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SELF-DUALITY CONDITIONS FOR SPIN SYSTEMS
DEFINED ON A NEW CLASS OF NON-ABELIAN SOLVABLE FINITE GROUPS

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A B S T R A C T

A new class of finite non-Abelian groups is presented. We work out explicitly the conditions under which the corresponding spin systems are self-dual.

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1. - INTRODUCTION

The duality transformation ^{1),2)} is an important tool which allows us to get information on the phase diagram of various models. This transformation has been applied to many statistical mechanics systems as well as to lattice field theories shedding a new and interesting light on their phase structure. Usually the transformation relates the high temperature phase of one theory to the low temperature phase of another theory. It may of course happen that the dual theory is simpler to handle than the original one. Such is, for example, the case with the $Z(N)$ theories in three dimensions which are dual to Ising-like theories ^{2),3)}.

In the particular case in which the theory and the dual theory are the same, we refer to the theory as self-dual. If self-duality can be supplemented with the knowledge that only two phases exist in the system, then the critical point can be determined. Otherwise duality just relates a given critical point to another one. There is an interesting relationship between the original degrees of freedom and the dual ones, i.e., the dual variables are related to topological excitations of the original ones. The topological excitations tend to disorder the system. The picture which emerges allows us to view the phase transition in terms of condensation of these topological excitations ²⁾. The various phases which appear can be characterized in terms of the order (magnetization, Wilson loop) and disorder operators.

Almost all the applications of the duality transformation were done in the framework of theories with an Abelian symmetry ⁴⁾. The technical reason is that in the Abelian case the characters of the irreducible representations form a group which is known as the dual group. In the duality transformation we go from the group to the dual group by means of a Fourier transform. While in the Abelian case, a product of two irreducible representations is also an irreducible representation, this is not anymore true for the non-Abelian case.

The interest in theories with a non-Abelian symmetry is self-evident. There are important statistical mechanics theories with non-Abelian symmetry. The Heisenberg $O(3)$ model is just one example.

A particular class of non-Abelian theories where the duality transformation can be done is that of solvable groups. Drouffe, Itzykson and Zuber ⁵⁾ and Monastyrsky and Zamolodchikov ⁶⁾ have shown that such a theory can be treated like an Abelian theory. The argument goes as follows.

Consider a given solvable group G . A function $f(g)$ over the group G ($g \in G$) can be written in terms of the characters $\chi_P(g)$:

$$f(g) = \sum_P a_P \chi_P(g) \quad (1.1)$$

where a_P are arbitrary constants and P labels the irreducible representations of G .

Since the group G is solvable, each element g can be mapped into an element g' of an Abelian group A and the character function $\chi_P(g)$ can be written as a linear combination of the characters $\chi'_Q(g')$ of the group A (Q labels the irreducible representations of A) :

$$f(g) = f'(g') = \sum_Q a'_Q \chi'_Q(g') \quad (1.2)$$

where the coefficients a'_Q are not anymore linear independent since they are expressed in terms of a_P (the number of irreducible representations of G is smaller than that of A).

Of course, we could have started with the group A , then by imposing appropriate relations among the coupling constants the resulting Hamiltonian will have the G group symmetry built in. We could have also arranged the couplings in such a way that the real symmetry of the Hamiltonian would correspond to yet another solvable group H of the same order. We can now use Eq. (1.2) and reduce the problem to the study of the dual transformation for the Abelian group A (this is a known procedure ^{4),6)} and is presented for completeness in Section 3). Since the dual group $\hat{A}$ of a finite Abelian group A is isomorphic to A one ends with a function defined on A

$$f^D(g') = \sum_Q a'^{*}_Q \chi'_Q(g') \quad (1.3)$$

where the coefficients a'^{*}_Q depend on a_P through a'_Q .

If the coefficients a_Q^* are such that the expression (1.3) can be brought back to a form expressed in terms of the characters of the original group G :

$$f^D(g') = \sum_P a_P^* \chi_P(g) \quad (1.4)$$

then our system is self-dual (this may occur if we have certain additional relations among the original coefficients a_P). It could happen for some other choices of the parameters a_P , that the expression (1.3) could be written as

$$f^D(g') = \sum_R b_R^* \chi_R''(g'') \quad (1.5)$$

where $\chi_R''(g'')$ are the characters of another solvable group H (with irreducible representations labelled by R). In this case the system with the original symmetry G is dual to the system with symmetry H .

For a few examples of solvable groups (the permutation groups S_3 and S_4 ^{5),6)}, the tetrahedron ^{6),7)} and the cube ⁶⁾) the duality transformations have been worked out.

Consider now G as a finite group of order N which can be Abelian or non-Abelian, solvable or non-solvable, and assume that the class function (1.1) has the very special expression :

$$f(g) = a_0 + a_1 \delta(g, e) \quad (1.6)$$

where

$$\delta(g, e) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases} \quad (1.7)$$

e is the unit element of G .

The choice (1.6) corresponds to a combination of the scalar and regular representations of the group G . In this case the dual transformation can be done in a trivial way : one maps the group G into any Abelian

group A of order N , mapping the unity of G into the unity of A and the other group elements of G into any choice we wish of the group elements of A . We then perform the duality transformation on A . The result depends of course only on N and does not depend on the specific group G we consider,

The physical system with the Hamiltonian (1.6) will be called an order N system. The N component Potts model is precisely an order N system, it does not only have a Z_N symmetry but that of any group of order N . For example, an order 8 system has the symmetry Z_8 , $Z_4 \otimes Z_2$, $Z_2 \otimes Z_2 \otimes Z_2$, D_4 and Q (D_4 is the dihedral group of order 8, Q is the quaternionic group) which are all groups of order 8.

In the present paper we study the duality transformations for a whole new class of solvable groups that we call $\mathcal{D}(n)$. A detailed presentation of these groups will be given elsewhere⁸⁾. Here we use their important feature that all their properties can be written in a closed form. For n odd, the groups $\mathcal{D}(n)$ are finite subgroups of $SU(n)$.

The paper is organized as follows. In Section 2 we give the definition of the groups $\mathcal{D}(n)$. These groups are of order n^3 . We give their representations and their characters. We present the expression of the Clebsch-Gordan coefficients in closed form. We also bring to the attention of the reader the groups $Q(n)$ also of order n^3 which are subgroups of $SU(n)$. This section can be read independently of the others, these groups being of interest on their own.

For the sake of completeness we present in Section 3 the duality transformation for an Abelian group derived previously in Refs. 4) and 6). We also consider as an example the order N systems.

In Section 4 we answer the question under which conditions the $\mathcal{D}(n)$ spin system is self-dual. We have studied the case $n=p$ where p is a prime number and $n=4$. Our result is the following : the $\mathcal{D}(p)$ systems are self-dual if and only if they correspond to order p^3 systems. The $\mathcal{D}(4)$ system is self-dual for a two-parameter duality transformation. From the structure of the self-duality condition one can see how the last situation generalizes for n non-prime. We did not solve the general $\mathcal{D}(n)$ situation since this would imply a knowledge of number theory which is beyond

our own means. An open and most interesting question that we did not consider is which solvable non-Abelian groups of order n^3 could be dual to $\mathcal{D}(n)$? In Section 5, we summarize our results and present our conclusions.

2. - THE $\mathcal{D}(n)$ AND $Q(n)$ GROUPS

The finite groups that we consider in this paper are very special in the sense that it is possible to give analytic expressions for all their representations and for all the Clebsch-Gordan coefficients. Because of this property they are suitable for applications in statistical mechanics of spin systems. We will give here a general review of their properties, but for a complete analysis, see Ref. 8).

We consider here two different, although related, classes of groups : the class $\mathcal{D}(n)$ and the class $Q(n)$. In both classes the elements are characterized by a triplet (α, β, γ) of elements of $\mathbb{Z}_n$, and the multiplication rules are

$$\mathcal{D}(n): g_{\alpha, \beta}^{\gamma} g_{\alpha', \beta'}^{\gamma'} = g_{\alpha+\alpha', \beta+\beta'}^{\gamma+\gamma'+\alpha\beta'} \quad (2.1a)$$

$$Q(n): g_{\alpha, \beta}^{\gamma} g_{\alpha', \beta'}^{\gamma'} = g_{\alpha+\alpha', \beta+\beta'}^{\gamma+\gamma'+\alpha\beta'+\alpha\alpha'+\beta\beta'} \quad (2.1b)$$

All the operations among elements of $\mathbb{Z}_n$ are made mod(n). The order of both groups $\mathcal{D}(n)$ and $Q(n)$ is n^2 .

Let us consider some simple examples.

$\mathcal{D}(2)$ can be realized in terms of the following eight 2×2 matrices

$$g_{0,0}^{\gamma} = (-1)^{\gamma} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-1)^{\gamma} \mathbb{I} ; \quad g_{1,0}^{\gamma} = (-1)^{\gamma} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = (-1)^{\gamma} \sigma_3 ;$$

$$g_{0,1}^{\gamma} = (-1)^{\gamma} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (-1)^{\gamma} \sigma_1 ; \quad g_{1,1}^{\gamma} = (-1)^{\gamma} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = (-1)^{\gamma} i \sigma_2 ;$$

where σ_i 's are Pauli matrices. One realizes immediately that $\mathcal{D}(2)$ is isomorphic to D_4 (the dihedral group of order $2 \times 4 = 8$).

$\mathcal{D}(3)$ can be realized in terms of 27 3×3 matrices. For instance, we have

$$g_{0,0}^\gamma = \omega^\gamma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad g_{1,0}^\gamma = \omega^\gamma \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} ;$$

$$g_{0,1}^\gamma = \omega^\gamma \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ; \quad \text{etc.}$$

here $\omega = e^{2\pi i/3}$. This representation is relevant for the three-component Potts model ²⁾.

$Q(2)$ can be realized in terms of eight 2×2 matrices :

$$g_{0,0}^\gamma = (-1)^\gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-1)^\gamma \mathbb{I} ; \quad g_{1,0}^\gamma = (-1)^\gamma \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = (-1)^\gamma i \sigma_3 ;$$

$$g_{0,1}^\gamma = (-1)^\gamma \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = (-1)^\gamma i \sigma_1 ; \quad g_{1,1}^\gamma = (-1)^\gamma \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = (-1)^\gamma i \sigma_2 ;$$

For n odd, $Q(n)$ and $\mathcal{D}(p)$ are isomorphic. Furthermore, $\mathcal{D}(n)$ is a subgroup of $SU(n)$ for odd n , whereas $Q(n)$ is a subgroup of $SU(n)$ for any n . Here, we will be mainly concerned with $\mathcal{D}(n)$. The study of $\mathcal{D}(n)$ can be completely reduced to the study of its subgroups of order p^ℓ , with p prime and ℓ an integer (Sylow p subgroups ⁹⁾). In fact, one can show that for $n = p_1^{s_1} \dots p_k^{s_k}$, $\mathcal{D}(n)$ factorizes in the following way :

$$\mathcal{D}(n) = \mathcal{D}(p_1^{s_1}) \otimes \dots \otimes \mathcal{D}(p_k^{s_k}) \quad (2.2)$$

but the order of $\mathcal{D}(p^s)$ is p^{3s} and it is indeed a Sylow p subgroup of $\mathcal{D}(n)$. The same factorization property holds for $Q(n)$. Since Sylow p groups are solvable the $\mathcal{D}(n)$ and $Q(n)$ groups are solvable. We recall

the definition of solvability⁹⁾ : the sequence $G^{(i)}$ of the derived groups $G \supseteq G^{(1)} \supseteq \dots \supseteq G^{(k)} = G^{(i+1)}$ is obtained from the commutators of the elements of $G^{(i)}$ ends after a finite number of steps (i.e., $G^{(k)} = e$). We can immediately check the solvability of $\mathcal{D}(n)$; in fact, from the commutator of two arbitrary elements

$$\begin{aligned} & (g_{\alpha, \beta}^{\gamma})^{-1} (g_{\alpha', \beta'}^{\gamma'})^{-1} (g_{\alpha, \beta}^{\gamma}) (g_{\alpha', \beta'}^{\gamma'}) = \\ & = g_{-\alpha, -\beta}^{-\gamma + \alpha\beta} g_{-\alpha', -\beta'}^{-\gamma' + \alpha'\beta'} g_{\alpha, \beta}^{\gamma} g_{\alpha', \beta'}^{\gamma'} = \\ & = g_{0,0}^{\alpha\beta' - \alpha'\beta} \end{aligned}$$

(2.3)

we see that $G^{(1)}$ is isomorphic to the Abelian group Z_n , and therefore the sequence ends after another step. It is also possible to show that $\mathcal{D}(n)$ can be obtained as a semi-direct product of Z_n with $Z_n \otimes Z_n$.

Because of Eq. (2.2) we need to analyze only the case in which $n = p^s$ (p prime number). The irreducible representations of $\mathcal{D}(n)$ ($n = p^s$) are labelled by a triplet of elements of Z_n , and their matrix elements will be denoted by $T_{a,b}^{[i,j;k]}(g_{\alpha,\beta}^{\gamma})$. Here, $i, j, k \in Z_n$, k is arbitrary and $i, j = 0, 1, \dots, (\bar{k}-1)$, where $\bar{k} = \varphi(k, n)$ is the greatest common divisor of k and n . The dimensionality of the representation $[i, j; k]$ is $n/\bar{k}$, and therefore $a, b = 0, 1, \dots, (n/\bar{k})-1$. The matrix elements of all the representations of $\mathcal{D}(n)$ are given by the following expression¹⁰⁾

$$\begin{aligned} T_{a,b}^{[i,j;k]}(g_{\alpha,\beta}^{\gamma}) &= \delta_{ka, k(b+\beta)} \omega^{k(\gamma + \alpha b) + i\alpha + j\beta} \\ i, j &= 0, 1, \dots, \bar{k}-1 ; a, b = 0, 1, \dots, (n/\bar{k})-1 \\ \dim[i, j; k] &= n/\bar{k} \\ i, j, k &\in Z_n ; \omega = e^{2\pi i/n} \end{aligned}$$

(2.4)

From this formula, one gets immediately the expression for the characters

$$\chi^{[i,j;k]}(g_{\alpha,\beta}^\gamma) = \frac{n}{\bar{k}} \omega^{i\alpha + j\beta + k\gamma} \delta_{k\alpha,0} \delta_{k\beta,0} \quad (2.5)$$

The situation is particularly simple for $n=p$, that is $s=1$. In fact, there are only two possibilities :

- i) $k=0 \rightarrow \bar{k}=p$, and therefore $i,j=0,1,\dots,(p-1)$ and $\dim[i,j;0]=1$,
- ii) $k \neq 0 \rightarrow \bar{k}=1$, and therefore $i,j=0$ and $\dim[0,0;k]=p$.

We see that there are p^2 one-dimensional representations and $p-1$ p dimensional representations.

The rule for getting the conjugated representations is very simple and reads

$$[i,j;k]^* = [n-i, n-j; n-k] \quad (2.6)$$

Having analytic expressions for the representations, one can get an analytic form also for the Clebsch-Gordan coefficients. The Clebsch-Gordan series are given by

$$\begin{aligned} [i_1, j_1; k_1] \otimes [i_2, j_2; k_2] &= \\ &= \sum_{u,v,\eta} \oplus [i_1+i_2-\kappa u, j_1+j_2-\kappa v; k_1+k_2]_\eta \\ &\quad 0 \leq i_1+i_2-\kappa u \leq \overline{(k_1+k_2)}-1 \\ &\quad 0 \leq j_1+j_2-\kappa v \leq \overline{(k_1+k_2)}-1 \end{aligned} \quad (2.7)$$

Here $\kappa = \varphi(\bar{k}_1, \overline{k_1+k_2})$ is the greatest common divisor of $\bar{k}_1$ and $\overline{k_1+k_2}$. Because the same representation $[i,j;k]$ occurs more than once in the product [for example, think to the representations $\underline{8}_a$ and $\underline{8}_s$ occurring in the Clebsch-Gordan series of $SU(3)$], we need an extra index η to specify the product (2.7) completely. It turns out that $0 \leq \eta \leq (n/\lambda)-1$,

where $\lambda = \psi(\bar{k}_1, \overline{k_1+k_2})$ is the least common multiplier of $\bar{k}_1$ and $\overline{k_1+k_2}$. Therefore, the multiplicity of a given representation is n/λ . Because in our case $n = p^S$, we can always write

$$\varphi(\bar{k}_1, \overline{k_1+k_2}) = c_0 \bar{k}_1 + c_1 \overline{(k_1+k_2)} \quad (2.8)$$

where c_0, c_1 are 0 or 1. Then, the expression for the Clebsch-Gordan coefficients is given by

$$\begin{aligned} & C_{\begin{matrix} [i_1+i_2-xu, i_1+i_2-xv; k_1+k_2]_{\eta, c} \\ [i_1, i_1; k_1], a; [i_2, i_2; k_2], b \end{matrix}} = \\ & = \sqrt{\frac{x}{(k_1+k_2)}} \delta_{k_1 a + k_2 b - (k_1+k_2), xu} \cdot \\ & \cdot \delta_{\lambda(q-c-\eta), 0} \omega^{\nu(c_0 \bar{k}_1 a + c_1 \overline{(k_1+k_2)} c)} \\ & 0 \leq q \leq \frac{n}{k_1} - 1; 0 \leq b \leq \frac{n}{k_2} - 1; 0 \leq c \leq \frac{n}{k_1+k_2} - 1; \\ & 0 \leq u, v \leq \frac{\overline{k_1+k_2}}{x} - 1 \end{aligned} \quad (2.9)$$

where c_0 and c_1 are implicitly defined by Eq. (2.8).

It is interesting to consider explicitly the case $\mathcal{D}(3)$. Here we have nine one-dimensional representations $[i, j; k]$, $0 \leq i, j \leq 2$, and two three-dimensional representations $[0, 0; 1]$ and $[0, 0; 2]$. Notice that we have

$$[0, 0; 1]^* = [0, 0; 2]$$

Therefore we can put

$$\underline{3} = [0, 0; 1]; \quad \underline{3}^* = [0, 0; 2]$$

The Clebsch-Gordan series for these representations is

$$\begin{aligned}\underline{3} \otimes \underline{3}^* &= [0, 0; 1] \otimes [0, 0; 2] = \\ &= \sum_{u, v} \oplus [-u, -v; 0] = \sum_{u, v} \oplus [3-u, 3-v; 0] = \\ &= \sum_{i, j=0}^2 \oplus [i, j; 0]\end{aligned}$$

where we have used the fact that we are working mod(3). Therefore, $\underline{3} \otimes \underline{3}^*$ gives the nine one-dimensional representations. Analogously we get

$$\begin{aligned}\underline{3} \otimes \underline{3} &= [0, 0; 1] \otimes [0, 0; 1] = \\ &= \sum_{\eta=0}^2 \oplus [0, 0; 2]_{\eta} = \sum_{\eta=0}^2 \oplus \underline{3}^*_{\eta}\end{aligned}$$

Because $\mathcal{D}(3)$ is a subgroup of $SU(3)$, this means that the $\underline{8}$ of $SU(3)$ decomposes into one-dimensional representations of $\mathcal{D}(3)$, whereas the $\underline{6}$ of $SU(3)$ decomposes into two three-dimensional representations.

3. - THE DERIVATION OF THE DUALITY TRANSFORMATION

In order to make this paper self-contained we will review here some of the most relevant properties of spin systems on groups (we will follow, including notations, the paper by Monastyrsky and Zamolodchikov ⁶⁾). A spin system on a two-dimensional lattice is defined by attaching to the vertices of the lattice a "spin" variable taking values in a manifold S . Therefore, from a mathematical point of view, a spin system is nothing but a fibre bundle ξ_S with base space B (the lattice) and fibre S (the spin manifold). We will be interested only in the case in which S consists of a finite number of points. We define a co-ordinate system on the lattice by assigning to each vertex the pair n_{μ} ($\mu = 1, 2$), n_1 and n_2 integers. The basic vectors of the lattice will be denoted by $e_{\mu}^{\alpha} = \delta_{\mu}^{\alpha}$ ($\alpha = 1, 2$). The simplest Hamiltonian for such a system takes into account only the nearest neighbour interactions

$$H = \sum_{n_{\mu}, \alpha} C_{n_{\mu}, \alpha} f(s_{n_{\mu}}, s_{n_{\mu}+e_{\mu}^{\alpha}}) \quad (3.1)$$

where $C_{n_{\mu}, \alpha}$ are the strengths of the interaction among neighbours. The spin variables $s_{n_{\mu}}$ can be regarded as cross-sections of the fibre bundle ξ_S . The case in which we are presently interested (and one of the most important) is the case in which ξ_S is a principal fibre bundle ; that is, when S is a group manifold G (finite in our case). We also require the Hamiltonian to be invariant under both right and left translations. This is easily satisfied by assuming f of the form

$$f(g_1, g_2) = f(g_1 g_2^{-1}) \quad (3.2)$$

and requiring

$$f(h g h^{-1}) = f(g) \quad (3.3)$$

which implies that f is a class function (from now on, we will denote the spin variables g_i , to remember that they take values in the group manifold G).

The partition function of the system is given by

$$Z = \sum_{g_{n_{\mu}} \in G} \prod_{n_{\mu}, \alpha} W_{n_{\mu}, \alpha} (g_{n_{\mu}} g_{n_{\mu}+e_{\mu}^{\alpha}}^{-1}) \quad (3.4)$$

where

$$W_{n_{\mu}, \alpha} (g_{n_{\mu}} g_{n_{\mu}+e_{\mu}^{\alpha}}^{-1}) = e^{-\beta C_{n_{\mu}, \alpha} f(g_{n_{\mu}} g_{n_{\mu}+e_{\mu}^{\alpha}}^{-1})} \quad (3.5)$$

From a geometrical point of view, the expression $g_{n_{\mu}} g_{n_{\mu}+e_{\mu}^{\alpha}}^{-1}$ represents the parallel transport of the element $g_{n_{\mu}+e_{\mu}^{\alpha}}$, belonging to the fibre at $n_{\mu}+e_{\mu}^{\alpha}$, to the element $g_{n_{\mu}}$ belonging to the fibre at n_{μ} ; furthermore, this particular parallel transport is the one occurring in a flat space. It turns out to be extremely convenient to introduce the G group valued variables $g_{n_{\mu}, \alpha}$ depending on the line connecting the two

adjacent vertices n_μ and $n_\mu + e_\mu^\alpha$. Their geometrical interpretation is to provide parallel transport between the corresponding fibres. Since in our case we have a parallel transport in a flat space we have to require that the link variables satisfy such a condition. Recalling that flat space means coming back to the original point when performing a close path, we see that this condition can be expressed by requiring

$$g_{n_\mu,1} g_{n_\mu+e_\mu^1,2} g_{n_\mu+e_\mu^2,1}^{-1} g_{n_\mu,2}^{-1} = e, \forall n_\mu \quad (3.6)$$

where e is the identity of the group. In fact, it is easy to show that the general solution of this constraint is

$$g_{n_\mu,\alpha} = g_{n_\mu} g_{n_\mu+e_\mu^\alpha}^{-1}$$

The left-hand side of Eq. (3.6) depends only on the plaquette which is uniquely defined by giving the co-ordinates of its centre

$$\tilde{n}_\mu = n_\mu + \frac{1}{2} (e_\mu^1 + e_\mu^2) \quad (3.7)$$

In other words, the plaquettes are completely identified by the vertices of the dual lattice. It follows from these considerations that we can re-write the partition function in the following form ⁶⁾

$$Z = \sum_{g_{n_\mu,\alpha} \in G} \left(\prod_{n_\mu,\alpha} W_{n_\mu,\alpha}(g_{n_\mu,\alpha}) \right) \left(\prod_{\tilde{n}_\mu} \delta(Q_{\tilde{n}_\mu}, e) \right) \quad (3.8)$$

where

$$Q_{\tilde{n}_\mu} = g_{n_\mu,1} g_{n_\mu+e_\mu^1,2} g_{n_\mu+e_\mu^2,1}^{-1} g_{n_\mu,2}^{-1} \quad (3.9)$$

and the delta function is defined by

$$\delta(g, e) = \begin{cases} 1 & , g = e \\ 0 & , g \neq e \end{cases} \quad (3.10)$$

This function can be expressed in terms of the character of the regular representation of G :

$$\delta(g, e) = \frac{1}{N} \sum_P \chi_P(g) d_P \quad (3.11)$$

where d_P is the dimensionality of the irreducible representation labelled by P .

A very special situation arises if G is an Abelian group. In such a case, the characters form an Abelian group $\hat{G}$ (called the dual group) with respect to the multiplication

$$\begin{aligned} \chi_{\hat{g}_1 \hat{g}_2}(g) &= \chi_{\hat{g}_1}(g) \chi_{\hat{g}_2}(g) \\ \chi_{\hat{g}^{-1}}(g) &= \chi_{\hat{g}}^{-1}(g) \end{aligned} \quad (3.12)$$

(We have changed the notation for the characters to stress that now the index $\hat{g} \in \hat{G}$.) For finite Abelian groups, the dual group is isomorphic to the group itself. This is due to the fact that the representations of G are one-dimensional. Therefore, the sum over P in Eq. (3.11) is nothing but a sum over the dual group $\hat{G}$ and we get

$$\begin{aligned} Z &= \frac{1}{N} \sum_{g_{n_\mu, \alpha} \in G} \left(\prod_{n_\mu, \alpha} W_{n_\mu, \alpha}(g_{n_\mu, \alpha}) \right) \cdot \\ &\cdot \prod_{\tilde{n}_\mu} \sum_{\hat{g}_{\tilde{n}_\mu} \in \hat{G}} \chi_{\hat{g}_{\tilde{n}_\mu}}(g_{n_\mu, 1}) \chi_{\hat{g}_{\tilde{n}_\mu}}(g_{n_\mu + e_{\mu, 2}^1}) \cdot \\ &\cdot \chi_{\hat{g}_{\tilde{n}_\mu}}^{-1}(g_{n_\mu + e_{\mu, 1}^2}) \chi_{\hat{g}_{\tilde{n}_\mu}}^{-1}(g_{n_\mu, 2}) \end{aligned} \quad (3.13)$$

To each link $(n_\mu + e_\mu^\alpha, n_\mu + e_\mu^\beta + e_\mu^\alpha) \ (\alpha \neq \beta)$

the following three factors are associated

$$W_{n_\mu, \alpha}(g_{n_\mu + e_\mu^\alpha, \beta}) \chi_{\hat{g}_{\tilde{n}_\mu}}(g_{n_\mu + e_\mu^\alpha, \beta}) \chi_{\hat{g}_{\tilde{n}_\mu + e_\mu^\alpha}}^{-1}(g_{n_\mu + e_\mu^\alpha, \beta})$$

(see the Figure). Hence, it is easy to show that the various terms in Eq. (3.13) can be rearranged in the following way :

$$Z = \sum_{\hat{g}_{\tilde{n}_\mu} \in \hat{G}} \prod_{\tilde{n}_\mu, \alpha} \widetilde{W}_{\tilde{n}_\mu, \alpha}(\hat{g}_{\tilde{n}_\mu} \hat{g}_{\tilde{n}_\mu + e_\mu^\alpha}^{-1}) \quad (3.14)$$

where

$$\begin{aligned} \widetilde{W}_{\tilde{n}_\mu, \alpha}(\hat{g}_{\tilde{n}_\mu} \hat{g}_{\tilde{n}_\mu + e_\mu^\alpha}^{-1}) &= \frac{1}{N} \sum_{g \in G} W_{n_\mu, \alpha}(g) \chi_{\hat{g}_{\tilde{n}_\mu}}(g) \chi_{\hat{g}_{\tilde{n}_\mu + e_\mu^\alpha}}^{-1}(g) = \\ &= \frac{1}{N} \sum_{g \in G} W_{n_\mu, \alpha}(g) \chi_{\hat{g}_{\tilde{n}_\mu} \hat{g}_{\tilde{n}_\mu + e_\mu^\alpha}^{-1}}(g) \end{aligned} \quad (3.15)$$

We see that the expression (3.14) is formally the same as the original one (3.4). We can describe briefly these operations in the following way : first of all we define a new bundle $\hat{\xi}_{\hat{G}}$ starting from ξ_G , by taking as a base space the dual lattice and as a fibre the dual group. Next, on this new spin system we define a new W function which is the Fourier transform with respect to the group G of the old W function. This set of operations is the duality transformation. We notice also that the Hamiltonian for the dual system is

$$e^{-\tilde{H}(\hat{g})} = \frac{1}{N} \sum_{g \in G} e^{-H(g)} \chi_{\hat{g}}(g) \quad (3.16)$$

We illustrate the method by the example of an order N model. In this system, the spin variable s_i belongs to Z_N and the Hamiltonian is

$$H(s_i, s_j) = H(s_i s_j^{-1}) = \beta_0 + N \beta_1 \delta(s_i s_j^{-1}, e) \quad (3.17)$$

where $\beta_1 < 0$ and the delta function is defined by Eq. (3.11). We notice that $H(g)$ ($g \in \mathbb{Z}_N$) is expanded in terms of the scalar and of the regular representations. Furthermore, we have $\hat{\mathbb{Z}}_N = \mathbb{Z}_N$. The W function is

$$W(g) = e^{-\beta_0} + e^{-\beta_0} (e^{-N\beta_1} - 1) \delta(g, e) \quad (3.18)$$

Therefore, its Fourier transform is

$$\begin{aligned} \widetilde{W}(\hat{g}) &= \frac{1}{N} \sum_{g \in \mathbb{Z}_N} W(g) \chi_{\hat{g}}(g) = \\ &= e^{-\beta_0} \delta(\hat{g}, e) + \frac{1}{N} e^{-\beta_0} (e^{-N\beta_1} - 1) \end{aligned} \quad (3.19)$$

and we see that the scalar and the regular representations go one into each other through the Fourier transformation. In this way $\widetilde{W}(\hat{g})$ has the same structure as $W(g)$. One asks if it is possible to introduce new parameters β_0^* and β_1^* as functions of the old parameters β_0 and β_1 such that

$$\widetilde{W}(g; \beta_0, \beta_1) = W(g; \beta_0^*, \beta_1^*) \quad (3.20)$$

From this self-duality condition we get

$$\begin{aligned} e^{-\beta_0} &= e^{-\beta_0^*} (e^{-N\beta_1^*} - 1) \\ e^{-\beta_0^*} &= \frac{1}{N} e^{-\beta_0} (e^{-N\beta_1} - 1) \end{aligned} \quad (3.21)$$

The second equation defines the dual zero point of the energy. Combining together the two equations we get the self-duality equation for β_1 and β_1^*

$$(e^{-N\beta_1} - 1)(e^{-N\beta_1^*} - 1) = N \quad (3.22)$$

4. - SELF-DUALITY CONDITIONS ON THE $\mathcal{D}(n)$ GROUPS

If a spin system is defined on a finite-dimensional Abelian group G (and therefore $\hat{G} = G$), its W function has the following expression in terms of the characters

$$W[g; b_i] = \sum_i b_i \chi_{\hat{g}_i}^*(g) \quad (4.1)$$

In order to have a self-dual system, there must exist a new set of parameters $b_i^* = b_i^*(b_i)$ such that the Fourier transform of W is equal to W itself in terms of the new parameters ; that is

$$\tilde{W}[\hat{g}; b_i] = W[\hat{g}; b_i^*] = \sum_i b_i^* \chi_{\hat{g}_i}^*(\hat{g}) \quad (4.2)$$

However, we can evaluate directly $\tilde{W}$ and we get

$$\begin{aligned} \tilde{W}[\hat{g}; b_i] &= \frac{1}{N} \sum_{g \in G} W[g; b_i] \chi_{\hat{g}}(g) = \\ &= \sum_i b_i \widetilde{(\chi_{\hat{g}_i}^*)}(\hat{g}) \end{aligned} \quad (4.3)$$

where

$$\widetilde{(\chi_{\hat{g}_i}^*)}(\hat{g}) = \frac{1}{N} \sum_{g \in G} \chi_{\hat{g}_i}^*(g) \chi_{\hat{g}}(g) \quad (4.4)$$

is the Fourier transform of the characters. Comparing the two expressions we get the condition the system should satisfy in order to be self-dual

$$\sum_i b_i \widetilde{(\chi_{\hat{g}_i}^*)}(\hat{g}) = \sum_i b_i^* \chi_{\hat{g}_i}^*(\hat{g}) \quad (4.5)$$

All what we have said up to now is valid for an Abelian group. If we have non-Abelian groups such that their characters can be expanded in terms of the characters of an Abelian group, then all the previous considerations hold true. This is just the case for the groups $\mathcal{D}(n)$, in fact we have the following relation

$$\chi^{[i,j];k]}(g_{\alpha,\beta}^\gamma) = \frac{1}{n\bar{k}} \sum_{p,q=0}^{n-1} \chi^{(pk+i, qk+j), k)}(h_{\alpha,\beta,\gamma}) \quad (4.6)$$

where

$$\begin{aligned} \chi^{(i,j),k)}(h_{\alpha,\beta,\gamma}) &= \omega^{i\alpha+j\beta+k\gamma} \\ \omega &= e^{2\pi i/n} \end{aligned} \quad (4.7)$$

are the characters of $Z_n \otimes Z_n \otimes Z_n$ corresponding to the element $h_{\alpha,\beta,\gamma}$. Of course we have

$$h_{\alpha,\beta,\gamma} h_{\alpha',\beta',\gamma'} = h_{\alpha+\alpha', \beta+\beta', \gamma+\gamma'}$$

Equation (4.6) illustrates the theorem ⁵⁾ mentioned in Section 1, that any spin system on a solvable group can be expressed in terms of a spin system on a related Abelian group. At this point we are ready to attack the problem of solving the self-duality conditions for $\mathcal{D}(n)$. The most general W function on the group can be written in terms of the characters of $\mathcal{D}(n)$. We then expand the characters of $\mathcal{D}(n)$ in terms of the characters of $Z_n \otimes Z_n \otimes Z_n$ and we can apply all the previous considerations. Therefore, we write

$$W[g_{\alpha,\beta}^\gamma; b_{[i,j];k}] = \sum_{[i,j];k} \frac{n}{\bar{k}} b_{[i,j];k} \chi^{[i,j];k]}(g_{\alpha,\beta}^\gamma)^* \quad (4.8)$$

where we have inserted the factor $n/\bar{k}$ for later convenience. The W function can be regarded as a function on $Z_n \otimes Z_n \otimes Z_n$:

$$\begin{aligned} W[h_{\alpha,\beta,\gamma}; b_{[i,j];k}] &= \\ &= \sum_{[i,j];k} \frac{n}{\bar{k}} b_{[i,j];k} \frac{1}{n\bar{k}} \sum_{u,v=0}^{n-1} \omega^{-(uk+i)\alpha - (vk+j)\beta - k\gamma} = \end{aligned} \quad (4.9)$$

$$\begin{aligned}
 &= \sum_{[i,j];k} \frac{n^2}{k^2} b_{[i,j];k} \delta_{\alpha k,0} \delta_{\beta k,0} \omega^{-\alpha i - \beta j - \gamma k} = \\
 &= \sum_{[i,j];k} \left(\frac{n^2}{k^2} b_{[i,j];k} \delta_{\alpha k,0} \delta_{\beta k,0} \right) \chi_{(h_{\alpha,\beta,\gamma})}^{(i,j),k}{}^* \quad (4.9 \text{ Cont.})
 \end{aligned}$$

Hence, by taking its Fourier transform with respect to $\underbrace{Z_n \otimes Z_n \otimes Z_n}_{(Z_n \otimes Z_n \otimes Z_n) = Z_n \otimes Z_n \otimes Z_n}$, we get the function $\tilde{W}$ defined on the dual group $(Z_n \otimes Z_n \otimes Z_n) = Z_n \otimes Z_n \otimes Z_n$:

$$\begin{aligned}
 \tilde{W}[h_{m,n,e}; b_{[i,j];k}] &= \\
 &= \frac{1}{n^3} \sum_{\alpha,\beta,\gamma} W[h_{\alpha,\beta,\gamma}; b_{[i,j];k}] \chi_{(h_{\alpha,\beta,\gamma})}^{(m,n,e)} = \\
 &= \sum_{\substack{i,j=0 \\ i,j=0}}^{\bar{e}-1} b_{[i,j];e} \sum_{u,v=0}^{n-1} \frac{\delta_{ne+i,m} \delta_{ve+j,n}}{\bar{e}^2} \quad (4.10)
 \end{aligned}$$

Now we can apply to these two functions the duality condition

$$\tilde{W}[h_{\alpha,\beta,\gamma}; b_{[i,j];k}] = W[h_{\alpha,\beta,\gamma}; b_{[i,j];k}^*] \quad (4.11)$$

where the dual parameters $b_{[i,j];k}^*$ have to be determined as functions of the old ones. The analysis of this condition can be performed, in principle, for any n , but for simplicity we will consider the cases $n=p$ (p =prime number) and $n=4$.

1) $\mathcal{D}(p)$: p =prime number.

We recall from Section 2 that in this case the structure of the representations simplifies, and we get

$$W[h_{\alpha,\beta,\gamma} ; b^*_{[i,j];k}] = \\ = \sum_{i,j=0}^{p-1} b^*_{[i,j];0} \omega^{-\alpha i - \beta j} + p^2 \delta_{\alpha,0} \delta_{\beta,0} \sum_{k=1}^{p-1} b^*_{[0,0];k} \omega^{-\gamma k}$$

$$\widetilde{W}[h_{\alpha,\beta,\gamma} ; b_{[i,j];k}] = \\ = \delta_{\gamma,0} b_{[\alpha,\beta];0} + (1 - \delta_{\gamma,0}) b_{[0,0];\gamma} \quad (4.12)$$

Now, from the self-duality condition, taking $\gamma \neq 0$, multiplying with $\omega^{u\alpha+v\beta}$ and summing over α and β we get

$$\delta_{u,0} \delta_{v,0} b_{[0,0];\gamma} = \\ = b^*_{[u,v];0} + \sum_{k=1}^{p-1} b^*_{[0,0];k} \omega^{-\gamma k} \quad (4.13)$$

By taking u or v different from zero and summing over γ we get

$$b^*_{[u,v];0} = b^*_{[0,0];k} \equiv X_1^* \\ (u,v) \neq (0,0) \quad (4.14)$$

For $u=v=0$ we get

$$b_{[0,0];\gamma} = X_0^* - X_1^* \\ X_0^* \equiv b^*_{[0,0];0} \quad (4.15)$$

independent of γ ; therefore we have

$$X_1 = X_0^* - X_1^* \quad (4.16)$$

with

$$X_1 \equiv b_{[0,0];\gamma} \quad (4.17)$$

For $\gamma = 0$ we have

$$b_{[\alpha, \beta; 0]} = X_0^* - X_1^* + p^3 X_1^* \delta_{\alpha, 0} \delta_{\beta, 0} \quad (4.18)$$

We thus get

$$\begin{aligned} X_0 &= X_0^* + (p^3 - 1) X_1^* \\ X_0 &\equiv b_{[0, 0; 0]} \end{aligned} \quad (4.19)$$

and for α or $\beta \neq 0$ we get that $b_{[\alpha, \beta; 0]}$ does not depend on its indices. For $(\alpha, \beta) \neq (0, 0)$ we use Eq. (4.18) to obtain

$$b_{[\alpha, \beta; 0]} = X_0^* - X_1^* = X_1 \quad (4.20)$$

We conclude that our system depends only on the parameters X_0 and X_1 , related to the dual ones by

$$\begin{aligned} X_0 &= X_0^* + (p^3 - 1) X_1^* \\ X_1 &= X_0^* - X_1^* \end{aligned} \quad (4.21)$$

By using the original expression for W and $\tilde{W}$ we get

$$\begin{aligned} W &= X_0 - X_1 + p^3 X_1 \delta(g_{\alpha, \beta}^{\tau}, e) \\ \tilde{W} &= X_1 + (X_0 - X_1) \delta(g_{\alpha, \beta}^{\tau}, e) \end{aligned} \quad (4.22)$$

and we see that the self-duality condition forces our system to be an order p^3 spin system. The W function (4.22) corresponds to the Hamiltonian

$$H = \beta_0 + \beta_1 p^3 \delta(g_{\alpha, \beta}^{\gamma}, e) \quad (4.23)$$

with β_0 and β_1 given by

$$\begin{aligned} e^{-\beta_0} &= X_0 - X_1 \\ e^{-p^3 \beta_1} &= \frac{X_0 - X_1 + p^3 X_1}{X_0 - X_1} \end{aligned} \quad (4.24)$$

From the relations defining the dual transformation we get

$$\frac{X_0}{X_1} = 1 + p^3 \frac{X_1^*}{X_0^* - X_1^*} = e^{-p^3 \beta_1^*} \quad (4.25)$$

and

$$(e^{-p^3 \beta_1} - 1)(e^{-p^3 \beta_1^*} - 1) = p^3 \quad (4.26)$$

This should be compared with Eq. (3.22) obtained for the order N model (recall that in this case the order of the group was N).

2) $\mathcal{D}(4)$

This example is rather interesting because it shows that there are solutions more general than the order N spin systems. In fact, by solving the self-duality condition, we find

$$\begin{aligned} W(g_{\alpha, \beta}^{\gamma}) &= Y_0 + 4^3 Y_1 \delta(g_{\alpha, \beta}^{\gamma}, e) + \\ &+ 2^3 Y_2 (\delta_{\alpha, 0} + \delta_{\alpha, 2})(\delta_{\beta, 0} + \delta_{\beta, 2})(\delta_{\gamma, 0} + \delta_{\gamma, 2}) \end{aligned} \quad (4.27)$$

with the related Hamiltonian

$$H = \beta_0 + 4^3 \beta_1 \delta(q_{\alpha,\beta}^\gamma, e) + \\ + 2^3 \beta_2 (\delta_{\alpha,0} + \delta_{\alpha,2})(\delta_{\beta,0} + \delta_{\beta,2})(\delta_{\gamma,0} + \delta_{\gamma,2}) \quad (4.28)$$

The following relations among the parameters hold

$$e^{-\beta_0} = Y_0 \\ e^{-(\beta_0 + 2^3 \beta_2)} = Y_0 + 2^3 Y_2 \\ e^{-(\beta_0 + 4^3 \beta_1 + 2^3 \beta_2)} = \\ = Y_0 + 4^3 Y_1 + 2^3 Y_2 \quad (4.29)$$

This time one has two essential parameters defining the system, (β_1, β_2) , and the relation among these parameters and their dual is

$$(e^{-4^3 \beta_1} - 1)(e^{-4^3 \beta_1^*} - 1)e^{-2^3(\beta_2 + \beta_2^*)} = 4^3$$

$$e^{-2^3 \beta_2^*} - 1 = 2^3 \frac{e^{-2^3 \beta_2} - 1}{e^{-2^3 \beta_2}(e^{-4^3 \beta_1^*} - 1)}$$

(4.30)

We notice that for $\beta_2 = 0$ we get $\beta_2^* = 0$ and the dual transformation (4.30) reduces to the corresponding transformation of an order 4^3 system.

5. - CONCLUSIONS

As we have seen in the previous sections, the solvable $\mathcal{D}(n)$ groups have the important property that we can co-ordinate the group elements by the Abelian $\mathbb{Z}_n \otimes \mathbb{Z}_n \otimes \mathbb{Z}_n$ group in a way very similar to that used to parametrize Lie groups. We conjecture that close form expressions for the irreducible representations and their Clebsch-Gordan coefficients can be obtained from many families of solvable groups (most certainly for the nilpotent ones⁹⁾). Once this knowledge is acquired one could find which solvable groups are dual to each other. The question arises for groups of order larger than 8 because for groups of order 2, 3, 5 and 7 one has only Abelian groups, for order 6 the single non-Abelian group is S_3 and for order 8 the only two non-Abelian groups D_4 and Q have identical characters.

As for the problem of self-duality, we have seen that for n prime the self-duality condition for $\mathcal{D}(n)$ reduces the problem to an order n^3 system and that for $\mathcal{D}(4)$ (group of order 4^3) the self-duality condition reduces the problem to a superposition of an order 4^3 system and an order 2^3 system.

It appears to us that we can go even one step further and look for the classification of self-dual systems. We conjecture that this problem is deeply related to the classification of Abelian groups in the following way. Consider groups of order N and write

$$N = p_1^{s_1} \cdot p_2^{s_2} \cdots p_e^{s_e} \quad (5.1)$$

where p_i are prime numbers. Consider now a partition

$$s_i = \sum_m r_{i,m} \quad (5.2)$$

where $r_{i,m}$ are positive numbers. For each partition consider the Abelian group :

$$\mathbb{Z}(p_1^{r_{i,1}}) \otimes \mathbb{Z}(p_1^{r_{i,2}}) \otimes \cdots \otimes \mathbb{Z}(p_1^{r_{i,s}}) \quad (5.3)$$

and solve the self-duality conditions for this group. The solution to this problem will be a superposition of δ functions over the subgroups of (5.3). A further restriction on the solutions can be obtained considering the solvable groups which are obtained taking semi-direct products instead of direct products in Eq. (5.3).

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FIGURE CAPTION

The lattice and the dual lattice for a two-dimensional spin system.

FIG. 1