.2$.
- Fig. 5 : Comparison of $q_Y^u(X,t)$ (a) with the corresponding result (b) in Ref. 5) for $Q^2 = 200 \text{ GeV}^2/c^2$.
- Fig. 6 : Up-quark distribution $Xq_Y^u(X,t)$ of the hadron-like part for $Q^2 = 200 \text{ GeV}^2/c^2$ (a) and $Q^2 = 20 \text{ GeV}^2/c^2$ (b).
- Fig. 7 : Down-quark distribution $Xq_Y^d(X,t)$ of the hadron-like part for $Q^2 = 200 \text{ GeV}^2/c^2$ (a) and $Q^2 = 20 \text{ GeV}^2/c^2$ (b).
- Fig. 8 : s quark distribution $Xq_Y^s(X,t)$ of the hadron-like part for $Q^2 = 200 \text{ GeV}^2/c^2$ (a) and $Q^2 = 20 \text{ GeV}^2/c^2$.
- Fig. 9 : Gluon distribution $XG_Y(X,t)$ of the hadron-like part of $Q^2 = 200 \text{ GeV}^2/c^2$ (a) and $Q^2 = 20 \text{ GeV}^2/c^2$ (b).

FIG.1

FIG.2

FIG.3

FIG. 4

FIG. 5

FIG.6

FIG. 7

FIG. 8

FIG. 9