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TWO THEOREMS ON APPROXIMATION METHODS
IN PERTURBATIVE FIELD THEORY

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A B S T R A C T

We present two theorems on approximations in perturbative field theories.

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Before discussing the two theorems of this paper, we want to make the following remarks. First, Theorem 1 has already been demonstrated¹⁾, but with Eq. (2) not given explicitly. It is derived from Eq. (1) of the Theorem. Second, Theorem 2 might be useful in the optimization procedures of Ref. 1).

Theorem 1

Let $R^{(i)}(\alpha)$ be an approximant to the physical quantity $R(\alpha)$ obtained in expanding $R(\alpha)$ up to the i^{th} order in α , α being a solution of the renormalization group equation with the complete beta function:

$$\frac{d\alpha}{dt} = \beta(\alpha) ; \quad (\beta(\alpha) \equiv \beta_c(\alpha)).$$

Designating by $R_A^{(i)}$ the approximant $R^{(i)}$ in the renormalization scheme A and by $R_B^{(i)}$ the same approximant calculated in scheme B; then $R_A^{(i)}$ and $R_B^{(i)}$ differ only by terms which are at least $O(\alpha^{i+1})$, α being either one of the couplings α_A, α_B .

Proof

$$R_A(\alpha_A) = R_A^{(i)}(\alpha_A) + O(\alpha_A^{i+1})$$

$$R_B(\alpha_B) = R_B^{(i)}(\alpha_B) + O(\alpha_B^{i+1})$$

As all $R_N(\alpha_N)$, ($N=A,B$) satisfy the full renormalization group equations involving the famous j -indexed β functions $\beta_j(\alpha)$ ²⁾ which, by suitably modifying α , guarantee the invariance of physical quantities against arbitrary changes of scheme,

$$\frac{\partial R}{\partial c_j} = \left\{ \frac{\partial}{\partial c_j} \Big|_{\alpha} + \beta_j(\alpha) \frac{\partial}{\partial \alpha} \right\} R = 0 \quad (1)$$

whatever the scheme specified by the set $\{c_j\}$ may be (A,B,...,etc), then

$$R_A = R_B \equiv R \quad (2)$$

Moreover, since $\alpha_A = \alpha_B + c\alpha_B^2 + \dots$, then

$$\alpha^{i+1} \equiv \alpha_A^{i+1} = \alpha_B^{i+1} + \dots \quad (3)$$

Therefore, from (1) and (2)

$$R_A^{(i)} = R_B^{(i)} + O(\alpha^{i+1})$$

Corollary

The cancellation of the scheme dependence at i^{th} order in the expansion in the effective coupling implies a cancellation between the scheme dependence of the perturbative coefficients r_k of $R^{(i)}$ and the scheme dependence of the coupling. As we know the differential transformations $\beta_{i\alpha} \equiv \beta_i(\alpha) \equiv (\partial\alpha/\partial c_i)$ of the coupling with respect to the c_i which parametrize the full renormalization group, one can deduce from these cancellations the analytic dependence of the r_k on the c_i . This can be achieved either by using the straightforward expressions for the beta functions $\beta_{i\alpha}$ or by the formula

$$\beta_{i\alpha}(\alpha) = -\beta(\alpha) \int_0^\alpha x^{i+2} [\beta(x)]^{-2} dx ; i \geq 2 \quad (4)$$

Note: The case $i = 1$ needs special care in the normalization and Eq. (4) is then invalid.

Theorem 2

Let $R^{(i,j)}$ be an approximant to a physical quantity R obtained in the following way.

$\alpha^{(j)}$ being the solution of the renormalization group equation with a truncated beta function $\beta^{(j)}$ at order $j+1$ in α , $R^{(i,j)}$ is the approximant obtained by expanding R in powers of $\alpha^{(j)}$ and truncated at the i^{th} order in $\alpha^{(j)}$.

One can then state the following theorem :

A necessary condition for $R^{(i,j)}$ to yield a trustable ^{*)} approximation of the exact value R is that it satisfies the set of equations

$$\frac{\partial R^{(i,j)}}{\partial c_k} \ll \epsilon , \quad 2 \leq k \leq j-1 ,$$

the range of ϵ being $\ll 1$. The c_k are the beta function coefficients which parametrize the full renormalization group and therefore the renormalization scheme.

^{*)} By trustable we mean: if $\Delta R \equiv R_{ij} - R \leq \eta$, the smaller η is, the more trustable are the approximants R_{ij} (η small compared to unity).

Practical example

$$R^{(2,3)} = \alpha^{(3)} [1 + r_1 \alpha^{(3)}]$$

with $\beta^{(3)}$ is a solution of

$$\beta^{(3)} = b \alpha^2 (1 + c_1 \alpha + c_2 \alpha^2)$$

Proof

Suppose that there exists a scheme S' , parametrized by the parameters c'_k , such that $R^{(i,j)}(c'_k)$ is a trustable approximant to R . If this $R^{(i,j)}$ does not satisfy the condition of Theorem 2, then there exist schemes S'' such that

$$R^{(i,j)}(c'_k) - R^{(i,j)}(c''_k) = (c'_k - c''_k) \cdot O(K)$$

where K is a large number ($>$ or $\gg 1$), $\Delta c_k = c'_k - c''_k$ being a finite difference. Therefore, by shifting from c'_k to c''_k , $R^{(i,j)}(c''_k)$ becomes very different from $R^{(i,j)}(c'_k)$ which is, by hypothesis, a trustable approximant to R . But R , by full renormalization group arguments, is not sensitive to the change $c'_k \rightarrow c''_k$ and $R^{(i,j)}$ cannot be a trustable approximation to R since a change in the unphysical c_k 's modifies drastically its value.

On the other hand, if $R^{(i,j)}$ is situated in the range $0 - \epsilon$, then

$$R^{(i,j)}(c'_k) - R^{(i,j)}(c''_k) \leq \Delta c_k \cdot \epsilon$$

and $R^{(i,j)}(c''_k)$ remains a trustable approximation to R since the variation is of order ϵ , and will do so in a whole domain $\mathcal{D}$ around S' in the space of renormalization schemes. In order to avoid that all domains $\mathcal{D}$ are of infinitesimally small measure, the condition of Theorem 2 is therefore a necessary condition.

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