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HEAVY QUARK-ANTIQUARK SYSTEMS FROM EXPONENTIAL MOMENTS IN QCD

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A B S T R A C T

We present a detailed analysis for heavy $Q\bar{Q}$ systems, of how they emerge from the moment procedure of Shifman, Vainshtein and Zakharov. We work with exponential moments which we calculate as limits of power moments presented by Reinders, Rubinstein and Yazaki. Application to charmonium reproduces the results of these authors very well. We are able to treat bottonium states too, and predict the centre-of-mass of the p states at 9.80 GeV with a bottom on-shell quark mass of $\bar{m}_b = 4.71$ GeV. Finally, we show that non-relativistic approximations to the moments, which provide extremely simple formulae, yield results very close to the relativistic moments, for both s and p waves.

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1. - INTRODUCTION

The moment procedure of Shifman, Vainshtein and Zakharov¹⁾ (SVZ hereafter) is based essentially on the good old duality²⁾⁻⁵⁾ in e^+e^- annihilation into hadrons. There the cross-section of resonances can be calculated when averaged over energy, which corresponds to concentration on small times, by asymptotically free QCD perturbation theory. SVZ, however, go further. They add some small non-perturbative contribution to allow for larger times. What they have found is that one can allow for times small enough to keep this modified perturbation valid, while, on the other hand, large enough to permit determination of the properties of a single resonance.

The starting point is the vacuum expectation of two heavy quark currents which will be expanded via the operator product expansion. Then its Fourier transform, the polarization tensor in the true QCD vacuum, picks up a term proportional to $G_{\mu\nu}^a G_{\mu\nu}^a$, the slow gluonic fluctuations, in addition to the pure perturbative contributions. For the vector case, for instance

$$i \int d^4x e^{iqx} \langle \Omega | T j_\mu(x) j_\nu(0) | \Omega \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \quad (1)$$

with

$$\Pi(q^2) = C_{\text{pert.}}(q^2) + C_G(q^2) \langle \Omega | G_{\mu\nu}^a G_{\mu\nu}^a | \Omega \rangle .$$

Whereas the functions $C_{\text{pert.}}(q^2)$ and $C_G(q^2)$ can be calculated perturbatively in QCD, the vacuum expectation of GG cannot. It represents the confinement mechanism and will be important in determining the properties of the bound states.

Using a dispersion relation for the polarization function

$$\Pi(Q^2) = \frac{1}{\pi} \int \frac{\text{Im} \Pi(s)}{s + Q^2} ds, \quad Q^2 = -q^2 \quad (2)$$

the resonances in the physical $\text{Im} \Pi(s)$ will be related to the QCD calculation (1), in this way they are determined by the vacuum structure of QCD.

The procedure is now to apply derivatives on (2) which define the power moments

$$\mathcal{M}_n(Q^2) = \frac{1}{\pi} \int \frac{\text{Im} \Pi(s)}{(s + Q^2)^{n+1}} ds . \quad (3)$$

The increase in n emphasizes the low energy region, probes the large distances. Moments (3) at arbitrary Q^2 have been calculated for various currents by Reinders, Rubinstein and Yazaki⁶⁾⁻⁸⁾ (RRY hereafter). Their application to charmonium produced beautiful results.

We are interested in a certain limit of (3), $n, Q^2 \rightarrow \infty$ with n/Q^2 fixed, which defines the exponential moments. Whereas other authors^{1),9),10)} apply exponential moments only to light quark systems, Bell and myself¹¹⁾ also use them for heavy quark states. We see an advantage in the exponential procedure for reasons we shall explain below.

We present in Section 2 the exponential version of the moment procedure, how to derive the formulae from the QCD calculations of RRY, and how to treat ratios of moments which finally yield the masses of the ground states. The formulae for all s and p states are collected in the Appendix.

In Section 3 we explain why we can use off- or on-shell quark mass definitions, and why the latter is advantageous for us. The running coupling constant α_s will be defined in Section 4.

Next we discuss in detail the application to charmonium systems (Section 5). We reproduce nicely the results of RRY, except for the 3P_0 state which lies about 60 MeV too high. In a recent letter¹²⁾ we have briefly reported on that.

The exponential procedure also allows us to treat bottomonium systems (see Section 6). We predict the centre-of-mass (c.o.m.) of the p states at 9.80-9.81 GeV with a value of the on-shell bottom quark mass at $\bar{m} = 4.71$ GeV.

Finally we show in Section 7 that for c.o.m. positions it is sufficient to use the non-relativistic approximations to the moments; then all formulae become extremely simple and easy to handle.

2. - EXPONENTIAL MOMENTS IN QCD

We prefer to work with exponential moments

$$\mathcal{M}(\sigma) = \int e^{-\sigma s} \operatorname{Im} \bar{\Pi}(s) ds . \quad (4)$$

They have, in a non-relativistic limit, the advantage of being directly related to an imaginary time Green function. Its perturbation, calculated within potential models, has always been closer to the exact result than the corresponding power perturbation¹³⁾. Furthermore the exponential weight cuts off large s contributions in $\text{Im}\pi(s)$ more sharply than a power weight; at relevant σ we just pick up the ground-state contribution.

We obtain the mass of a ground state from the logarithmic derivative of $\mathcal{M}(\sigma)$, a ratio of moments

$$R(\sigma) = -\frac{d}{d\sigma} \log \mathcal{M}(\sigma) = \frac{-\mathcal{M}'(\sigma)}{\mathcal{M}(\sigma)} . \quad (5)$$

Given the exact (or experimental) value, $R(\sigma)$ approaches for $\sigma \rightarrow \infty$ just the squared mass of the ground state

$$R(\sigma) \xrightarrow{\sigma \rightarrow \infty} M_{\text{gr.st.}}^2 . \quad (6)$$

On the other hand, the theoretical ratio, which we calculate in a first order approximation, has a minimum at finite σ and we regard this minimum as an approximation to the ground state

$$\min_{\sigma} R(\sigma) = M_{\text{gr.st.}}^2 . \quad (7)$$

We get the exponential moments by a limiting process

$$\lim_{\substack{n \rightarrow \infty \\ Q^2 \rightarrow \infty \\ n/Q^2 = \sigma \text{ fixed}}} (Q^2)^{n+1} \pi \mathcal{M}_n(Q^2) = \mathcal{M}(\sigma) \quad (8)$$

from the power moments (3).

RRY have calculated these power moments for various currents by using QCD in a first approximation in α_s and $\langle GG \rangle$. They present formulae for the pseudoscalar 1S_0 and vector 3S_1 , for the scalar 3P_0 , axial vector 3P_1 and 1P_1 , and for the tensor current 3P_2 . We are going to rely on their results listed in Table 1 of Ref. 8).

For reasons of convenience, SVZ and RRY separate perturbative and non-perturbative contributions in the following way:

$$\mathcal{M}_n(Q^2) = A_n(Q^2) \left[1 + \alpha_s a_n(Q^2) + \phi b_n(Q^2) \right], \quad (9)$$

where A_n is the free quark contribution, $\alpha_s a_n$ corresponds to the first perturbative gluon correction, ϕb_n represents the non-perturbative correction. ϕ is called the gluon condensate parameter and is related to $\langle GG \rangle$ by

$$\phi = \phi_1 / (4m^2)^2 \quad (10)$$

and

$$\phi_1 = \frac{4\pi^2}{9} \left\langle \frac{\alpha_s}{\pi} GG \right\rangle. \quad (11)$$

We want to emphasize that it is ϕ_1 which is the flavour (mass) independent quantity¹⁾.

Analogous to (9) we also split the exponential moments into pieces

$$\mathcal{M}(\sigma) = e^{-4m^2\sigma} \pi A(\sigma) \left[1 + \alpha_s a(\sigma) + \frac{\phi_1}{(4m^2)^2} b(\sigma) \right], \quad (12)$$

and calculate the free quark term $\pi A(\sigma)$, the perturbative $\alpha_s a(\sigma)$ and non-perturbative $\phi b(\sigma)$ corrections by applying the limit (8) in the corresponding formulae of RRY.

To evaluate the masses we use ratio (5) and perturb this ratio rather than the moments themselves

$$R(\sigma) = 4m^2 - \frac{d}{d\sigma} \left[\log \pi A(\sigma) + \alpha_s a(\sigma) + \frac{\phi_1}{(4m^2)^2} b(\sigma) \right]. \quad (13)$$

Defining a dimensionless variable

$$\omega = 4m^2 \sigma \quad (14)$$

we want to write finally the ratio in the following form

$$R(\sigma) \Big|_{\sigma = \frac{\omega}{4m^2}} = 4m^2 F(\omega) \left[1 + \alpha_s P(\omega) + \frac{\phi_1}{(4m^2)^2} Q(\omega) \right], \quad (15)$$

where

$$F(\omega) = 1 - \frac{\pi A'(\omega)}{\pi A(\omega)} \quad (16)$$

is the free part, and

$$P(\omega) = - \frac{a'(\omega)}{F(\omega)} \quad (17)$$

$$Q(\omega) = - \frac{b'(\omega)}{F(\omega)} \quad (18)$$

correspond to perturbative and non-perturbative corrections, respectively. F , P and Q are dimensionless functions of the dimensionless variable ω ; the mass dependence of the ratio is shown explicitly in (15).

Bell and the author¹¹⁾ have performed this exponential procedure already for the 3S_1 current, now we calculate the formulae for all the other currents and list them in the Appendix. We remark that for all s and p waves, the limit (8) exists and can be calculated analytically in terms of Whittaker functions^{*)}.

3. - MASS RENORMALIZATION

In the formulae for $\alpha_s a(\omega)$ and $-\alpha_s a'(\omega)$ (see Appendix) there occurs a term

$$\alpha_s \frac{4 \ln 2}{\pi} 4m^2 f(\omega) \quad , \quad (19)$$

where

$$m = m(p^2 = -m^2) \quad (20)$$

is the Euclidean off-shell mass.

This term is a mass renormalization term originally introduced by SVZ in order to reduce the gluon corrections in the power moments (9). In our perturbed ratio (15) this term, however, does not lower the corrections. It just adds to the quark-mass term $4m^2$ in (15), effectively replacing m by the on-shell mass

*) RRY¹⁴⁾ always calculate with a dimensionless $\text{Im}\pi(s)$ in moments (3), and not with formulae as they appear in their paper⁸⁾.

$$\bar{m} = m(p^2 = m^2) , \quad (21)$$

which is related to (20) by

$$\bar{m}^2 = m^2 \left(1 + \alpha_s \frac{4 \ln 2}{\pi} \right) . \quad (22)$$

This enables us to work with an on-shell quark mass (21) by replacing m^2 by $\bar{m}^2$ [including the coefficient of $Q(\omega)$ in (15); although this is a distinction of higher order in $\alpha_s \phi_1$] and dropping the terms (19) in all the formulae.

4. - DEFINITION OF THE RUNNING COUPLING CONSTANT

We define the running coupling constant in power moments (9) like *)

$$\alpha_s = \alpha_s \left(\frac{Q^2 + 4m^2}{n} \right) . \quad (23)$$

The reason is simply that (23) is meaningful in the limit (8)

$$\alpha_s \left(\frac{Q^2 + 4m^2}{n} \right) \xrightarrow[n \rightarrow \infty]{Q^2 \rightarrow \infty, n/Q^2 = \sigma \text{ fixed}} \alpha_s \left(\frac{1}{\sigma} \right) , \quad (24)$$

and corresponds to what SVZ¹⁾ [see also Refs. 10), 15)] use in their Borel transformations. Therefore α_s in moments (12) and ratios (15) is given by the right-hand side of (24). Furthermore, we ignore a variation of α_s with respect to σ , we take its value at the minimum of (15), the relevant σ region.

The quantity $1/\sigma_{\min}$ should be related to some level splitting rather than to the particle mass itself¹³⁾. Indeed for charmonium the minimum in the ratio (15) occurs at

$$\frac{1}{\sigma_{\min}} \approx 1.11 \text{ GeV}^2 , \quad (25)$$

and definitely not at

*) RRY⁸⁾ use $\alpha_s = \alpha_s(Q^2 + 4m^2)$ in (9); we do not understand why.

$$\frac{1}{\sigma} = M_{J/\psi}^2 = 9.61 \text{ GeV}^2. \quad (26)$$

Then we deduce the numerical value of the coupling constant from the standard QCD formula

$$\alpha_s(Q^2) = \frac{12\pi}{b_n \ln \frac{Q^2}{\Lambda^2}} \quad (27)$$

with

$$b_n = 33 - 2n_f \quad \text{and} \quad \Lambda \approx 0.1 \text{ GeV}. \quad (28)$$

We obtain

$$\alpha_s\left(\frac{1}{\sigma_{\min}}\right) = 0.30 \quad (29)$$

which is about a factor 1.35 larger than the value at J/ψ

$$\alpha_s(M_{J/\psi}^2) = 0.22. \quad (30)$$

Analogously, for bottonium $\mathcal{R}(\sigma)$ minimizes at

$$\frac{1}{\sigma_{\min}} \approx 2.63 \text{ GeV}^2. \quad (31)$$

Again, (31) is far from the T mass squared

$$\frac{1}{\sigma} = M_T^2 = 89.49 \text{ GeV}^2. \quad (32)$$

Therefore, the coupling constant in $\mathcal{R}(\sigma)$ will be

$$\alpha_s\left(\frac{1}{\sigma_{\min}}\right) = 0.25 \quad (33)$$

and not

$$\alpha_s(M_T^2) = 0.18. \quad (34)$$

5. - THE CHARMONIUM SYSTEM

The charmonium spectrum is indeed the best think to look at, for both experimental and theoretical reasons. Since we finally want to compare our results with those of RRY, it is important to point out common and different features of both versions.

RRY perturb the moments (9) and calculate the ratio

$$r_n(Q^2) = \frac{\mathcal{M}_n(Q^2)}{\mathcal{M}_{n-1}(Q^2)} \xrightarrow{n \rightarrow \infty} \frac{1}{M_{gr.st.}^2} \quad (35)$$

to obtain the mass of the ground state. At low n the whole spectrum contributes and the corrections are small in (9). By increasing n the ground state dominates but also the corrections become very large. They can be reduced again by increasing Q^2 , which is still free to be chosen. In this way they achieve a stability plateau in ratio (35) at n high enough for the ground state to dominate but still small enough to keep the corrections small. Now an important difference to our version is that RRY have chosen Q^2 quite differently for s and p waves -- $Q^2 = 4m^2$ for s and $Q^2 = 2.5 \cdot 4 m^2$ for p waves -- which implies with their philosophy a significant change in the running mass m_c and coupling constants α_s .

In our approach, however, by using exponential moments as a limit of power moments, the Q^2 dependence disappears, the quark mass and the coupling constant remain fixed in moving from s to p states. Furthermore, we perturb the ratio (15) since we consider it as the more stable quantity and demonstrate below, while $\alpha_s a(\omega)$ and $(\phi_1/(4m^2)^2) \cdot b(\omega)$ blow up, their derivatives stay small. Ratio (15) allows us to work with an on-shell mass definition (21) and (22) instead of the off-shell mass (20) preferred by previous authors. We calculate ratio (15) with both mass versions (20) and (21), and notice that the results for all currents differ only by a few MeV!

The parameters in the moments are very few, just the quark mass m or $\bar{m}$, the coupling constant α_s , and the gluon condensate ϕ_1 . If we use values which previous authors obtained from a power moment analysis, the 3S_1 states always fall below $J/\psi(3.10)$. Assuming that m and α_s have been determined correctly from low moments, we have to increase ϕ_1 to reach the correct $^3S_1(3.10)$. The amount of this increase is somewhat dependent on the

size of m and α_s . For parameter values of SVZ¹⁾ we have to increase ϕ_1 by a factor of about 2.5, for parameter values of RRY⁸⁾ of about 1.4^{*)}.

Within potential models, Bell and myself¹³⁾ observed that the perturbed ratio for exponential moments approximated the exact results better than for power moments. Furthermore, a perturbed exponential ratio like (15) [power ratio like (35)] was always bigger (smaller) than the exact answer. Therefore, from adjusting ϕ_1 to the exact mass, as it is done in the moment procedure, we have conjectured that the value of ϕ_1 has been underestimated. This is also what happens in QCD₂ model calculations¹⁶⁾. We finally choose the following parameter set:

$$\begin{aligned}\bar{m}_c &= 1.44 \text{ GeV}, \text{ or } m = 1.28 \text{ GeV} \\ \alpha_s &= 0.30 \\ \phi_1 &= 0.07 \text{ GeV}^4, \end{aligned} \quad (36)$$

and illustrate the behaviour of the ratios (15) for all currents -- 3S_1 , 1S_0 , 3P_0 , 3P_1 , 3P_2 , 1P_1 -- on Figs. 1-6.

In the vector case, 3S_1 (Fig. 1), where the experimental spectrum is known quite accurately, we also plot the experimental ratio (5). There we approximate the resonances¹⁷⁾ by δ functions and the continuum just by a step

$$\begin{aligned}c \mathcal{M}^{\text{exper.}}(\sigma) &= \frac{9\pi}{\alpha^2} \sum_V \Gamma_{ee}^V M_V e^{-\sigma M_V^2} + \\ &+ 3e_q^2 \left(1 + \frac{\alpha_s}{\pi}\right) \sigma^{-1} e^{-\sigma M_{\text{thresh.}}^2}, \end{aligned} \quad (37)$$

and

$$\begin{aligned}-c \mathcal{M}'^{\text{exper.}}(\sigma) &= \frac{9\pi}{\alpha^2} \sum_V \Gamma_{ee}^V M_V^3 e^{-\sigma M_V^2} + \\ &+ 3e_q^2 \left(1 + \frac{\alpha_s}{\pi}\right) (\sigma M_{\text{thresh.}}^2 + 1) \sigma^{-2} e^{-\sigma M_{\text{thresh.}}^2}, \end{aligned} \quad (38)$$

*) This differs from the value of 2.3 which Bell and myself claimed in a previous publication¹¹⁾. The reason is that we previously relied on RRY⁷⁾, used $m_c = 1.26 \text{ GeV}$ and overlooked the fact that this is its value at $Q^2 = 4m^2$. But now from RRY⁸⁾ it became clear to us that for the Euclidean mass definition (20) the value has to be $m_c = 1.28 \text{ GeV}$. This causes an appreciable change in ϕ_1 .

where

$$C = 4\pi 3e_q^2 \quad . \quad (39)$$

We also show the pure perturbative curve: it deviates from experiment for increasing σ . It is the non-perturbative term which narrows the gap between theory and experiment in the allowed region $\sigma \leq \sigma_{\min}$. For $\sigma > \sigma_{\min}$ the whole approximation breaks down. At the rather broad minimum J/ψ dominates the experimental ratio by 99%! We expect a similar behaviour in all other currents, therefore we are confident to predict only the properties of a single ground state.

Numerically we summarize all level positions in Table 1. We agree very nicely with the results of RRY, however, our p states are equally spaced, or alternatively 3P_0 lies about 60 MeV too high. Choosing parameter values of SVZ¹⁾ and Miller-Olsson¹⁸⁾ the s wave splitting is perfect; however, all p states emerge about 150 MeV too high.

In closing this section we want to show how we can trust that the perturbations are small. We demonstrate the rôle that the corrections play in (12) and (15). In Table 2, we present two typical examples of s and p states and compare off-shell -- and on-shell quark mass calculations. The corrections in the ratio (15) always stay acceptably small, less than 18% for both versions, while the corrections in the moments (12) become too large.

6. - THE BOTTONIUM SYSTEM

Bottonium systems are already quite different from charmonium. Since the bottom quark is much heavier than the charm, the $b\bar{b}$ system is concentrated on a shorter range, the Coulomb force is enhanced and confinement is felt only weakly. In moments (12) or ratios (15), the non-perturbative terms are suppressed by the mass term $1/m^4$. Since

$$\left(\frac{m_c}{m_b}\right)^4 \approx 0.01 \quad (40)$$

the pure perturbative terms dominate. But it happens that at large ω the non-perturbative terms regain their importance and provide in ratio (15) a balance à la SVZ.

Coupling constants α_s and gluon condensate ϕ_1 are given by (33) and (36); we only adjust the bottom quark mass

$$\bar{m}_b = 4.71 \text{ GeV} \quad (41)$$

or

$$m_b = 4.26 \text{ GeV} \quad (42)$$

by the T state. Again, we obtain for all currents nice minima in (15), and plot 3S_1 and 3P_1 for illustration on Figs. 7 and 8. In the case of 3S_1 we compare theoretical ratios -- we also show the pure perturbative curve -- with the experimental one. At the minimum of (15) T dominates the experimental ratio by 99.8%!

The masses of s and p states are listed in Table 3. We observe that the splitting within a particular wave is maximal at 10 MeV. We are more confident in the centre-of-mass of the states

$$M_{s\text{-wave}}^{\text{com}} = 9.455 \text{ GeV} \quad (43)$$

$$M_{p\text{-wave}}^{\text{com}} = 9.803 \text{ GeV} \quad (44)$$

than in the fine splittings. Note that the mass of the p wave (44) is close to a prediction of Ref. 19); it is certainly less (about 60-100 MeV) than potential models usually predict.

Let us finally show the size of the corrections for two typical examples 3S_1 and 3P_1 in Table 4. We observe that for both quark mass versions the perturbative corrections in the moments (12) are certainly too large. But the corrections in the ratio (15), especially for the on-shell quark mass definition (21), stay reasonably small.

7. - THE LARGE σ LIMIT - THE NON-RELATIVISTIC APPROXIMATION

Large σ (or ω) expansions provide already a good estimate for ratio (15) ¹¹⁾. Let us illustrate it in more detail.

Looking at moments (4) we see that large σ (or ω) values weigh the low s region. There a non-relativistic approximation will be relevant. Expanding the moments in σ , retaining in each term only the leading powers, then the formulae for moments (12) appear extremely simple (see Appendix).

For s waves, we obtain

$$\pi A(\sigma) \Big|_{\sigma = \frac{\omega}{4\bar{m}^2}} = \frac{3}{16\sqrt{\pi}} 4\bar{m}^2 \omega^{-\frac{3}{2}} \quad (45)$$

$$a(\sigma) \Big|_{\sigma = \frac{\omega}{4\bar{m}^2}} = \frac{4\sqrt{\pi}}{3} \omega^{\frac{1}{2}} - c_2 \quad (46)$$

$$b(\sigma) \Big|_{\sigma = \frac{\omega}{4\bar{m}^2}} = -\frac{\omega^3}{2} , \quad (47)$$

with

$$c_2 = \frac{\pi}{2} - \frac{3}{4\pi} . \quad (48)$$

Now, non-relativistically

$$s = (2\bar{m} + E)^2 , \quad (49)$$

so that (neglecting terms proportional to E^2)

$$e^{4\bar{m}^2\sigma} \int e^{-\sigma s} \text{Im} \pi(s) ds \xrightarrow{\sigma \text{ large}} 4\bar{m} \int e^{-\tau E} \text{Im} \pi(E) dE \quad (50)$$

with

$$4\bar{m}\sigma = \tau . \quad (51)$$

The right-hand side of (50) is called a non-relativistic moment

$$M(\tau) = \int e^{-\tau E} \text{Im} \pi(E) dE , \quad (52)$$

and we define a non-relativistic ratio by

$$R(\tau) = -\frac{d}{d\tau} \log M(\tau) . \quad (53)$$

Then we achieve correspondence between large σ , or ω , and non-relativistic expressions by setting

$$\omega \longrightarrow \bar{m} \tau \quad [\text{recall (14) and (51)}] \quad (54)$$

$$e^{4\bar{m}^2\sigma} \mathcal{U}(\sigma) \longrightarrow 4\bar{m} M(\tau) \quad [\text{from (50)}] \quad (55)$$

$$\frac{1}{4\bar{m}} (R(\sigma) - 4\bar{m}^2) \longrightarrow R(\tau) \quad . \quad (56)$$

This implies a non-relativistic ratio

$$R(\tau) = \frac{3}{2\tau} - \alpha_s \frac{2\sqrt{\pi}}{3} \left(\frac{\tau}{\bar{m}}\right)^{-\frac{1}{2}} + \phi_1 \frac{3}{32} \bar{m} \left(\frac{\tau}{\bar{m}}\right)^2 \quad (57)$$

or, presented in the form of (15)

$$Q(\tau) \Big|_{\tau = \frac{\rho}{2\bar{m}}} = 2\bar{m} + R(\tau) \Big|_{\tau = \frac{\rho}{2\bar{m}}} \quad (58)$$

$$= 2\bar{m} F(\rho) \left[1 + \alpha_s P(\rho) + \frac{\phi_1}{(4\bar{m}^2)^2} Q(\rho) \right] , \quad (59)$$

with

$$F(\rho) = 1 + \frac{3}{2\rho} \quad (60)$$

$$P(\rho) = -\frac{\sqrt{2\pi}}{3} \rho^{-\frac{1}{2}} / F \quad (61)$$

$$Q(\rho) = \frac{3}{16} \rho^2 / F \quad (62)$$

and

$$\rho = 2\bar{m}\tau \quad . \quad (63)$$

The ground state mass is determined by the minimum of (59)

$$M_{gr.st.} = \min_{\tau} Q(\tau) \quad (64)$$

$$\approx \min_{\sigma} \sqrt{R(\sigma)} , \quad (65)$$

and corresponds to the expansion of the square root of $R(\sigma)$. Equations (57) or (59) clearly represent spin averaged waves. Note that the "equivalent" potential which simulates the term proportional to ϕ_1 is r^4 and proportional to the quark mass¹¹⁾.

Now, we apply (59) to charm, we use parameter values (36), and obtain

$$M_{s\text{-wave}} = 3.088 \text{ GeV}. \quad (66)$$

When we compare (66) with the spin averaged relativistic calculation

$$M_{s\text{-wave}}^{\text{c.a.m.}} = \frac{3 M(1^-) + M(0^-)}{4} \quad (67)$$

$$= 3.080 \text{ GeV} , \quad (68)$$

we notice that the difference is only 8 MeV!

Let us now turn to p waves. In the large σ (or ω) limit we infer from the formulae of the Appendix

$$\pi A(\sigma) \Big|_{\sigma = \frac{\omega}{4\bar{m}^2}} = \frac{3}{2} \cdot \frac{3}{16\sqrt{\pi}} 4\bar{m}^2 \omega^{-\frac{5}{2}} \quad (69)$$

$$a(\sigma) \Big|_{\sigma = \frac{\omega}{4\bar{m}^2}} = \frac{2}{3} \left(\frac{4\sqrt{\pi}}{3} \omega^{\frac{1}{2}} - c_2 \right) \quad (70)$$

$$b(\sigma) \Big|_{\sigma = \frac{\omega}{4\bar{m}^2}} = -\frac{3\omega^3}{2} . \quad (71)$$

It implies a non-relativistic ratio

$$R(\tau) = \frac{5}{2\tau} - \alpha_s \frac{4\sqrt{\pi}}{9} \left(\frac{\tau}{\bar{m}} \right)^{-\frac{1}{2}} + \phi \frac{9}{32} \bar{m} \left(\frac{\tau}{\bar{m}} \right)^2 \quad (72)$$

and, in the presentation (59), the free and correction terms are given by

$$F(\rho) = 1 + \frac{5}{2\rho} \quad (73)$$

$$P(\rho) = -\frac{2\sqrt{2}\pi}{9} \rho^{-\frac{1}{2}} / F \quad (74)$$

$$Q(\rho) = \frac{9}{16} \rho^2 / F . \quad (75)$$

Finding the minimum of ratio (72) for charm with parameter values (36) provides the mass of the mean p waves

$$M_{p\text{-wave}} = 3.556 \text{ GeV}. \quad (76)$$

Again, we compare it with the spin averaged relativistic calculation

$$M_{p\text{-wave}}^{\text{c.a.m.}} = \frac{5 M(2^+) + 3 M(1^+) + M(0^+)}{9} \quad (77)$$

$$= 3.549 \text{ GeV} , \quad (78)$$

and find the difference is only 7 MeV!

Finally, performing the analogous calculations for the bottonium system, the result

$$M_{s\text{-wave}} = 9.455 \text{ GeV} \quad (79)$$

$$M_{p\text{-wave}} = 9.804 \text{ GeV} \quad (80)$$

just coincides with the relativistic calculations (43) and (44).

For illustration, we have plotted relativistic (15) and non-relativistic (59) ratios on Figs. 9-12. Already for charmonium, the relativistic ratio for both s and p waves becomes quickly asymptotic in σ near the minimum, and for bottom (15) and (59), it practically coincides. Being interested just in the centre-of-mass, it is quite sufficient to use the non-relativistic approximations where the formulae (57) and (72) are extremely simple and easy to handle.

8. - CONCLUDING REMARKS

We have investigated heavy quark systems by using exponential moments. They have been calculated via a limit from the corresponding formulae of RRY⁸⁾. Comparing exponential moments with power moments, our parameters are fewer but more restrictive. We reproduce the charmonium spectrum very well, but our p states are equally spaced. RRY, however, in varying Q^2 , obtain a perfect splitting, especially for the p states.

We also treat bottonium systems where we are more confident in the centre-of-mass positions of the states. Finally, we investigate non-relativistic approximations which are simple and close to relativistic results.

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α_s $\overline{m}_c$ $\overline{m}_c$ ϕ_1	0.30 1.28 1.44 0.06 0.07	0.27 1.28 1.42 0.06 0.08	0.2 1.26 1.37 0.05 0.14	0.2 1.24 1.34 0.13 0.19	RRY ⁸⁾ 3.10 $\pm$ 0.01	Experi- ment ¹⁷⁾
3S_1	3.09 3.10	3.07 3.10	2.99 3.10	3.04 3.10	3.10 $\pm$ 0.01	3.10
1S_0	3.03 3.04	3.02 3.03	2.94 3.00	2.95 2.98	3.01 $\pm$ 0.02	2.98
3P_0	3.46 3.48	3.44 3.49	3.35 3.58	3.51 3.63	3.40 $\pm$ 0.01	3.41
3P_1	3.49 3.52	3.47 3.53	3.38 3.63	3.55 3.68	3.50 $\pm$ 0.01	3.51
3P_2	3.53 3.56	3.51 3.57	3.41 3.68	3.61 3.75	3.56 $\pm$ 0.01	3.55
1P_1	3.48 3.51	3.46 3.52	3.37 3.61	3.53 3.65	3.51 $\pm$ 0.01	

Table 1 - Charmonium results from the exponential moment ratio $\mathcal{R}(\sigma)$, Eq. (15), for several parameter sets. In the first column we list results for parameter values (36), in the second we change α_s to 0.27. Both columns correspond to parameters advocated by RRY⁸⁾. In the third and fourth columns, we collect results for parameters used by SVZ¹⁾ and Miller-Olsson¹⁸⁾. The first value of ϕ_1 is determined by the original authors, the second by adjusting the minimum of the ratio $\mathcal{R}(\sigma)$, Eq. (17), to the mass of J/ψ (3.10).

$2s+1 L_J$	α_s^a	$\frac{\phi_1}{(4m^2)^2} b$	α_s^P	$\frac{\phi_1}{(4m^2)^2} Q$
3S_1	-0.127	-0.178	0.107	0.071
	1.633	-0.181	-0.109	0.063
3S_1	-0.306	-0.490	0.126	0.177
	1.312	-0.454	-0.069	0.148

Table 2 - We compare the size of the corrections in the moments (12) with those in the ratio (15) for two typical examples in charmonium: 3S_1 and 3P_1 . The first line corresponds to off-shell, the second to on-shell quark mass calculations at $\sigma_{\min}$.

$2s+1 L_J$	3S_1	1S_0	3P_0	3P_1	3P_2	1P_1
M(GeV)	9.46	9.45	9.79	9.80	9.81	9.80

Table 3 - The spectrum of bottomium states as given by ratio (15).

$2s+1 L_J$	α_s^a	$\frac{\phi_1}{(4m^2)^2} b$	α_a^P	$\frac{\phi_1}{(4m^2)^2} Q$
3S_1	-3.109	-0.121	0.156	0.013
	2.984	-0.123	-0.050	0.011
3P_1	-3.704	-0.343	0.167	0.036
	2.188	-0.341	-0.033	0.032

Table 4 - We compare the size of the corrections in the moments (12) with those in the ratio (15) for two typical examples in bottomium: 3S_1 and 3P_1 . The first line corresponds to off-shell, the second to on-shell quark mass calculations at $\sigma_{\min}$.

APPENDIX

RRY list their formulae for moments (9) in Table 1 of Ref. 8). The free term $A_n(Q^2)$, the perturbative $a_n(Q^2)$ and non-perturbative $b_n(Q^2)$ corrections consist of some n factorials, or gamma functions, of hypergeometric functions and some Q^2 dependence. We are interested in the limit

$$\begin{aligned} n &\longrightarrow \infty \\ Q^2 &\longrightarrow \infty \end{aligned} \quad (A.1)$$

with

$$n/Q^2 = \sigma \text{ fixed} .$$

Then their variables change to

$$1 + \xi = 1 + \frac{Q^2}{4m^2} = 1 + \frac{n}{\omega} \quad (A.2)$$

and

$$\rho = \frac{\xi}{1+\xi} = 1 - \frac{\omega}{n} \quad (A.3)$$

with

$$\omega = 4m^2 \sigma . \quad (A.4)$$

Using asymptotics for gamma functions

$$\frac{\Gamma(n+c)}{\Gamma(n)} \underset{n \rightarrow \infty}{=} n^c \quad (A.5)$$

the hypergeometric functions

$$F(n, b, n+c; \rho) = \frac{\Gamma(n+c)}{\Gamma(n) \Gamma(c)} \int_0^1 (1-t)^{n-1} t^{c-1} (1-\rho+te)^{-b} dt \quad (A.6)$$

transform into Whittaker functions

$$F(n, b, n+c; \rho) \underset{n \rightarrow \infty}{=} n^b G(b, c, \omega) \quad (A.7)$$

where

$$G(b, c, \omega) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-t} t^{c-1} (\omega+t)^{-b} dt . \quad (A.8)$$

These have a nice derivative - and asymptotic properties

$$-\frac{d}{d\omega} G(b, c, \omega) = b G(b+1, c, \omega) \quad (A.9)$$

$$G(b, c, \omega) \underset{\omega \rightarrow \infty}{=} \omega^{-b} \quad (A.10)$$

Using these formulae, moments (12) and ratios (15) are given by the following functions ^{*)}.

Vector 3S_1

$$\pi A(\omega) = \frac{3}{16\sqrt{\pi}} \frac{4m^2}{\omega} G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (A.11)$$

$$a(\omega) = \frac{4}{3\sqrt{\pi}} G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left[\pi - c_1 G(1, 2, \omega) + \frac{c_2}{3} G(2, 3, \omega) \right] - c_2 - \frac{4 \ln 2}{\pi} h(\omega) \quad (A.12)$$

where

$$c_2 = \frac{\pi}{2} - \frac{3}{4\pi} \quad (A.13)$$

$$c_1 = \frac{\pi}{3} + \frac{c_2}{2} \quad (A.14)$$

$$h(\omega) = \omega G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (A.15)$$

$$b(\omega) = -\frac{\omega^2}{2} G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (A.16)$$

$$\frac{-\pi A'(\omega)}{\pi A(\omega)} = \frac{1}{\omega} + \frac{1}{2} G\left(\frac{3}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \quad (A.17)$$

^{*)} If the reader replaces, in the formulae of Table 1 of RRY⁸⁾

$$\frac{2+\xi}{1+\xi} \ln(2+\xi) \longrightarrow 2 \ln 2$$

according to RRY's Eq. (6.3), she/he will obtain a quark mass definition (20) in the free contributions which remains in limit (8).

$$\begin{aligned}
 -\alpha'(\omega) = & \frac{4}{3\sqrt{\pi}} G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left\{ -c_1 G(2, 2, \omega) + \frac{2c_2}{3} G(3, 3, \omega) \right. \\
 & \left. - \frac{1}{2} G\left(\frac{3}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left[\pi - c_1 G(1, 2, \omega) + \frac{c_2}{3} G(2, 3, \omega) \right] \right\} \\
 & + \frac{4\ell n^2}{\pi} f(\omega)
 \end{aligned} \tag{A.18}$$

with

$$\begin{aligned}
 f(\omega) = & G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left[G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) - \frac{\omega}{2} G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) \right. \\
 & \left. + \frac{\omega}{2} G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) G\left(\frac{3}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \right]
 \end{aligned} \tag{A.19}$$

$$\begin{aligned}
 -b'(\omega) = & \omega G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \left[G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) + \frac{\omega}{4} G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \right. \\
 & \left. + \frac{\omega}{4} G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) G\left(\frac{3}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \right]
 \end{aligned} \tag{A.20}$$

Pseudoscalar 1S_0

$$\pi A(\omega) = \frac{3}{16\sqrt{\pi}} \frac{4m^2}{\omega} G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \tag{A.21}$$

$$\begin{aligned}
 a(\omega) = & \frac{4}{3\sqrt{\pi}} G^{-1}\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \left[\pi - \frac{c_2}{2} G(1, 2, \omega) \right] - c_2 \\
 & - \frac{4\ell n^2}{\pi} h(\omega) + \frac{1}{\pi} \left[\frac{8}{3} + S(\omega) \right]
 \end{aligned} \tag{A.22}$$

with

$$h(\omega) = \omega G(\frac{1}{2}, \frac{1}{2}, \omega) G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \quad (\text{A.23})$$

$$S(\omega) = -G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \left[4 G(\frac{3}{2}, \frac{3}{2}, \omega) + \frac{5}{6} G(\frac{3}{2}, \frac{5}{2}, \omega) \right] \quad (\text{A.24})$$

$$b(\omega) = -\frac{\omega}{2} G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \left[G(-\frac{3}{2}, \frac{3}{2}, \omega) - 6 G(-\frac{1}{2}, \frac{3}{2}, \omega) \right] \quad (\text{A.25})$$

$$\frac{-\pi A'(\omega)}{\pi A(\omega)} = \frac{1}{\omega} + \frac{1}{2} G(\frac{3}{2}, \frac{3}{2}, \omega) G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \quad (\text{A.26})$$

$$\begin{aligned} -\alpha'(\omega) = & \frac{2}{3\sqrt{\pi}} G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \left\{ -c_2 G(2, 2, \omega) \right. \\ & \left. - G(\frac{3}{2}, \frac{3}{2}, \omega) G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \left[\pi - \frac{c_2}{2} G(1, 2, \omega) \right] \right\} \\ & + \frac{4 \ln 2}{\pi} f(\omega) - \frac{1}{\pi} S'(\omega) \end{aligned} \quad (\text{A.27})$$

with

$$\begin{aligned} f(\omega) = & G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \left[G(\frac{1}{2}, \frac{1}{2}, \omega) - \frac{\omega}{2} G(\frac{3}{2}, \frac{1}{2}, \omega) \right. \\ & \left. + \frac{\omega}{2} G(\frac{1}{2}, \frac{1}{2}, \omega) G(\frac{3}{2}, \frac{3}{2}, \omega) G^{-1}(\frac{1}{2}, \frac{3}{2}, \omega) \right] \end{aligned} \quad (\text{A.28})$$

$$\begin{aligned}
 -\frac{1}{\pi} \mathcal{S}'(\omega) = & -\frac{1}{\pi} G^{-1}\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \left\{ 6 G\left(\frac{5}{2}, \frac{3}{2}, \omega\right) + \frac{15}{12} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) \right. \\
 & \left. - \frac{1}{2} G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \left[4 G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) + \frac{5}{6} G\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \right] \right\}
 \end{aligned}
 \tag{A.29}$$

$$\begin{aligned}
 -\mathcal{L}'(\omega) = & \frac{1}{2} G^{-1}\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \left\{ \frac{3\omega}{2} G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) - 3\omega G\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \right. \\
 & \left. + \left[1 + \frac{\omega}{2} G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{1}{2}, \frac{3}{2}, \omega\right) \right] \left[G\left(-\frac{3}{2}, \frac{3}{2}, \omega\right) - 6 G\left(-\frac{1}{2}, \frac{3}{2}, \omega\right) \right] \right\}
 \end{aligned}
 \tag{A.30}$$

Scalar 3P_0

$$\pi A(\omega) = \frac{9}{32\sqrt{\pi}} \frac{4m^2}{\omega} G\left(\frac{3}{2}, \frac{5}{2}, \omega\right)
 \tag{A.31}$$

$$\alpha(\omega) = \tilde{\alpha}(\omega) + \frac{1}{\pi} \left[\frac{16}{3} + \mathcal{S}(\omega) \right]
 \tag{A.32}$$

where

$$\begin{aligned}
 \tilde{\alpha}(\omega) = & \frac{8}{9\sqrt{\pi}} G(1, 2, \omega) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left[\pi - 2c_3 G(2, 3, \omega) G^{-1}(1, 2, \omega) \right] \\
 & - \frac{2c_3}{3} - \frac{4\ln 2}{\pi} h(\omega)
 \end{aligned}
 \tag{A.33}$$

$$c_3 = \frac{\pi}{2} - \frac{3}{\pi} \quad (\text{A.34})$$

$$h(\omega) = \omega G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \quad (\text{A.35})$$

$$\begin{aligned} S(\omega) = & -\frac{4}{3} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left[5 G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) + \frac{3}{2} G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) \right. \\ & \left. + \left(5 - \frac{1}{\omega}\right) G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right] \end{aligned} \quad (\text{A.36})$$

$$b(\omega) = -\frac{3\omega}{2} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left[G\left(-\frac{1}{2}, \frac{5}{2}, \omega\right) - \frac{2}{3} G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \right] \quad (\text{A.37})$$

$$\frac{-\pi A'(\omega)}{\pi A(\omega)} = \frac{1}{\omega} + \frac{3}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \quad (\text{A.38})$$

$$-a'(\omega) = -\tilde{\alpha}'(\omega) - \frac{1}{\pi} S'(\omega) \quad (\text{A.39})$$

with

$$\begin{aligned} -\tilde{\alpha}'(\omega) = & \frac{8}{9\sqrt{\pi}} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left\{ -2c_3 \left[2 G(3, 3, \omega) \right. \right. \\ & \left. - G(2, 2, \omega) G(2, 3, \omega) G^{-1}(1, 2, \omega) \right] + \left[\pi - 2c_3 G(2, 3, \omega) G^{-1}(1, 2, \omega) \right] \\ & \left. \cdot \left[G(2, 2, \omega) - \frac{3}{2} G(1, 2, \omega) G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \right] \right\} + \frac{4 \ln 2}{\pi} f(\omega) \end{aligned} \quad (\text{A.40})$$

$$f(\omega) = G^{-1}(\frac{3}{2}, \frac{5}{2}, \omega) \left[G(\frac{3}{2}, \frac{3}{2}, \omega) - \frac{3\omega}{2} G(\frac{5}{2}, \frac{3}{2}, \omega) \right. \\ \left. + \frac{3\omega}{2} G(\frac{3}{2}, \frac{3}{2}, \omega) G(\frac{5}{2}, \frac{5}{2}, \omega) G^{-1}(\frac{3}{2}, \frac{5}{2}, \omega) \right] \quad (A.41)$$

$$-\frac{1}{\pi} S'(\omega) = -\frac{4}{3\pi} G^{-1}(\frac{3}{2}, \frac{5}{2}, \omega) \left\{ \frac{25}{2} G(\frac{7}{2}, \frac{5}{2}, \omega) - \frac{1}{\omega^2} G(\frac{5}{2}, \frac{7}{2}, \omega) \right. \\ \left. + \frac{5}{2} (5 - \frac{1}{\omega}) G(\frac{7}{2}, \frac{7}{2}, \omega) + \frac{21}{4} G(\frac{9}{2}, \frac{9}{2}, \omega) - \frac{3}{2} G(\frac{5}{2}, \frac{5}{2}, \omega) G^{-1}(\frac{3}{2}, \frac{5}{2}, \omega) \right. \\ \left. \cdot [5 G(\frac{5}{2}, \frac{5}{2}, \omega) + (5 - \frac{1}{\omega}) G(\frac{5}{2}, \frac{7}{2}, \omega) + \frac{3}{2} G(\frac{7}{2}, \frac{9}{2}, \omega)] \right\} \quad (A.42)$$

$$-b'(\omega) = \frac{3}{2} G^{-1}(\frac{3}{2}, \frac{5}{2}, \omega) \left\{ \frac{\omega}{2} G(\frac{1}{2}, \frac{5}{2}, \omega) + \frac{\omega}{3} G(\frac{3}{2}, \frac{5}{2}, \omega) \right. \\ \left. + \left[1 + \frac{3\omega}{2} G(\frac{5}{2}, \frac{5}{2}, \omega) G^{-1}(\frac{3}{2}, \frac{5}{2}, \omega) \right] \cdot \left[G(-\frac{1}{2}, \frac{5}{2}, \omega) - \frac{2}{3} G(\frac{1}{2}, \frac{5}{2}, \omega) \right] \right\} \quad (A.43)$$

Axial vector 3P_1

$$\pi A(\omega) \Big|_{^3P_1} = \frac{2}{3} \pi A(\omega) \Big|_{^3P_0} \quad (A.44)$$

$$\alpha(\omega) = \tilde{\alpha}(\omega) + \frac{1}{\pi} \left[\frac{4}{3} + S(\omega) \right] \quad (A.45)$$

with

$$S(\omega) = -\frac{4}{3} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left[\frac{3}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) + 2 G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) \right. \\ \left. + \left(1 - \frac{3}{2\omega}\right) G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right] \quad (\text{A.46})$$

$$b(\omega) = -\frac{3\omega}{2} G\left(-\frac{1}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \quad (\text{A.47})$$

$$\frac{-\pi A'(\omega)}{\pi A(\omega)} \Big|_{3p_1} = \frac{-\pi A'(\omega)}{\pi A(\omega)} \Big|_{3p_0} \quad (\text{A.48})$$

$$-a'(\omega) = -\tilde{a}'(\omega) - \frac{1}{\pi} S'(\omega) \quad (\text{A.49})$$

with

$$-\frac{1}{\pi} S'(\omega) = -\frac{4}{3\pi} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left\{ \frac{15}{4} G\left(\frac{7}{2}, \frac{5}{2}, \omega\right) - \frac{3}{2\omega^2} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right. \\ \left. + \frac{5}{2} \left(1 - \frac{3}{2\omega}\right) G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) + 7 G\left(\frac{9}{2}, \frac{9}{2}, \omega\right) - \frac{3}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \right. \\ \left. \cdot \left[\frac{3}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) + \left(1 - \frac{3}{2\omega}\right) G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) + 2 G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) \right] \right\} \quad (\text{A.50})$$

$$-b'(\omega) = \frac{3}{2} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left\{ \frac{\omega}{2} G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) \right. \\ \left. + \left[1 + \frac{3\omega}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \right] G\left(-\frac{1}{2}, \frac{5}{2}, \omega\right) \right\} \quad (\text{A.51})$$

Axial vector 1P_1

$$\pi A(\omega) \Big|_{^1P_1} = \frac{1}{3} \pi A(\omega) \Big|_{^3P_0} \quad (A.52)$$

$$\alpha(\omega) = \tilde{\alpha}(\omega) + \frac{1}{\pi} S(\omega) \quad (A.53)$$

with

$$S(\omega) = -\frac{4}{3} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left[\frac{1}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) + \frac{1}{4} G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) - \left(\frac{1}{2} + \frac{1}{\omega}\right) G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right] \quad (A.54)$$

$$b(\omega) \Big|_{^1P_1} = b(\omega) \Big|_{^3P_0} \quad (A.55)$$

$$\frac{-\pi A'(\omega)}{\pi A(\omega)} \Big|_{^1P_1} = \frac{-\pi A'(\omega)}{\pi A(\omega)} \Big|_{^3P_0} \quad (A.56)$$

$$-\alpha'(\omega) = -\tilde{\alpha}'(\omega) - \frac{1}{\pi} S'(\omega) \quad (A.57)$$

where

$$\begin{aligned}
 -\frac{1}{\pi} \int'(\omega) = & -\frac{4}{3\pi} G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \left\{ \frac{5}{4} G\left(\frac{7}{2}, \frac{5}{2}, \omega\right) - \frac{1}{\omega^2} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right. \\
 & -\frac{5}{2} \left(\frac{1}{2} + \frac{1}{\omega}\right) G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) + \frac{7}{8} G\left(\frac{9}{2}, \frac{9}{2}, \omega\right) - \frac{3}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) G^{-1}\left(\frac{3}{2}, \frac{5}{2}, \omega\right) \\
 & \cdot \left[\frac{1}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) - \left(\frac{1}{2} + \frac{1}{\omega}\right) G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) + \frac{1}{4} G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) \right] \left. \right\}
 \end{aligned}
 \tag{A.58}$$

$$-b'(\omega) \Big|_{^1P_1} = -b'(\omega) \Big|_{^3P_0}
 \tag{A.59}$$

Tensor 3P_2

$$\pi A(\omega) = \frac{3}{8\sqrt{\pi}} \frac{4m^2}{\omega} \tilde{G}(\omega)
 \tag{A.60}$$

with

$$\tilde{G}(\omega) = G\left(\frac{3}{2}, \frac{5}{2}, \omega\right) - G\left(\frac{5}{2}, \frac{7}{2}, \omega\right)
 \tag{A.61}$$

$$\alpha(\omega) = \tilde{\alpha}_T(\omega) + \frac{1}{\pi} \left[-\frac{4}{3} + S(\omega) \right]
 \tag{A.62}$$

with

$$\begin{aligned}
 \tilde{\alpha}_T(\omega) = & \frac{8}{9\sqrt{\pi}} G(1,2,\omega) \tilde{G}^{-1}(\omega) \left[\pi + \frac{12c_3}{5} G(3,4,\omega) G^{-1}(1,2,\omega) \right. \\
 & \left. - \left(\frac{4\pi}{5} + 2c_3 \right) G(2,3,\omega) G^{-1}(1,2,\omega) \right] - \frac{2c_3}{3} - \frac{4\ln 2}{\pi} h(\omega)
 \end{aligned}
 \tag{A.63}$$

$$h(\omega) = \omega \tilde{G}^{-1}(\omega) \left[G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) - G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) \right] \quad (\text{A.64})$$

$$\begin{aligned} S(\omega) = \tilde{G}^{-1}(\omega) \left\{ \frac{16}{9} \left[G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) - G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) \right] \right. \\ \left. + \frac{24}{7} \left[G\left(\frac{5}{2}, \frac{9}{2}, \omega\right) + \frac{1}{2} G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) \right] - \frac{286}{525} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right\} \end{aligned} \quad (\text{A.65})$$

$$\begin{aligned} b(\omega) = -\frac{3\omega}{2} \tilde{G}^{-1}(\omega) \left[G\left(-\frac{1}{2}, \frac{5}{2}, \omega\right) + G\left(\frac{1}{2}, \frac{7}{2}, \omega\right) \right. \\ \left. - \frac{13}{6} G\left(\frac{3}{2}, \frac{7}{2}, \omega\right) + \frac{8}{3} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right] \end{aligned} \quad (\text{A.66})$$

$$\frac{-\pi A'(\omega)}{\pi A(\omega)} = \frac{1}{\omega} + R \tilde{G}(\omega) \quad (\text{A.67})$$

with

$$\begin{aligned} R \tilde{G}(\omega) &= -\frac{d}{d\omega} \log \tilde{G}(\omega) = -\tilde{G}'(\omega) \tilde{G}^{-1}(\omega) \\ &= \tilde{G}^{-1}(\omega) \left[\frac{3}{2} G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) - \frac{5}{2} G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) \right] \end{aligned} \quad (\text{A.68})$$

$$-\alpha'(\omega) = -\tilde{\alpha}'_T(\omega) - \frac{1}{\pi} S'(\omega) \quad (\text{A.69})$$

where

$$\begin{aligned}
 -\tilde{\alpha}'_T(\omega) = & \frac{8}{9\sqrt{\pi}} \tilde{G}^{-1}(\omega) \left\{ \left[G(2,2,\omega) - G(1,2,\omega) R \tilde{G}(\omega) \right] \right. \\
 & \cdot \left[\pi - \left(\frac{4\pi}{5} + 2c_3 \right) G(2,3,\omega) \tilde{G}^{-1}(1,2,\omega) + \frac{12c_3}{5} G(3,4,\omega) \tilde{G}^{-1}(1,2,\omega) \right] \\
 & - \left(\frac{4\pi}{5} + 2c_3 \right) \left[2G(3,3,\omega) - G(2,2,\omega) G(2,3,\omega) \tilde{G}^{-1}(1,2,\omega) \right] \\
 & \left. + \frac{12c_3}{5} \left[3G(4,4,\omega) - G(2,2,\omega) G(3,4,\omega) \tilde{G}^{-1}(1,2,\omega) \right] \right\} \\
 & + \frac{4 \ln 2}{\pi} f(\omega)
 \end{aligned} \tag{A.70}$$

$$\begin{aligned}
 f(\omega) = & \tilde{G}^{-1}(\omega) \left\{ -\frac{3\omega}{2} G\left(\frac{5}{2}, \frac{3}{2}, \omega\right) + \frac{5\omega}{2} G\left(\frac{7}{2}, \frac{5}{2}, \omega\right) \right. \\
 & \left. + \left[1 + \omega R \tilde{G}(\omega) \right] \cdot \left[G\left(\frac{3}{2}, \frac{3}{2}, \omega\right) - G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) \right] \right\}
 \end{aligned} \tag{A.71}$$

$$\begin{aligned}
 -\frac{1}{\pi} S'(\omega) &= \frac{1}{\pi} \tilde{G}^{-1}(\omega) \left\{ \frac{40}{9} G\left(\frac{7}{2}, \frac{5}{2}, \omega\right) - \frac{56}{9} G\left(\frac{9}{2}, \frac{7}{2}, \omega\right) \right. \\
 &\quad + \frac{60}{7} G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) + 6 G\left(\frac{9}{2}, \frac{9}{2}, \omega\right) - \frac{143}{105} G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) \\
 &\quad - R \tilde{G}(\omega) \left[\frac{16}{9} \left[G\left(\frac{5}{2}, \frac{5}{2}, \omega\right) - G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) \right] \right. \\
 &\quad \left. \left. + \frac{24}{7} \left[G\left(\frac{5}{2}, \frac{9}{2}, \omega\right) + \frac{1}{2} G\left(\frac{7}{2}, \frac{9}{2}, \omega\right) \right] - \frac{286}{525} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right] \right\} \\
 &\hspace{15em} (A.72)
 \end{aligned}$$

$$\begin{aligned}
 -b'(\omega) &= \frac{3}{2} \tilde{G}^{-1}(\omega) \left\{ \left[1 + \omega R \tilde{G}(\omega) \right] \right. \\
 &\quad \cdot \left[G\left(-\frac{1}{2}, \frac{5}{2}, \omega\right) + G\left(\frac{1}{2}, \frac{7}{2}, \omega\right) - \frac{13}{6} G\left(\frac{3}{2}, \frac{7}{2}, \omega\right) + \frac{8}{3} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) \right] \\
 &\quad \left. + \frac{\omega}{2} \left[G\left(\frac{1}{2}, \frac{5}{2}, \omega\right) - G\left(\frac{3}{2}, \frac{7}{2}, \omega\right) + \frac{13}{2} G\left(\frac{5}{2}, \frac{7}{2}, \omega\right) - \frac{40}{3} G\left(\frac{7}{2}, \frac{7}{2}, \omega\right) \right] \right\} \\
 &\hspace{15em} (A.73)
 \end{aligned}$$

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FIGURE CAPTIONS

- Fig. 1 The theoretical ratios (15) with parameters (36), curve a), and with $\phi_1 = 0$, curve c), are compared with the experimental ratios (5), (37) and (38), curve b), for the charmonium case 3S_1 .
- Fig. 2 Ratio (15) for charmonium 1S_0 with parameters (36).
- Fig. 3 Ratio (15) for charmonium 3P_0 with parameters (36).
- Fig. 4 Ratio (15) for charmonium 3P_1 with parameters (36).
- Fig. 5 Ratio (15) for charmonium 3P_2 with parameters (36).
- Fig. 6 Ratio (15) for charmonium 1P_1 with parameters (36).
- Fig. 7 The theoretical ratios (15) with parameters $\bar{m}_b = 4.71$ GeV, $\alpha_s = 0.25$, $\phi_1 = 0.07$ GeV, curve a), and with $\phi_1 = 0$, curve c), are compared with the experimental ratio, Eqs. (5), (37) and (38), curve b), for the bottonium case 3S_1 .
- Fig. 8 Ratio (15) for bottonium 3P_1 with parameter values as for 3S_1 .
- Fig. 9 The spin averaged relativistic ratio (15) is compared with the non-relativistic ratio (59) for s waves in charmonium.
- Fig. 10 The spin averaged relativistic ratio (15) is compared with the non-relativistic ratio (59) for p waves in charmonium.
- Fig. 11 The spin averaged relativistic ratio (15) is compared with the non-relativistic ratio (59) for s waves in bottonium.
- Fig. 12 The spin averaged relativistic ratio (15) is compared with the non-relativistic ratio (59) for p waves in bottonium.

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