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CHARGES AND CURRENT INDUCED BY MOVING IONS IN MULTIWIRE CHAMBERS

ABSTRACT

A method for calculating the charges induced on the grid wires, and on cathode strips parallel to the grid wires, by a point charge in a multiwire chamber is described. The method is applied to the calculation, as a function of time, of the charge and current induced by a small group of positive ions moving in accordance with the drift equation $v = \mu E$ where v is the velocity. An appendix lists a number of formulae relating to the electrostatic field of a multiwire chamber.

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1. Coordinate system and single-wire potential

We consider an idealized multiwire proportional chamber consisting of a grid of equally-spaced thin wires lying between, and parallel to, two plane zero-potential cathodes. The grid and cathodes are assumed to extend to infinity. A cross-section of such a chamber is shown in fig. 1. The cathodes are situated at $y = \pm L$ and the wires at $z_k = z_0 + ks$, $k = 0, \pm 1, \pm 2, \dots$. The wire radius is ρ . Electrostatic units are used throughout.

For a *single* infinitely-thin wire carrying unit charge per unit length situated at z' , the complex potential at an arbitrary point z is [ref. 1, p. 90]:

$$\phi(z, z') = -2 \ln \left\{ \frac{\sinh [(\pi/4L)(z - z')] }{\cosh [(\pi/4L)(z - \bar{z}')] } \right\}, \quad (1)$$

where $\bar{z}'$ is the complex conjugate of z' . The corresponding physical potential is

$$v(z, z') = \text{Re } \phi(z, z').$$

We adopt the convention that $v(z', z')$ denotes the potential on the surface of the wire at z' . Therefore, if this wire has radius $\rho \ll L$,

$$v(z', z') = -2 \ln \left\{ \frac{\pi \rho / 4L}{\cos (\pi y' / 2L)} \right\}.$$

2. The effect of introducing a point charge

2.1 INDUCED CHARGE AND CURRENT ON THE WIRES

A small group of positive ions drifting along a field line in a multiwire chamber will be replaced, for the purpose of calculation, by a single point charge. In order to simplify the calculation of the supplementary charges Q_m induced on the wires by the presence of this point charge we shall assume that the wires are maintained at a constant potential, i.e. that the impedance between each wire and the constant-voltage source is negligible. In this case the charges Q_m are equal to the charges which would be induced on the wires of a chamber whose grid was maintained at zero potential.

If the point charge has magnitude Q and is situated at z_Q , we may obtain equations determining the values of the Q_m by means of Green's reciprocity theorem. To do this, we replace the point charge by a small conducting sphere and consider two distinct, but physically permissible, electrostatic configurations of the system of conductors consisting of the sphere, the grid wires, and the two cathodes. Configuration 1 is the physical situation of interest to us. Configuration 2 is one in which only a single wire, the k th, carries a charge, the sphere and the remaining wires being uncharged.

Configuration 1. Sphere: Charge Q , potential V_Q .

Wires: Charge Q_m , potential $V_m = 0$ ($m = 0, \pm 1, \dots$).

Configuration 2. Sphere: Charge $Q' = 0$, potential V'_Q .

Wire k : Charge Q'_k , potential V'_k .

Wires $m \neq k$: Charge $Q'_m = 0$, potential V'_m ($m = 0, \pm 1, \dots$).

The cathodes are at zero potential in both configurations, and therefore need not be taken into account in applying the reciprocity theorem [ref. 2, section 2.12], which states that

$$QV'_Q + \sum Q_m V'_m = Q'V_Q + \sum Q'_m V_m = 0.$$

Since the wires are thin and the sphere is small, V'_Q is the potential at z_Q , and V'_m ($m \neq k$) the potential at z_m , in the unperturbed field of a single charged wire at z_k . If the wires are assumed to have length ℓ , where ℓ is large but finite, the charge per unit length on the k th wire is Q'_k/ℓ . Therefore,

$$V'_Q = (Q'_k/\ell)v(z_Q, z_k), \quad V'_m = (Q'_k/\ell)v(z_m, z_k),$$

and hence

$$\sum_{m=-\infty}^{\infty} Q_m v(z_m, z_k) = -Qv(z_Q, z_k),$$

this equation being valid even when the point charge Q is arbitrarily close to the surface of one of the wires.

For a *symmetric* chamber with $z_0 = 0$, we have $v(z, z_m) = v(|z - z_m|, 0)$ for all m . Therefore, allowing k to assume successively all integer values, we obtain the following system of simultaneous linear equations determining the induced wire charges Q_m :

$$\sum_{m=-\infty}^{\infty} a_{k-m} u_m = c_k, \quad -\infty < k < \infty, \quad (2)$$

where

$$\left. \begin{aligned} u_m &= Q_m/Q, \\ a_0 &= -2 \ln (\pi \rho / 4L), \\ a_m &= -2 \ln \{ \tanh |m\pi s / 4L| \}, \quad m \neq 0, \\ c_k &= 2 \operatorname{Re} \ln \{ \tanh [(\pi / 4L)(z_Q - ks)] \} \\ &= \ln \left\{ \frac{\cosh [(\pi / 2L)(x_Q - ks)] - \cos (\pi y_Q / 2L)}{\cosh [(\pi / 2L)(x_Q - ks)] + \cos (\pi y_Q / 2L)} \right\}. \end{aligned} \right\} \quad (3)$$

If the charge Q is moving with velocity $\dot{z}_Q = dz_Q/dt$, the wire currents $\dot{Q}_m$ are determined by the following system of equations obtained by differentiating (2) with respect to t :

$$\sum_{m=-\infty}^{\infty} a_{k-m} \dot{u}_m = \dot{c}_k, \quad -\infty < k < \infty, \quad (4)$$

where

$$\dot{u}_m = \dot{Q}_m/Q, \quad \dot{c}_k = (\dot{x}_Q \partial / \partial x_Q + \dot{y}_Q \partial / \partial y_Q) c_k.$$

The numerical solution of (2) and (4) is discussed in section A4 of the appendix.

2.2 INDUCED CHARGE AND CURRENT ON CATHODE STRIPS

In this section we shall consider a multiwire chamber whose cathodes have been divided into strips the edges of which are parallel to the wires. We shall show that, for this particular orientation of the strips, the calculation of the charge induced on any strip when a point charge is introduced into the chamber may be formulated as a two-dimensional electrostatic problem. The more general problem of calculating the induced charge on strips which are not parallel to the wires is inherently three-dimensional, and will not be discussed here.

Consider first a chamber consisting only of two plane cathodes whose surfaces have been divided into strips the edges of which are perpendicular to the z -plane, where it is assumed that the gaps between the strips are of negligible width so as to allow the cathodes to be regarded as uniform zero-potential conductors. If a point charge Q is introduced at any position on a fixed line lying perpendicular to the z -plane, the total induced charge on any selected cathode strip will be independent of the position of Q along the line. Therefore, if the point charge is replaced by a quantity of charge Q distributed arbitrarily along the line, the induced charge on the strip will be independent of the form of this distribution, which may therefore be replaced by a *uniformly* charged line.

Given a cathode strip $[X_1, X_2]$, bounded in the z -plane by the points $X_1 + iL$ and $X_2 + iL$, and given a uniformly charged line passing through the point z , we define $p(z)$ to be the ratio of the induced charge per unit length on the strip to the charge per unit length on the line. Therefore [ref. 2, section 4.11], with ϕ defined by (1),

$$p(z) = p_L(X_2, z) - p_L(X_1, z) ,$$

where

$$p_L(X, z) = -(1/4\pi) \operatorname{Im} \phi (X+iL, z) .$$

For a *symmetric* chamber,

$$p_L(X, z) = -(1/\pi) \arctan \left\{ \tanh \left[(\pi/4L)(X-x) \right] \right. \\ \left. \times \tan \left[(\pi y/4L) + (\pi/4) \right] \right\} ,$$

and hence

$$\dot{p}_L(X, z) = \frac{\dot{x} \cos (\pi y/2L) - \dot{y} \sinh \left[(\pi/2L)(X-x) \right]}{4L \{ \cosh \left[(\pi/2L)(X-x) \right] - \sin (\pi y/2L) \}} .$$

We now consider a chamber containing grid wires at the points z_m shown in fig. 1, and suppose that a point charge of magnitude Q is introduced at some position z_Q . By applying the principle of the preceding paragraph, and noting that a point charge is a particular case of a line distribution, we find that the total charge $Q_S(z_Q)$ induced on the strip $[X_1, X_2]$ is given by

$$Q_S(z_Q) = Q_P(z_Q) + \sum_m Q_m p(z_m) ,$$

where the summation, in this and subsequent equations, is from $m = -\infty$ to $m = +\infty$.

The current flowing to the strip is

$$\dot{Q}_S(z_Q) = Q \dot{P}(z_Q) + \sum_m \dot{Q}_m p(z_m) .$$

These equations can be written in abbreviated form for future reference as

$$\begin{aligned} Q_S &= Q \left[P_Q + \sum_m u_m p_m \right] , \\ \dot{Q}_S &= Q \left[\dot{P}_Q + \sum_m \dot{u}_m p_m \right] , \end{aligned} \tag{5}$$

where $Q_S = Q_S(z_Q)$, $P_Q = P(z_Q)$ and $p_m = p(z_m)$, $m = 0, \pm 1, \pm 2, \dots$. It is shown in section A4 of the appendix how these expressions may be evaluated without explicitly computing the terms u_m and $\dot{u}_m$.

3. Ion drift motion

3.1 DRIFT EQUATIONS

We wish to derive formulae for the position $z_Q(t)$ and velocity $\dot{z}_Q(t)$ of a small group of positive ions drifting along a field line in a multiwire chamber, neglecting any perturbation of the electrostatic field of the chamber caused by the ions themselves. We shall consider only a symmetric chamber, and shall suppose that $L > s$. If the grid of such a chamber is at potential V_0 , the physical potential at z may be represented with sufficient accuracy (appendix, section A1) by the expression

$$V(z) = cV_0 \operatorname{Re} \psi(z) , \tag{6}$$

where

$$\psi(z) = (2\pi L/s) - 2 \ln \{2 \sin (\pi z/s)\} \tag{7}$$

is the complex potential when the wires carry unit charge per unit length, and where the dimensionless constant

$$c = \left[(2\pi L/s) - 2 \ln (2\pi \rho/s) \right]^{-1} \tag{8}$$

is the capacitance per unit length. (Expressions for the potential in multiwire chambers with arbitrary values of L , s and z_0 are given in section A1 of the appendix.)

We shall suppose that a small group of positive ions moving outwards from the wire $z_0 = 0$ along a field line may be assigned a single vector coordinate $\underline{z}(t)$ whose value is determined by a drift equation of the form

$$\dot{\underline{z}} = \mu \underline{E} , \quad (9)$$

where $\dot{\underline{z}}$ is the velocity, $\underline{E}$ is the local field intensity, and μ is a constant which depends on the gas in the chamber, its value being proportional to the gas pressure p . It should be noted, however, that for certain pure gases at sufficiently high field intensity (E/p of order $100 \text{ V cm}^{-1} \text{ Torr}^{-1}$ and upwards) the appropriate drift equation [3] is of the form

$$\dot{\underline{z}} = \lambda |\underline{E}|^{-1/2} \underline{E} , \quad (10)$$

where λ is constant.

3.2 WIRE REGION

The field in the neighbourhood of any wire of a multiwire chamber for which $L > s$ is isotropic out to a distance of approximately $0.1 s$, at which distance the departure from isotropy is approximately 3%. Near the wire $z_0 = 0$ we have $E = 2cV_0 r^{-1}$ where $r = |z|$. Therefore the solution of (9) corresponding to ions leaving the surface of the wire at time $t = 0$ along a field line making an angle α with the x-axis is given by

$$z_Q(t, \alpha) = \{1 + (4\mu cV_0/\rho^2)t\}^{1/2} \rho e^{i\alpha} . \quad (11)$$

The corresponding solution of (10) is

$$z_Q(t, \alpha) = \{1 + (9\lambda^2 cV_0/2\rho^3)t\}^{2/3} \rho e^{i\alpha} . \quad (12)$$

3.3 CHAMBER REGION

The field intensity at any point z in the chamber is $E = E_x + iE_y = -cV_0 \overline{\psi'(z)}$, where ψ is given by (7). The drift equation (9) takes the form

$$\dot{z} = (2\pi\mu cV_0/s) \cot(\pi\bar{z}/s) . \quad (13)$$

Once the ions have left the immediate neighbourhood of the wire, it is no longer necessary to distinguish between the surface of the wire and the point $z_0 = 0$. Therefore, for ions which leave the origin at angle α , the appropriate

initial conditions for (13) are that $z \rightarrow 0$ and $\arg z \rightarrow \alpha$ as $t \rightarrow 0$. These conditions determine the following unique solution (appendix, section A3):

$$\left. \begin{aligned} x(t, \alpha) &= (s/2\pi) \arccos \{ (1 + \cos 2\alpha) e^{-\kappa t} - \cos 2\alpha \} \\ y(t, \alpha) &= (s/2\pi) \operatorname{arccosh} \{ (1 - \cos 2\alpha) e^{\kappa t} + \cos 2\alpha \} \end{aligned} \right\} \quad (14)$$

where

$$\kappa = 4\pi^2 \mu c V_0 / s^2 . \quad (15)$$

For a given value of t , the points (x, y) specified by (14) for $0 \leq \alpha \leq 2\pi$ constitute a drift envelope, on which lie all ions for which $z_0 = 0$ at $t = 0$. Drift envelopes for some equally-spaced values of κt are shown in fig. 2, together with field lines plotted at 10° intervals in α .

4. The shape of the induced pulse

4.1 METHOD OF CALCULATION

If a point charge Q , representing a small group of ions, leaves the wire $z_0 = 0$ at time $t = 0$ and drifts towards the cathode $y = L$ along a field line which enters the wire at angle α , the induced wire charges $Q_m[z_Q(t, \alpha)]$ at time t may be calculated by substituting the appropriate expression $z_Q(t, \alpha)$ in (3) and then solving the system of equations (2). The wire currents $\dot{Q}_m[z_Q(t, \alpha)]$ may be calculated by differentiating z_Q with respect to t and solving (4). The charge and current, $Q_S[z_Q(t, \alpha)]$ and $\dot{Q}_S[z_Q(t, \alpha)]$, respectively, induced on a given cathode strip may be calculated from (5).

If at time $t = 0$ a quantity of charge Q is distributed around the surface of the wire in such a way that the charge contained in angular interval $d\alpha$ is $Q\omega(\alpha) d\alpha$, the averaged induced charge and current on the m th wire are given by

$$\hat{Q}_m(t) = \int_0^{2\pi} Q_m[z_Q(t, \alpha)] \omega(\alpha) d\alpha, \quad \hat{I}_m(t) = \int_0^{2\pi} \dot{Q}_m[z_Q(t, \alpha)] \omega(\alpha) d\alpha. \quad (16)$$

Similarly, the averaged induced charge and current on a cathode strip are given by

$$\hat{Q}_S(t) = \int_0^{2\pi} Q_S[z_Q(t, \alpha)] \omega(\alpha) d\alpha, \quad \hat{I}_S(t) = \int_0^{2\pi} \dot{Q}_S[z_Q(t, \alpha)] \omega(\alpha) d\alpha. \quad (17)$$

If t_L is the time of arrival of the first ions at the cathode, the integrals (16) and (17) are valid only for $t < t_L$. The modifications required when $t > t_L$ are discussed in section 4.3.

4.2 WIRE REGION PULSE

For ions moving in the isotropic field surrounding the wire $z_0 = 0$, $z_Q(t, \alpha)$ is given by (11) or (12). For sufficiently small values of $r_Q = |z_Q|$ within the isotropic region, further simplification of the charge calculation is possible. Thus, letting $r_Q \rightarrow 0$ in (3),

$$c_k \approx -a_k + \gamma_k (r_Q/s) \cos \alpha + 2\delta_{k0} \ln (r_Q/\rho),$$

where δ_{k0} is the Kronecker symbol (equal to 1 if $k = 0$, otherwise zero), and where

$$\gamma_0 = 0,$$

$$\gamma_k = \frac{-(\pi s/L)}{\sinh (k\pi s/2L)}, \quad (k \neq 0).$$

Substituting this approximation for c_k in (2), and assuming that r_Q is sufficiently small, we obtain:

$$\begin{aligned} Q_m/Q &\approx -\delta_{m0} + 2\alpha_m \ln (r_Q/\rho) + \beta_m (r_Q/s) \cos \alpha, \\ \dot{Q}_m/Q &\approx 2\alpha_m (\dot{r}_Q/r_Q) + \beta_m (\dot{r}_Q/s) \cos \alpha, \end{aligned} \quad (18)$$

where the coefficients α_m and β_m satisfy the equations

$$\begin{aligned} \sum_m a_{k-m} \alpha_m &= \delta_{k0}, \quad -\infty < k < \infty, \\ \sum_m a_{k-m} \beta_m &= \gamma_k, \quad -\infty < k < \infty, \end{aligned} \quad (19)$$

which may be solved in the same way as (2). Since $a_{-m} = a_m$ and $\gamma_{-m} = -\gamma_m$ for all m , we have $\alpha_{-m} = \alpha_m$ and $\beta_{-m} = -\beta_m$ for all m . In particular, $\alpha_{-1} = \alpha_1$ and $\beta_{-1} = -\beta_1$. Hence, for sufficiently small r_Q , the right-left difference between the induced charges and currents on the wires z_1 and z_{-1} adjacent to the active wire z_0 are given by

$$\begin{aligned} (Q_1 - Q_{-1})/Q &\approx 2\beta_1 (r_Q/s) \cos \alpha, \\ (\dot{Q}_1 - \dot{Q}_{-1})/Q &\approx 2\beta_1 (\dot{r}_Q/s) \cos \alpha. \end{aligned} \quad (20)$$

Fig. 3 shows the coefficient $2\beta_1$ as a function of ρ/s for three values of L/s . It can be seen that for typical values of the geometrical parameters the value of $2\beta_1$ lies between -0.2 and -0.3.

A similar calculation shows that the induced charge on the cathode strip $[X_1, X_2]$, for sufficiently small r_Q , is given by

$$Q_S \approx 2A \ln (r_Q/\rho) + (B+f)(r_Q/s) \cos \alpha + g(r_Q/s) \sin \alpha ,$$

where

$$A = \sum_m \alpha_m p_m , \quad B = \sum_m \beta_m p_m ,$$

$$f = (s/4L) \{ \operatorname{sech} (\pi X_2/2L) - \operatorname{sech} (\pi X_1/2L) \} ,$$

$$g = -(s/4L) \{ \tanh (\pi X_2/2L) - \tanh (\pi X_1/2L) \} .$$

~~The region around the wire within which these approximations are likely to be adequate is considerably smaller than the region of isotropic field.~~

4.3 CHAMBER REGION PULSE

If the drift equation $\dot{z} = \mu E$ is assumed to hold throughout the chamber, then $z_Q(t, \alpha) = x_Q + iy_Q$ is given by (14), and ions moving on field line α reach the cathode at time

$$t_a(\alpha) = \frac{1}{K} \ln \left\{ \frac{\cosh (2\pi L/s) - \cos 2\alpha}{1 - \cos 2\alpha} \right\} .$$

If $t < t_L$, where $t_L = t_a(\frac{1}{2}\pi)$ is the time of arrival of the first ions, the averaged induced charge and current on the wires is given by the integrals (16).

If $t > t_L$, ions which leave the origin at time $t = 0$ will reach the cathode exactly at time t if they lie on the field line whose angle is obtained by setting $y = L$ in (14), namely

$$\alpha_L(t) = \frac{1}{2} \arccos \left\{ \frac{1 - e^{-Kt} \cosh (2\pi L/s)}{1 - e^{-Kt}} \right\} .$$

At time t , ions for which $|\alpha| > \alpha_L(t)$ or $|\pi - \alpha| > \alpha_L(t)$ will have already been collected on the cathode and will make no contribution to the induced wire charges $Q_m[z_Q(t, \alpha)]$. Therefore, for $t > t_L$ the averaged wire charges are

$$\hat{Q}_m(t) = \left\{ \int_{-\alpha_L}^{\alpha_L} + \int_{\pi-\alpha_L}^{\pi+\alpha_L} \right\} Q_m[z_Q(t, \alpha)] \omega(\alpha) d\alpha. \quad (21)$$

Because α_L is a function of t , the time derivative of (21) will contain terms such as $\dot{\alpha}_L Q_m[z_Q(t, \alpha_L)] \omega(\alpha_L)$. However, these terms vanish because $z_Q(t, \alpha_L)$ lies on the cathode. The expression for the averaged wire currents therefore reduces to

$$\hat{I}_m(t) = \left\{ \int_{-\alpha_L}^{\alpha_L} + \int_{\pi-\alpha_L}^{\pi+\alpha_L} \right\} \dot{Q}_m[z_Q(t, \alpha)] \omega(\alpha) d\alpha.$$

For $t > t_L$, the form of the integral representing the averaged induced charge on a cathode strip is similar to (21) provided the strip interval $[X_1, X_2]$ does not overlap the active interval $[-\frac{1}{2}s, \frac{1}{2}s]$ within which ions can be collected on the cathode, but the case in which these intervals do overlap requires further discussion. This occupies the remainder of this section.

If a point charge Q leaves the origin at time $t = 0$ and moves along a field line α until it is collected on the cathode at time t_a , we shall define the induced strip charge Q_S , for all t , to be the integral from time $t = 0$ to time t of the charge supplied by an external source in order to maintain the strip at zero potential. If the field line α intersects the strip, then $Q_S \rightarrow -Q$ as $t \rightarrow t_a$ and remains equal to $-Q$ for $t > t_a$. If the field line does not intersect the strip, then $Q_S \rightarrow 0$ as $t \rightarrow t_a$ and remains equal to zero for $t > t_a$.

For a fixed value of $t > t_L$, consider the term $Q_S[z_Q(t, \alpha)]$ in (17) as a function of α . If $|\alpha| \leq \alpha_L$ or $|\pi - \alpha| \leq \alpha_L$, an ion with coordinate $z_Q(t, \alpha)$ will not yet have reached the cathode, and Q_S will be given by (5). If $|\alpha| > \alpha_L$ or $|\pi - \alpha| > \alpha_L$, then Q_S will be equal to $-Q$ if the field line α intersects the strip $[X_1, X_2]$, and Q_S will be equal to zero if the field line does not intersect the strip. Therefore, if α_1 and α_2 identify the field lines passing through X_1 and X_2 respectively (fig. 4), and if $[\alpha_A, \alpha_B]$ is the intersection of the angular intervals $[\alpha_1, \alpha_2]$ and $[\pi - \alpha_L, \alpha_L]$, the integral (17) takes the form

$$\hat{Q}_S(t) = \left\{ \int_{-\alpha_L}^{\alpha_L} + \int_{\pi-\alpha_L}^{\pi+\alpha_L} \right\} Q_S[z_Q(t, \alpha)] \omega(\alpha) d\alpha - Q \int_{\alpha_A}^{\alpha_B} \omega(\alpha) d\alpha. \quad (22)$$

When differentiating (22) with respect to t in order to obtain the averaged strip current, the terms involving $\dot{\alpha}_L$, $\dot{\alpha}_A$ and $\dot{\alpha}_B$ make no contribution to the final result. For example, in the situation shown in fig. 4, the terms involving the derivatives of $-\alpha_L$, $\pi-\alpha_L$ and $\pi+\alpha_L$ vanish because $Q_S[z_Q(t, \alpha)]$ is zero for these values of α , and the term involving $\dot{\alpha}_B$ vanishes because $\alpha_B = \alpha_1$ is a constant. The remaining term,

$$\dot{\alpha}_L Q_S[z_Q(t, \alpha_L)] \omega(\alpha_L) + \dot{\alpha}_A Q \omega(\alpha_A)$$

vanishes because $Q_S[z_Q(t, \alpha_L)] = -Q$ and $\alpha_A = \alpha_L$. The final expression for the averaged strip current for $t > t_L$ is always the same as for $t < t_L$, namely

$$\hat{I}_S(t) = \left\{ \int_{-\alpha_L}^{\alpha_L} + \int_{\pi-\alpha_L}^{\pi+\alpha_L} \right\} \dot{Q}_S[z_Q(t, \alpha)] \omega(\alpha) d\alpha.$$

The angles α_1 and α_2 may be computed from the relation

$$\alpha = (\frac{1}{2} - X/s) \pi,$$

obtained by setting $\tanh(\pi L/s) = 1$ in formula (28) of the appendix.

4.4 TOTAL CHARGE AND CURRENT ON THE GRID AND CATHODE FOR A UNIFORM ANGULAR DISTRIBUTION OF IONS

In general, the integrals containing the angular distribution $\omega(\alpha)$ must be evaluated numerically. However, if we consider only the *total* charge and current induced on the grid or cathode, and further suppose that the initial angular distribution is uniform, these integrals can be evaluated analytically. The resulting formulae are useful for checking computer programs written to handle the general case.

If $V(z)$ is the physical potential in a symmetric multiwire chamber whose grid is at potential V_0 , a simple application of the reciprocation theorem shows

that, for a point charge Q at z_Q , the total charge $Q_G = \sum Q_m$ induced on the grid, and the total charge Q_L induced on the cathode $y = L$, are given by

$$Q_G = -QV(z_Q)/V_0 ,$$

$$Q_L = -\frac{1}{2} Q[1 + (y_Q/L)] - \frac{1}{2} Q_G .$$

Multiplying these equations by $1/2\pi$ and integrating from $\alpha = 0$ to $\alpha = 2\pi$, with $V(z)$ given by (6) and z_Q by (14), leads after some algebra to

$$\hat{Q}_G(t) = -cQ \left\{ (2\pi L/s) - \ln \left[\left(4 + 4 \cosh \frac{1}{2} \kappa t \right) \sinh \frac{1}{2} \kappa t \right] \right\} ,$$

$$\hat{Q}_L(t) = -\frac{1}{2} [Q + \hat{Q}_G(t)] ,$$

where c and κ are defined by (8) and (15), respectively. The corresponding expressions for the current are

$$\hat{I}_G(t) = \frac{1}{2} \kappa c Q \left(2 \cosh \frac{1}{2} \kappa t - 1 \right) / \sinh \frac{1}{2} \kappa t ,$$

$$\hat{I}_L(t) = -\frac{1}{2} \hat{I}_G(t) .$$

5. Illustrative example

Assuming a drift equation of the form $\dot{z} = \mu E$, the induced pulse on selected wires and cathode strips of a symmetric multiwire chamber was computed as a function of time using the following physically realistic values of the parameters:

$$L = 0.5 \text{ cm} , \quad s = 0.254 \text{ cm} , \quad \rho = 7.5 \times 10^{-4} \text{ cm} ,$$

$$V_0 = 3 \text{ kV} , \quad \mu = 1 \text{ cm}^2 \text{ V}^{-1} \text{ s}^{-1} , \quad Q = 10^7 \text{ e} ,$$

$$(e = 1.602 \times 10^{-19} \text{ C}) .$$

The corresponding values of c and κ are:

$$c = 4.92 \times 10^{-2} \text{ (dimensionless)} , \quad \kappa = 9.02 \times 10^4 \text{ s}^{-1} .$$

The first few coefficients α_m and β_m occurring in the wire region approximation (18) for Q_m and $\dot{Q}_m$ are the following:

m	α_m	β_m
0	7.74×10^{-2}	0
1	-9.97×10^{-3}	-1.38×10^{-1}
2	-2.82×10^{-3}	-3.28×10^{-2}
3	-8.99×10^{-4}	-1.00×10^{-2}
4	-2.93×10^{-4}	-3.24×10^{-3}
5	-9.61×10^{-5}	-1.06×10^{-3}
6	-3.16×10^{-5}	-3.48×10^{-4}
7	-1.04×10^{-5}	-1.14×10^{-4}
8	-3.41×10^{-6}	-3.75×10^{-5}

The largest error in this approximation was found to occur at $m = \pm 1$:

t (μ s)	r_Q/s	Error in Q_1 (%)	Error in $\dot{Q}_1$ (%)
0.2	0.043	0.3	2
1.0	0.096	1	10

Pulse calculations for $0 \leq t \leq 220 \mu$ s were carried out using two different initial angular distributions of ions around the wire $z_0 = 0$: i) the delta-function distribution in which the entire charge Q leaves the wire at the same angle α , ii) the Gaussian distribution specified by

$$\omega(\theta) = (2\pi\sigma^2)^{-1/2} \exp \{-(\theta-\alpha)^2/2\sigma^2\}, \quad \sigma = 30^\circ.$$

For the Gaussian distribution, the integration over θ was performed numerically using a 16-point Gauss quadrature formula for each of the angular sub-intervals $[0, \alpha_L]$, $[\pi - \alpha_L, \pi]$, $[\pi, \pi + \alpha_L]$, $[-\alpha_L, 0]$. The discretization parameter N (appendix, section A4) was set equal to 16.

Fig. 5 shows the induced charges on the three central wires, with $Q_0 + Q$ and $\hat{Q}_0 + Q$ plotted instead of Q_0 and $\hat{Q}_0$ in order to ensure zero charge at $t = 0$. Fig. 6 shows the corresponding currents, plotted with reversed sign in order to give the voltage polarity which would be observed across a finite impedance. Fig. 7 shows the right-left differences $Q_1 - Q_{-1}$, etc., plotted with the same convention.

Fig. 8 shows the induced charge and (sign-reversed) current on a cathode strip running parallel to the wires and bounded by the x interval $[-s, s]$. All

charge leaving $z_0 = 0$ in the positive y direction ($0 \leq \alpha \leq \pi$) is eventually collected on this strip, but charge leaving the wire in the negative y direction ($-\pi \leq \alpha \leq 0$) is collected on the opposite cathode. Thus, for the Gaussian angular distribution with mid-point α , the averaged induced charge $\hat{Q}_S$ on the strip under consideration decreases steadily as α decreases from $\frac{1}{2}\pi$ towards zero.

Fig. 9 shows the induced charge and (sign-reversed) current on a cathode strip bounded by the x interval $[s, 3s]$ adjoining that shown in fig. 8.

I should like to thank Prof. G. Charpak and Dr. G. Petersen for helpful comments.

APPENDIX

A1. ACCURATE FIELD POTENTIALS

For a multiwire chamber with cathodes at $y = \pm L$ and thin wires carrying unit charge per unit length situated at $z_m = z_0 + ms$, $m = 0, \pm 1, \pm 2, \dots$, where z_0 is arbitrary, the complex potential $\psi(z, z_0)$ at a point z can be written in either of the two forms [ref. 1, p. 92]:

$$\psi(z, z_0) = -2 \ln \left\{ \frac{\vartheta_1[(\pi/s)(z-z_0), q]}{\vartheta_4[(\pi/s)(z-\bar{z}_0), q]} \right\} + i \frac{2\pi y_0}{Ls} (z-x_0) \pm i\pi \quad (23)$$

or

$$\psi(z, z_0) = -2 \ln \left\{ \frac{\vartheta_1[(i\pi/4L)(z-z_0), q']}{\vartheta_2[(i\pi/4L)(z-\bar{z}_0), q']} \right\}, \quad (24)$$

where

$$q = \exp(-4\pi L/s), \quad q' = \exp(-\pi s/4L).$$

For fast convergence of the theta function series [4], eq. (23) should be used when $L/s > 1/4$, and (24) when $L/s < 1/4$. (Expressions for $\psi(z, z_0)$ in terms of Jacobian elliptic functions are also available [5].)

If the wires of radius ρ (where $\rho \ll s$ and $\rho \ll L$) are at potential V_0 , the complex potential at z is $cV_0\psi(z, z_0)$, where $c = [\text{Re } \psi(z_0 + \rho, z_0)]^{-1}$ is the capacitance per unit length computed by neglecting terms of order ρ^2 in the expression for ψ .

For a symmetric chamber with $z_0 = 0$, the following approximations obtained by truncating the theta function series yield field lines and equipotentials whose position is correct to one part in 10^5 or better:

$$\psi(z, 0) = (2\pi L/s) - 2 \ln \{2 \sin(\pi z/s)\} \quad [1.5 < L/s] \quad (25)$$

$$\psi(z, 0) = \frac{2\pi L}{s} - 2 \ln \left\{ \frac{2 \sin(\pi z/s) - 2q^2 \sin(3\pi z/s)}{1 - 2q \cos(2\pi z/s)} \right\} \quad [0.3 < L/s < 1.5]$$

$$\psi(z, 0) = -2 \ln \left\{ \frac{\sinh(\pi z/4L) - (q')^2 \sinh(3\pi z/4L)}{\cosh(\pi z/4L) + (q')^2 \cosh(3\pi z/4L)} \right\} \quad [L/s < 0.3].$$

A2. THE SYMMETRIC CHAMBER WITH $L > s$

We summarize here some properties of the electrostatic field of a symmetric multiwire chamber whose complex potential may be represented to sufficient accuracy by (25). The physical potential is

$$V(z) = cV_0 \{ (2\pi L/s) - 2 \ln |2 \sin (\pi z/s)| \} , \quad (26)$$

and the equation ($\text{Im } \psi = \text{constant}$) of the field line which leaves the origin at angle α is

$$\arg \{ \sin (\pi z/s) \} = \alpha , \quad (27)$$

or equivalently,

$$\frac{\tanh (\pi y/s)}{\tan (\pi x/s)} = \tan \alpha . \quad (28)$$

Using (26) and (27), the equation of this field line can be written as

$$\sin (\pi z/s) = \xi^{1/2} e^{i\alpha} , \quad (29)$$

where

$$\xi = \frac{1}{4} \exp \{ (2\pi L/s) - (V/cV_0) \} .$$

To express (x,y) in terms of (V,α) for the purpose of field plotting we solve for z in equation (29) to obtain

$$\begin{aligned} x &= (s/\pi) \arcsin \{ (\xi - \eta + 1)/2 \}^{1/2} \\ y &= (s/\pi) \operatorname{arcsinh} \{ (\xi + \eta - 1)/2 \}^{1/2} , \end{aligned}$$

where

$$\eta = \{ (\xi - 1)^2 + 4\xi \sin^2 \alpha \}^{1/2} .$$

The calculation of (x,y) in terms of (V,α) for a symmetric chamber which is not necessarily subject to the restriction $L > s$ is discussed by Tomitani [6].

If an ionizing particle, moving in a straight line parallel to the y -axis, crosses the chamber at distance X from the origin (where $|X| < \frac{1}{2}s$), each electron released on the track will move towards the wire $z_0 = 0$ along a field line specified by an equation of the form (28). If the density of ionization along the track is uniform, the number of electrons approaching the origin between angles α and $\alpha + d\alpha$ will be proportional to $(\partial y/\partial \alpha) d\alpha$, where from (28),

$$\frac{\partial y}{\partial \alpha} = \frac{(s/\pi) \sin(2\pi X/s)}{\cos 2\alpha + \cos(2\pi X/s)} .$$

Very close to the wire, this angular distribution is likely to be blurred by the presence of avalanche electrons. The angle α is limited to the range $|\alpha| \leq \alpha_L$, $|\pi - \alpha| \leq \alpha_L$, where α_L is obtained by setting $x = X$ and $y = L$ in (28).

A3. SOLUTION OF THE DRIFT EQUATION $\dot{z} = \mu E$

The differential equation (13) which describes the drift motion of the ions in the chamber region can be written as

$$\dot{z} \sin(\pi \bar{z}/s) = \beta \cos(\pi \bar{z}/s) \quad (30)$$

where

$$\beta = 2\pi\mu cV_0/s .$$

Taking the real part of (30) gives

$$\dot{x} \tan(\pi x/s) + \dot{y} \tanh(\pi y/s) = \beta ,$$

or

$$\frac{d}{dt} \left[\ln \left\{ \frac{\cosh(\pi y/s)}{\cos(\pi x/s)} \right\} \right] = \frac{\pi\beta}{s} .$$

Imposing the condition $x = y = 0$ at $t = 0$ gives

$$\frac{\cosh(\pi y/s)}{\cos(\pi x/s)} = e^{\pi\beta t/s} . \quad (31)$$

Eliminating first y , then x , between (31) and (28) leads to the expressions (14) for x and y in terms of t and α .

A4. SOLUTION OF THE CONVOLUTION EQUATIONS

If we define the Fourier series

$$A(\theta) = \sum_m a_m e^{im\theta} , \quad C(\theta) = \sum_m c_m e^{im\theta} , \quad U(\theta) = \sum_m u_m e^{im\theta} , \quad (32)$$

where the summation is from $m = -\infty$ to $m = +\infty$, the system of convolution equations

$$\sum_m a_{k-m} u_m = c_k , \quad -\infty < k < \infty , \quad (2)$$

is seen to be equivalent to the relation $A(\theta)U(\theta) = C(\theta)$. Therefore, provided $A(\theta) > 0$ for all θ (for which it may be shown that a sufficient condition is $1 \leq L/s \leq s/2\rho$), the solution of (2) is

$$u_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta} \frac{C(\theta)}{A(\theta)} d\theta, \quad -\infty < m < \infty. \quad (33)$$

On defining the additional Fourier series

$$P(\theta) = \sum_m p_m e^{im\theta},$$

the infinite sum occurring in the expression (5) for Q_S can be written as

$$s_0 = \sum_m u_m p_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\overline{P(\theta)} C(\theta)}{A(\theta)} d\theta. \quad (34)$$

Since (33) and (34) are integrals, taken over a complete period, of periodic functions possessing a Fourier expansion, they may be approximated with good accuracy [7] by the trapezoidal rule. Thus, dividing the interval $[-\pi, \pi]$ into $2N$ equal subintervals, we obtain

$$u_m \approx \frac{1}{2N} \sum_{k=-N+1}^N \frac{C_k}{A_k} e^{-ikm\pi/N}, \quad -\infty < m < \infty, \quad (35)$$

$$s_0 \approx \frac{1}{2N} \sum_{k=-N+1}^N \frac{\overline{P}_k C_k}{A_k}, \quad (36)$$

where

$$A_k = A(k\pi/N), \quad C_k = C(k\pi/N), \quad P_k = P(k\pi/N). \quad (37)$$

For the numerical calculation of the terms A_k , C_k and P_k , the infinite sums (32) must be replaced by finite sums. If the range of summation is taken to be from $m = -N + 1$ to $m = N$, we obtain

$$A_k \approx \sum_{m=-N+1}^N a_m e^{ikm\pi/N}, \quad -N + 1 \leq k \leq N, \quad (38)$$

with similar expressions for C_k and P_k . Equation (38) shows that the computed approximations to the finite sequence $\{A_k\}$ consisting of the $2N$ terms A_k , $-N + 1 \leq k \leq N$, is the finite Fourier transform of the corresponding sequence $\{a_m\}$. It is therefore possible to use standard fast Fourier transform (FFT) computer programs to evaluate A_k , C_k and P_k . Further, it follows from (35) and (36) that the sequence $\{u_m\}$ is the finite Fourier transform of the sequence $\{C_k/A_k\}$ and that s_0 is the zero-order term in the finite Fourier transform of the sequence

$\{\bar{P}_k C_k / A_k\}$. Complex arithmetic may be avoided by making use of the properties

$A_{-k} = A_k$, $C_{-k} = \bar{C}_k$ and $P_{-k} = \bar{P}_k$, whereby (35) and (36) reduce to

$$u_m \approx \frac{1}{N} \sum_{k=0}^N \frac{(\text{Re } C_k) \cos (km\pi/N) + (\text{Im } C_k) \sin (km\pi/N)}{A_k},$$

$$s_0 \approx \frac{1}{N} \sum_{k=0}^N \frac{(\text{Re } P_k)(\text{Re } C_k) + (\text{Im } P_k)(\text{Im } C_k)}{A_k},$$

where the double prime indicates that the terms corresponding to $k = 0$ and $k = N$ are to be taken with weight one half.

The calculation of the terms $\dot{u}_m = \dot{Q}_m / Q$ defined by (4), and of the term $\dot{s}_0 = \sum \dot{u}_m p_m$ occurring in the expression (5) for $\dot{Q}_S$, may be carried out by the same method, the only difference being that a_m , c_m and p_m are replaced by $\dot{a}_m$, $\dot{c}_m$ and $\dot{p}_m$, respectively.

An alternative procedure, which is convenient when only a few of the wire charges are required, is to compute the required $u_k = Q_k / Q$ from the relation

$$u_k = \sum_{m=-\infty}^{\infty} \alpha_m c_{k-m},$$

where the terms

$$\alpha_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-im\theta}}{A(\theta)} d\theta$$

$$\approx \frac{1}{N} \sum_{k=0}^N \frac{\cos (km\pi/N)}{A_k}$$

are the solution of the first of the equations (19). These terms α_m depend only on the geometrical parameters L , s and ρ , and do not need to be recalculated for each change in the charge position z_Q .

REFERENCES

- [1] H. Buchholz, Elektrische und magnetische Potentialfelder (Springer, Berlin, 1957).
- [2] W.R. Smythe, Static and dynamic electricity (McGraw-Hill, New York, 1968).
- [3] E.W. McDaniel and E.A. Mason, The mobility and diffusion of ions in gases (Wiley, New York, 1973), sections 5-2 and 7-1.
- [4] M. Abramowitz and I.A. Stegun (eds.), Handbook of mathematical functions (National Bureau of Standards, Washington, 1966), section 16.27.
- [5] P.M. Morse and H. Feshbach, Methods of theoretical physics, Vol. 2 (McGraw-Hill, New York, 1953), section 10.2.
- [6] T. Tomitani, Nuclear Instr. and Meth. 100 (1972) 179.
- [7] R.W. Hamming, Numerical methods for scientists and engineers (McGraw-Hill, New York, 1962), section 21.5.

Figure captions

- Fig. 1 : Coordinate system.
- Fig. 2 : Drift envelopes for ions leaving $z_0 = 0$ at time $t = 0$, with field lines plotted at 10° intervals.
- Fig. 3 : Coefficient $2\beta_1$ in approximation (20) for right-left wire difference.
- Fig. 4 : Term Q_S in integral (17) for averaged strip charge when $t > t_L$.
- Fig. 5 : Induced charges on the three central wires (angle α as parameter).
- Fig. 6 : Induced currents on the three central wires (angle α as parameter).
- Fig. 7 : Right-left difference in induced charge and current on the central wires (angle α as parameter).
- Fig. 8 : Induced charge and current on the cathode strip $[-s,s]$ (angle α as parameter).
- Fig. 9 : Induced charge and current on the cathode strip $[s,3s]$ (angle α as parameter).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

δ -function angular distribution

Gaussian angular distribution with $\sigma = 30^\circ$

Fig. 7

Fig. 8

δ -function angular distribution

Gaussian angular distribution with $\sigma = 30^\circ$

Fig. 9