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ON THE REFORMULATION OF THE $d = 3$ ISING MODEL
IN TERMS OF RANDOM SURFACES

A. Casher ^{*)}

CERN - Geneva
and

Université Libre de Bruxelles, Campus de la Plaine
Bruxelles

D. Foerster

CERN - Geneva

and

P. Windey

CEN, Saclay, Gif-sur-Yvette

A B S T R A C T

We derive equations of motion for order-disorder variables in the $d = 2$ and $d = 3$ Ising models, working directly on the lattice. The integration of these equations leads to a reformulation of the $d = 2$ Ising model in terms of random paths (this is obvious) and to a reformulation of the $d = 3$ Ising model in terms of random surfaces (this is new).

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1. INTRODUCTION

For several years now, there has been the speculation that Z_2 gauge theory might be completely integrated in terms of random surfaces. This idea was originally inspired by the topological method of Kac and Ward¹⁾ of solving the $d = 2$ Ising model. The speculation that Z_2 gauge theory might represent spinning world surfaces was due to the $d = 2$ Ising model being equivalent to free fermions. It is well known that free fermions may be represented, in the continuum limit, as supersymmetric spinning world lines²⁾. Part of these speculations collapsed upon the discovery, by computer simulation, of a first order transition in $d = 4$ Z_2 gauge theory³⁾. This leaves us with $d = 3$ Z_2 gauge theory which is dual to the $d = 3$ Ising spin model⁴⁾ and which is therefore guaranteed to have a second order⁵⁾ phase transition (infinite range correlations are needed to wipe out lattice effects and to define a continuum theory). In the meantime, a fermion representation of the $d = 3$ Ising model (in its spin or gauge version) had been discovered⁶⁾. Very recently, A. Polyakov has tried to check these speculations by computing the critical index of the Ising model in terms of his version⁷⁾ of the theory of spinning world surfaces.

The approach we will be using in this paper is different from those of Ref. 7) and is based on the order-disorder variables due originally to Onsager⁸⁾. We shall derive equations of motion for order-disorder pointlike variables in $d = 2$ and for order-disorder loop variables in $d = 3$ ^{*)}. These equations are essentially linear and their integration generates, respectively, random paths and random surfaces in $d = 2$ and $d = 3$ dimensions. The reason we bother to discuss the $d = 2$ Ising model is mainly pedagogical and to emphasize the close analogy between random paths and random surfaces.

Our work is organized as follows: Section 2 gives the order-disorder variables in the $d = 2$ model, their equation of motion and the solution of the model. In Section 3 order-disorder variables are defined for the $d = 3$ theory and their equations of motion are discussed. In Section 4 we integrate these equations of motion in terms of random surfaces. Section 5 is on phase factors and their possible connection with a fermionic structure on the surfaces. Section 6 gives our conclusions. Our main result is summarized by Eqs. (26) and (27) in Section 4.

^{*)} These are presumably the equations referred to by A. Polyakov in Ref. 9).

2. RANDOM PATHS AND THE $d = 2$ ISING MODEL

i) Equations of motion

The $d = 2$ Ising model is defined by

$$Z = \sum_{\{\sigma_n\}} \prod_b e^{\beta \sigma_b} \quad (1)$$

where $\sigma_b = \sigma_n \sigma_{n'}$; n, n' denote nearest neighbours on a square lattice along a bond b and the σ 's are spin variables taking on values ± 1 . It is well known that one may "dualize" this theory into the same model but with $\beta \rightarrow \beta^*$ with $\sinh 2\beta \sinh 2\beta^* = 1$ and the new spins sitting on the dual lattice. Since spin variables change into disorder variables under this procedure it is natural to consider spin and disorder variables simultaneously in one and the same theory. To construct a disorder variable, draw a line starting from some point on the dual lattice and going all the way to $y = -\infty$, for convenience. Then flip the coupling β into $-\beta$ for all the bonds that are cut by this line, see Fig. 1. Only the starting point of this line matters since the theory is invariant with respect to a simultaneous change of $\sigma_n \rightarrow -\sigma_n$ and $\beta_b \rightarrow -\beta_b$ for all bonds emerging from n .

A plaquette with

$$\prod_{b \in p} (\text{sign } \beta_b) = -1 \quad (2)$$

will be called a frustrated plaquette or, equivalently, one will say that a frustration sits at the plaquette p . The basic objects to be considered are order-disorder pairs formed by a spin variable with one of its neighbouring frustrations, see Fig. 2. If we insist on using the old Boltzmann weight without flipped bonds, an order-disorder pair is given by

$$\langle \Psi_i(x, y) \dots \rangle = \langle \sigma(x, y) \prod_{\text{path}} e^{-2\beta \sigma_b} \dots \rangle \quad (3)$$

with the path going to $y = -\infty$ and starting near (x, y) . The dots stand for far away sources or other field operators.

It is well known how to obtain the equations of motion in a statistical system by exploiting the invariance of the statistical measure. Here we may replace an arbitrary spin σ by $-\sigma$ in the summation over spins. Applying this argument to the vacuum (no operators) we find

$$\left\langle \prod_{b \ni n} e^{-2\beta \sigma_b} \right\rangle = 1 \quad (4)$$

where the product runs over all bonds containing the spin σ_n .

Diagrammatically this reads as follows

$$\left\langle \text{---} \text{---} \text{---} \right\rangle = 1 \quad (5)$$

(with the notations of Fig. 2).

It is now easy to move spin frustration pairs. For example, for $\psi_2(n)$ in Fig. 2 we use the identity

$$\sigma e^{-2\beta \sigma \sigma'} = \sigma \cosh 2\beta - \sinh 2\beta \sigma' \quad (6)$$

to eliminate one flipped bond. In this case we have moved the frustration by one step in the $-y$ direction and we also get a motion of the spin in the $-x$ direction. Instead of annihilating a flipped bond we may also create an extra flipped bond by the identity

$$\sigma = \frac{\sigma}{\cosh 2\beta} e^{-2\beta \sigma \sigma'} + \sigma' \tanh 2\beta \quad (7)$$

After applying this identity, we still have to move the string of flipped plaquettes into a standard configuration as illustrated in Fig. 3. This can be done using Eq. (5).

An example of an equation obtained in this way is, in the notation of Fig. 2

$$\psi_{n,2} = -\frac{1}{\cosh 2\beta} \psi_{n,3} + \tanh 2\beta \psi_{n,1} \quad (8)$$

We list below all the equations that govern the motion of the pairs $\psi_{n,i}$ for fixed n and any i

$$\begin{aligned}
 & \text{Diagram 1} = c \text{ Diagram 2} + s \text{ Diagram 3} \\
 & \text{Diagram 4} = c \text{ Diagram 5} - s \text{ Diagram 6} \\
 & \text{Diagram 7} = -c \text{ Diagram 8} + s \text{ Diagram 9} \\
 & \text{Diagram 10} = c \text{ Diagram 11} + s \text{ Diagram 12}
 \end{aligned}$$

(9)

and

$$\begin{aligned}
 & \text{Diagram 13} = c \text{ Diagram 14} + s \text{ Diagram 15} \\
 & \text{Diagram 16} = -c \text{ Diagram 17} + s \text{ Diagram 18} \\
 & \text{Diagram 19} = c \text{ Diagram 20} - s \text{ Diagram 21} \\
 & \text{Diagram 22} = c \text{ Diagram 23} + s \text{ Diagram 24}
 \end{aligned}$$

(9')

where $c = 1/\cosh 2\beta$ and $s = \tanh 2\beta$. For the sake of clarity we have depicted the related spin frustration pairs by dotted lines.

Note that the equations are simply obtained by rotating the pair either around its spin or around its frustration, giving rise to factors $1/\cosh 2\beta$ and $\tanh 2\beta$, respectively. Also, there is a - sign each time a spin crosses a line of flipped bonds.

One can easily see that the above equations are not all independent. As independent set one may choose either the first four or the last four. In the first set the frustrations rotate anticlockwise around the spin while the spin rotates clockwise around the frustrations and vice versa for the second set.

ii) Solution of the equations of motion of the $d = 2$ Ising model
in terms of random paths

We now choose the first set of equations (9) and iterate them. Such an iteration generates a kind of random walk for the spin frustration pairs that can be represented on the spin frustration lattice by a path connecting midpoints of adjacent links (see Fig. 4). At each step of this walk one may choose one of either equation satisfied by the last pair reached by the path. With a definite choice such as that of the first set of equations (9) we notice that we generate paths that must leave at step $n+1$ the square they entered at step n . We therefore specify paths more concisely by recording the centre of each square entered by the path. The path of Fig. 4 then has a representation that is drawn in the same figure by a dotted line. In this way, we obtain a somewhat simpler description of the allowed paths with a weight S for each right turn, C for each left turn and $(-)^{\frac{1}{2}}$ for a rotation by 2π .

If we now assign the phase $(-)$ in portions of $e^{i\pi/4}$ by redefining the variable Ψ and denote by χ_μ the sum over-all paths arriving at the centre of square r from direction μ we find:

$$\begin{aligned}\chi_\mu(r) &= \sum_{\mu'} \Gamma_{\mu\mu'} \chi_{\mu'}(r + \epsilon_{\mu'}) \\ &\equiv \sum_{\mu'r'} \Gamma_{\mu r, \mu' r'} \chi_{\mu'}(r')\end{aligned}\tag{10}$$

where

$$\Gamma_{\mu\mu'} = c e^{i\pi/4} \delta_{\mu, \mu'-1} + s e^{-i\pi/4} \delta_{\mu, \mu'+1}\tag{11}$$

and μ enumerates directions

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (\hat{x}, \hat{y}, -\hat{x}, -\hat{y})\tag{12}$$

The sum over closed paths is then given by

$$\begin{aligned}\text{Log } Z &= \text{const.} \sum_{N, \mu, r} \frac{1}{N} (\Gamma)_{\mu r, \mu r}^N \\ &= -\text{const.} \text{Tr } \log(1 - \Gamma)\end{aligned}\tag{13}$$

The trace is taken after Fourier transforming and yields:

$$\text{Log } Z = \text{const.} \int_{-\pi \leq p_x, p_y \leq \pi} dp_x dp_y \text{Log det} (\delta_{\mu\mu'} - \tilde{\Gamma}_{\mu\mu'}(p_x p_y)) \quad (14)$$

where $\tilde{\Gamma}_{\mu\mu'}(p_x p_y)$ is the Fourier transform of $\Gamma_{\mu r, \mu' r'}$:

$$\tilde{\Gamma}_{\mu\mu'}(p_x, p_y) = e^{ip_{\mu'}} (c e^{i\pi/4} \delta_{\mu, \mu'+1} + s e^{-i\pi/4} \delta_{\mu, \mu'+i}) \quad (15)$$

We find:

$$\text{Log } Z = \text{const.} \int_{-\pi \leq p_x, p_y \leq \pi} dp_x dp_y \text{Log} \left(1 - \frac{\tanh 2\beta}{\cosh 2\beta} (\cos(p_x + p_y) + \cos(p_x - p_y)) \right) \quad (16)$$

To compare with the literature⁸⁾ we just notice the relation between our lattice of spins and frustrations and the original lattice of spins only. This gives $p_x + p_y \rightarrow p_x$, $p_x - p_y \rightarrow p_y$, and one obtains the known result after changing the Brillouin zone appropriately. The constant in Eq. (16) can be fixed by a more careful argument that expresses nearby spin correlations in terms of order-disorder variables.

Finally, we want to point out that our equations of motion (9) are manifestly self-dual. In other words:

$$\begin{aligned} \left(\tanh 2\beta, \frac{1}{\cosh 2\beta} \right) & \xrightarrow[\text{transformation}]{\text{dual}} \left(\tanh 2\beta^*, \frac{1}{\cosh 2\beta^*} \right) \\ & = \left(\frac{1}{\cosh 2\beta}, \tanh 2\beta \right) \end{aligned} \quad (17)$$

with the relation $\sinh 2\beta \sinh 2\beta^* = 1$. A dual transformation therefore amounts just to exchanging spins $\leftrightarrow$ frustrations and $c \leftrightarrow s$ in Eqs. (9).

3. THE EQUATIONS OF MOTION FOR THE ORDER-DISORDER LOOPS IN THE d = 3 DIMENSIONAL ISING MODEL

It is well known⁴⁾ that the d = 3 Ising model dualizes into a d = 3 Z_2 gauge theory defined by the partition function

$$Z = \sum_{\{S_b\}} \prod_P e^{\beta S_P} \quad (18)$$

where $S_b \in (1, -1)$ are gauge variables sitting on the bonds or links of a cubic lattice and where S_P denotes the product of bond variables around an elementary plaquette P .

The natural generalization of the Onsager order-disorder variables is now a Wegner-Wilson loop of link variables with disorder variables attached to each link of this loop. A discussion of order and disorder variables in the context of a continuum gauge theory has been given by 't Hooft¹⁰⁾. Physically, the disorder variables correspond to monopoles with a Dirac string running to infinity. On the lattice, they are localized at the centre of cubes that start a line on the dual lattice with every plaquette pierced by this line having its coupling flipped from β to $-\beta$, see Fig. 5. As in d = 2 dimensions, the location of the string is arbitrary and may be moved around by the invariance of the statistical measure (or by gauge transformations). However, the starting point of the string cannot be moved and corresponds to a monopole sitting inside a cube. This cube is characterized by the curvature

$$\prod_{P \in \text{cube}} (\text{sign } \beta_P) = -1 \quad (19)$$

We shall refer to such a cube as a frustrated cube with a frustration as its centre. An example of a loop made up of links with associated frustrations is given in Fig. 6.

The equations of motion are again derived via the invariance of the statistical measure which now implies

$$\left\langle \prod_{\ell \in P} e^{-2\beta \ell} \right\rangle = 1 \quad (20)$$

where the product is taken over all plaquettes attached to a given link ℓ .

Pictorially, this is represented by

$$\langle \text{Diagram of a square with a diagonal line and a wavy line} \rangle = 1 \quad (21)$$

Frustrations may now be moved by using the identities

$$\begin{aligned} \sigma e^{-2\beta \sigma_p} &= \sigma \cosh 2\beta - \sigma_{\square} \sinh 2\beta \\ \sigma &= \frac{\sigma}{\cosh 2\beta} e^{-\beta \sigma_p} + \tanh 2\beta \sigma_{\square} \end{aligned} \quad (22)$$

where $\sigma_{\square}$ denotes $\sigma \cdot \sigma_p$, i.e., the product of σ 's around a plaquette excluding the link variable σ considered. As in $d = 2$ dimensions, one has two equations per link frustration pair, i.e.

and

$$\begin{aligned} \text{Diagram 1} &= -s \text{Diagram 2} + c \text{Diagram 3} \\ \text{Diagram 4} &= -c \text{Diagram 5} + s \text{Diagram 6} \end{aligned} \quad (23)$$

where, as before, $c = 1/\cosh 2\beta$, $s = \tanh 2\beta$ and the minus sign results whenever a Dirac string crosses a link. The main difference of the $d = 3$ equations relative to their $d = 2$ counterparts is that in moving a bit of loop one creates

two extra bits of string (only later shall we specify how to annihilate bits of string). To remain in the set of link frustration loops one has to associate with the added string bits a pair of frustrations sitting on top of each other (which amounts to having no frustration). There is a twofold ambiguity in the choice of location of this extra pair of frustrations that will be solved below. We shall not write down all possible equations of link frustration pairs because they are quite similar to the $d = 2$ equations (9).

However, we must now discuss an apparent anisotropy in the equations that is caused by our choice of flipping plaquettes along strings running in the $\hat{z}$ direction.

In this gauge or convention, the links that are oriented in the $\hat{z}$ direction are singled out as special because they cannot be cut by any Dirac strings. However, their frustrations cut through the appended curve with their Dirac strings upon a 2π rotation. This is illustrated in Fig. 7a. Generally, a 2π rotation of a frustration connected to a $\hat{z}$ link will cut through the appended curve an odd number of times. There is a class of exceptions to this rule that is illustrated in Fig. 7b. But in these exceptional cases a $\hat{z}$ link with its associated frustration behaves precisely as an $\hat{x}$ or $\hat{y}$ link and picks up a $+$ sign upon 2π rotation of its frustration. Therefore, it appears that the equations are essentially isotropic when one considers the link frustration pairs as part of an order-disorder loop.

4. SOLVING THE LOOP EQUATION OF THE $d = 3$ Z_2 GAUGE THEORY

i) The choice of an independent set of equations

Now we want to solve the loop equation for $d = 3$ in a way that parallels our approach in $d = 2$, replacing points by curves and random paths by random surfaces.

First, we shall determine which equation to use in our iteration by requiring frustrations and links to be rotated about one another in a definite way. To set up such a convention (choice among the two equations) we first assign an orientation to the loop. We then require that frustrations must always be rotated in an anticlockwise fashion around their links (see Fig. 8). The completion of this rule is then dictated by our Eqs. (23): links must be rotated around frustrations in the opposite sense.

When we create new links in deforming the loop via our equations we assign the natural orientation that is compatible with the rest of the loop. Finally, we must decide where to place the pair of frustrations associated with the new links. This choice will influence the way the curve will be deformed upon further use of the equations. We decide to put them on the same side as the normal of

the bit of surface that got swept out by the deformation. This convention has the advantage of preventing backtracking or immediate backfolding of the surface (see Fig. 9). This prescription then completes our choice of a subset of independent equations.

ii) Computation of the weight in simple cases

We shall now show how to compute the weight factor associated with a surface in the simplest possible case, i.e., for a flat surface. For simplicity, we have taken the loop in the xy plane and the frustrations below and inside the loop, see Fig. 10. According to our equations and with our conventions we may take away a corner plaquette in the two ways indicated in the figure. This costs a factor S in both cases. One may now cut away the backtracking noses to obtain a surface of three plaquettes as indicated in the figure. It is gratifying that the weight factor associated with both procedures is the same and it reinforces our belief that the theory is a theory of surfaces rather than a theory of loops. Before taking away another plaquette one has to rotate frustrations yielding a factor C .

In general, a flat surface in one of the principal lattice planes gives rise to a weight factor (modulo boundary effects) of $(C^2S)^A$ where A is the area of the surface. To determine the weight, modulo signs and boundary effects, of a more general surface, we look at two simple examples, see Fig. 11. In both cases one plaquette has already been removed and one can easily see that one needs an extra factor C in a) and that one saves a factor C in b). The general result is that for bending there is an extra factor $(C)^{\Delta n}$ where $n = (\text{number of edges with diverging normals}) - (\text{number of edges with converging normals})$. For a closed elementary cube we find the following weight (modulo a sign)

$$\text{weight of a cube} = \begin{cases} (Sc^2)^6 C^{12} = S^6 C^{24} & \text{outward normal} \\ (Sc^2)^6 C^{-12} = S^6 & \text{inward normal} \end{cases} \quad (24)$$

iii) Renormalization of the surface weights

We still have to introduce some renormalization of the factor S due to possible extra rotations of the frustrations. To see this, it is easiest to go back to $d = 2$ and consider the random walk as defined by successive positions of the spin rather than by the midpoints of spin frustration pairs. The correspondence between both walks is illustrated in Fig. 12. We see from the figure that a rotation of a frustration around a spin may be accompanied by an

arbitrary number of 2π rotations around the same spin. This leads to the replacement

$$S \rightarrow S (1 - C^4 + C^8 - \dots) = \frac{S}{1 + C^4} \quad (25)$$

where s is the unrenormalized weight factor for the motion of a spin. This simply means that there is an extra weight factor that does not depend on the geometry of the path but only on its total length. The other factors associated with the path do not change. This generalizes immediately to $d = 3$ dimensions to give an extra weight factor $(1/1+C^4)^A$ with $A = \text{area of the surface}^*$.

iv) On the phase associated with a surface

We still have to collect the minus signs that result when Dirac strings intersect gauge variables in the process of sweeping out of surface. It is not very interesting to compute the over-all sign for open surfaces since it depends on our convention of sending flipped bonds (Dirac strings) to $Z = -\infty$. However, for closed surfaces, this sign appears to be convention independent. Our examples are depicted in Fig. 14. To compute the weight for each surface we shall cut a slit into each of them with frustrations attached to the direction of the normal of the surface to be swept out and in such a way that they cancel. This is illustrated in the case of a single cube in Fig. 13. In the examples we have drawn in Fig. 14 all phases are 1 except for $(-)^p$ and $(-)^n$ in the intersecting cases. Such a phase $(-)^{\ell}$, ℓ being the length of self-intersection, has already been noticed by Fradkin et al.⁶⁾ in their rewriting of the $d = 3$ Z_2 gauge theory in terms of Grassmann variables. They argued that this sign oscillation would prevent one from rewriting the theory as a string model. To motivate our disagreement with these authors consider Fig. 15 where we have drawn a particular way of sweeping out a self-intersecting surface in which the idea strongly suggests itself that the sign $(-)^{\ell}$ is due to a 2π rotation for ℓ bits of loop as a result of the intersection. Clearly, we have in mind here the idea that the surfaces should carry a spin structure with one spin $1/2$ object per plaquette. This topic will be discussed in the next section.

In this section we have only discussed surfaces of the type of a disc or of a sphere. Surfaces with handles are also generated by iterating the loop equations and carry the same weight as those without handles.

^{*}) We have checked explicitly that the renormalization procedure works also at self-intersections where a priori some difficulties might be expected.

To summarize, the average of an order-disorder loop is given by (up to boundary effects)

$$\langle C \rangle = k \sum_{M(C)} \left(\frac{S}{1+C^4} \right)^{A(M)} C^{N(M)} (-)^{L(M)} \quad (26)$$

where $\{M(C)\}$ are all oriented connected surfaces (with self-intersections and backfolding permitted) spanned by the loop C . $A(M)$ is the area (=number of plaquettes), $L(M)$ is the total length of self-intersections, and $N(M)$ is given by the following sum over adjacent normals:

$$N(M) = \sum_{(\hat{n}, \hat{n}')} \nu(\hat{n}, \hat{n}') ; \quad \Theta(\hat{n}, \hat{n}') = [\nu(\hat{n}, \hat{n}') - 1] \frac{\pi}{2} \quad (27)$$

Clearly, the angle Θ can take on only the values $-\pi/2, 0, \pi/2, \pi$ so that $\nu = 0, 1, 2, 3$. The factor k is universal and is determined by the requirement that degenerate loops should be equal to one.

5. THE INTRINSIC MEANING OF OPEN LINES OF INTERSECTION

In order to have a good description in terms of random surfaces we must try to understand the meaning of the sign $(-)^{\ell}$ where ℓ is the length of intersection.

Notice that intersections along open lines with end-points are lattice effects that do not occur for continuously differentiable surfaces in $d = 3$ dimensions^{*)}. We may, of course, hope that these lattice effects drop out in the continuum limit but it is best to separate the problem of the reformulation of the $d = 3$ Ising model from that of constructing a continuum theory of spinning surfaces. We shall show now that the end-points of an open line of intersection may be discussed by looking at the surface locally only [the argument is borrowed from Ref. 11)]. Consider a surface with an open line of self-intersections such as in Fig. 16a and draw a small sphere S_2 around one of the end-points. Then the intersection of S_2 with the surface produces a self-intersecting curve C on S_2 . For an ordinary point such as a corner of a cube the curve C on S_2 is not self-intersecting.

^{*)} Fig. 14d is the lattice analog of an intersection in the continuum.

We shall now relate the number of self-intersections of C to local properties of a frame constructed on the surface. A frame of orthogonal vectors on the surface is provided by the normal $\vec{n}$ and a tangential vector $\vec{t}$ in the surface: $\vec{n}, \vec{t}, \vec{n} \times \vec{t}$ specifies the frame. For the tangential vector field we may use the type of vector field that enters the proof of the Poincaré-Hopf theorem¹²⁾, i.e., a tangential vector field with singularities of index 1, 1, -1 at the vertices, centres of plaquettes and centres of segments, respectively. See Fig. 16b. Let us connect the properties of this frame at a vertex with the intersections of C on S_2 . On S_2 , the frame $(\vec{n}, \vec{t}, \vec{n} \times \vec{t})$ may also be expressed as $(t \times \dot{t}, t, \dot{t})$ where $\dot{t}$ denotes the derivative of t . As one follows the frame all through C one rotates it by 2π or 4π , as the case may be. To relate the over-all rotation to the self-intersection number deform the curve C slowly into a small curve around the north pole of S_2 , say, without modifying the 2π or 4π total rotation. At the north pole it is clear, however, that the total rotation must be 2π or 4π according to whether C has an even or odd number of self-intersections.

We conclude: with a frame constructed on the surface via Hopfs' vector field and the normal we find a 2π rotation for every regular point and a 4π rotation for every vertex that is the end-point of an open line of self-intersection. Finally, the distance between endpoints of self-intersection has to be known only mod 2. It is therefore sufficient to count the number of steps required to go from one end-point to the other along an arbitrary path. The over-all picture that emerges is the following: a frame on a surface in $d = 3$ determines an element of $SU(2)$ up to a 2π rotation. The end-points of self-intersection lines then become branch-points in this double covering. It is clear then that an extra fermionic structure might be effective in keeping track of the minus sign in Eq. (26).

6. OUTLOOK AND CONCLUSIONS

We have argued in this paper that the equations of motion of order-disorder loops in the $d = 3$ Z_2 gauge theory may be solved in terms of summing over random surfaces with appropriately chosen weights. We emphasized the analogy between random paths in the $d = 2$ Ising model and random surfaces for $d = 3$. We have also found indications that the phases which deterred previous authors are locally computable in terms of the properties of an attached frame or, possibly, via an extra fermionic structure.

The next step will be to try and identify this fermionic structure. Since there does not exist, at the moment, a full theory of spinning surfaces there are good reasons to do the reformulation on the lattice and to take the continuum

limit in a separate step. When the discrete formulation will have been finished it will be possible to do computer simulations and obtain some understanding of the structure of spinning surfaces.

One may hope that eventually the $d = 3$ Ising model might teach us something of value about a good line of attack on $d = 4$ Yang-Mills theory.

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APPENDIX

There are some difficulties with the use of order disorder variables that lead us to introduce a different formulation of the theory in terms of Grassmann variables. Such difficulties occur already at the level of $d = 2$ dimensions. Consider two order-disorder variables that differ only in their y co-ordinates such as in Fig. 17 and rotate the upper frustration or the lower spin by 2π . Then the 2π rotation of the frustration, for instance, cuts through the lower spin twice, i.e., there is no extra sign change per 2π rotation due to the lower spin.

Now consider what happens when the two spin frustration pairs are on top of each other and one frustration executes a 2π rotation. Then we cut two spins and get $+1$. So it seems that one obtains a renormalization factor $1/1-C^4$ rather than $1/1+C^4$. This problem can be solved by a (path dependent) redefinition of the correlation function with a $-$ sign per collision of order-disorder pairs but it may be better to consider this problem in a different formalism.

Consider a Grassmannian action

$$S = \bar{\Psi}(1-\Gamma)\Psi = \sum_{xx'} \bar{\Psi}_x(1-\Gamma)_{xx'} \Psi_{x'} \quad (A1)$$

with x, x' some (possibly composite) index. One obtains the following equation of motion:

$$\left\langle e^{-S} \frac{d}{d\bar{\Psi}(x)} [\bar{\Psi}(y)e^S] \right\rangle = 0 \quad (A2)$$

where $\langle \rangle$ denotes the average with respect to the Grassmann integration and e^S . One obtains:

$$\begin{aligned} \langle \bar{\Psi}(y) \Psi(x) \rangle &= \Gamma_{xz} \langle \bar{\Psi}(y) \Psi(z) \rangle + \delta(x-y) \\ \langle \bar{\Psi}(x) \Psi(y) \rangle &= \Gamma_{zx} \langle \bar{\Psi}(z) \Psi(y) \rangle + \delta(x-y) \end{aligned} \quad (A3)$$

To make contact with the equations for order-disorder variables we now specify the convention that one pair is treated with the "left-handed" equations (9) and the other with the equations (9'). Therefore, we read off Γ from the first group of Eq. (9). Then ψ moves according to Γ while $\bar{\psi}$ moves according to $(\Gamma)^T$. But Γ^T just generates the equations of the opposite boundedness, i.e., the second group of Eq. (9).

Now, in these equations, there are no more long range effects of the type encountered previously due to lines of flipped bonds cutting through a far away spin. However, moving a spin frustration pair once around a second pair gave a $+$ sign. Therefore, we argue that there were really no long range effects. The lines of flipped bonds essentially provide a cut in the plane and define the physical sheet of the correlation functions. Sweeping through a far away spin just amounts to redefining the physical sheet and is therefore a harmless effect. In the Grassmann language it is now quite clear what happens when two spin frustration pairs sit at the same point. This just gives rise to an annihilation term from the right-hand side of Eq. (A3). In $d = 3$ we have not yet proven the exact equivalence of the Grassmann action with the order-disorder formalism but we want to describe this action nonetheless. In the action we have oriented plaquettes with frustrations on the side of the normal of the plaquette:

$$\begin{aligned} \mathcal{L}^{(4)} &= S \times \text{diagram of a square plaquette with arrows on its edges and a central dot} \\ &= S \cdot \psi_{\ell, \alpha_1} \dots \psi_{\ell, \alpha_4} \end{aligned} \tag{A4}$$

We also need a piece in the action that rotates the direction of the frustrations:

$$\mathcal{L}^{(2)} = C \sum_{\alpha} \bar{\psi}_{\ell, \alpha} \psi_{\ell, \alpha+1} \tag{A5}$$

where $\alpha, \alpha + 1$ denotes a pair of directions differing by a $\pi/2$ rotation in an appropriate screw direction. We also add

$$\mathcal{L}^{(0)} = - \sum_{\ell\alpha} \bar{\psi}_{\ell\alpha} \psi_{\ell\alpha} \quad (\text{A6})$$

to complete the action which now reads

$$\begin{aligned} S \stackrel{\text{modulo}}{=} \sum_{\text{phases}} & - \sum_{\ell\alpha} \bar{\psi}_{\ell\alpha} \psi_{\ell\alpha} + c \sum \bar{\psi}_{\ell\alpha} \psi_{\ell\alpha+1} \\ & + s \sum \psi_{\ell_1\alpha_1} \dots \psi_{\ell_4\alpha_4} \end{aligned} \quad (\text{A7})$$

The signs are (tentatively) fixed by regarding the $\psi_{\ell,\alpha}$ as double valued and introducing a cut to define the physical sheet. A loop variable is then

$$\psi_c = \prod_{i=1}^N \psi_{\ell_i\alpha_i} (-1)^{(K_i+q_i+z_i)} \quad (\text{A8})$$

where the last factor is needed to make ψ_c independent of its starting point. The derivation of the loop equations is then straightforward but the equivalence with the old loop equations still has to be proven.

REFERENCES

- 1) M. Kac and J. Ward, Phys. Rev. 88 (1952) 1332;
S. Sherman, Journal of Math. Phys. 1 (1960) 202;
P.N. Burgoyne, Journal of Math. Phys. 1 (1960) 202;
R.P. Feynman, Statistical Mechanics Lecture notes, Benjamin Press, Reading Mass. 1974.
L. Landau and E. Lifshitz, Statistical Physics, Pergamon Press (1969);
An analytic (as opposed to combinatorial) proof of the Kac and Ward method has been found by C. Itzykson (private communication).
- 2) L. Brink, S. Deser, P. Di Vecchia, P. Howe and B. Zumino, Phys. Lett. 64B (1976) 435;
F.A. Berezin and M.S. Marinov, Ann. of Physics 104 (1977) 336.
- 3) M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. D20 (1979) 1915.
- 4) F. Wegner, J. Math. Physics 12 (1971) 2259.
- 5) C. Domb in C. Domb and M.S. Green "Phase Transitions and Critical Phenomena" Academic Press, Vol. 3 (1975).
- 6) E. Fradkin, M. Srednicki and L. Susskind, Phys. Rev. D21 (1980) 2885;
J. Groeneveld, unpublished;
S. Samuel at the John Hopkins Workshop, Bonn (1980).
- 7) A. Polyakov, Quantum geometry of bosonic and fermionic strings, Phys. Lett. 103B (1981) 207 and 211.
- 8) L. Onsager, Phys. Rev. 65 (1944) 117;
Order-disorder variables are discussed in detail by
L. Kadanoff and H. Ceva, Phys. Rev. B3 (1971) 3918;
An elementary solution of the Ising model is also given in
E. Lieb, D.C. Mattis and T. Schultz, Rev. Mod. Phys. 36 (1964) 856.
- 9) A. Polyakov, Phys. Lett. 82B (1978) 247.
- 10) 't Hooft, Nucl. Phys. B138 (1978) 1.
- 11) D. Foerster, Nucl. Phys. B170 (FS1) (1980) 107.
- 12) V. Guillemin and A. Pollack, Differential Topology, Prentice Hall (1974), p. 149.

FIGURE CAPTIONS

- Fig. 1 : A string of flipped bonds corresponding to a disorder variable at n^* .
- Fig. 2 : Order disorder pairs at site n . The dots represent spins and the crosses frustrations. The curvy line indicates which bonds are flipped and the bold lines denote flipped bonds.
- Fig. 3 : Moving the string of flipped bonds into the standard configuration after flipping a bond.
- Fig. 4 : A typical path generated by iterating the equations of the $d = 2$ Ising model.
- Fig. 5 : A chain of flipped plaquettes in $d = 3$ Z_2 gauge theory.
- Fig. 6 : A Wegner-Wilson loop with associated frustrations in $d = 3$ Z_2 gauge theory.
- Fig. 7 : 2π rotations of a frustration around a link pointing in the z direction.
- Fig. 8 : Specifying an orientation for the loop and a convention for rotating the frustrations.
- Fig. 9 : Convention about extra frustrations associated with newly generated bits of string.
- Fig. 10 : Removing a corner plaquette in two different ways.
- Fig. 11 : Extra factors for bending a surface.
- Fig. 12 : Renormalization of a path on the spin lattice.
- Fig. 13 : Cutting a slit into a closed surface with the frustrations arranged in the direction of the surface normal.

- Fig. 14 : Phase factors for various closed surfaces.
- Fig. 15 : Interpreting the phase factor $(-)^{\ell}$ as the result of a 2 rotation per link of loop.
- Fig. 16 : a) The local geometry near an endpoint of self-intersection.
b) Hopf's vector field can be used in the construction of a frame.
- Fig. 17 : a) Apparent long range effects due to the motion of a line of frustrations.
b) An apparent difficulty with the renormalization prescription.

Fig. 1

$\psi_1(n)$

$\psi_2(n)$

$\psi_3(n)$

$\psi_4(n)$

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

(a)

(b)

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17