Landau-Ginzburg Models in Real Mirror Symmetry

Johannes Walcher *

CERN Physics Department, Theory Division
Geneva, Switzerland

Abstract
In recent years, mirror symmetry for open strings has exhibited some new connections between symplectic and enumerative geometry (A-model) and complex algebraic geometry (B-model) that in a sense lie between classical and homological mirror symmetry. I review the rôle played in this story by matrix factorizations and the Calabi-Yau/Landau-Ginzburg correspondence.


December 2010

*After January 1, 2011: McGill University, Montréal, Canada
We begin with a brief overview of the (pre-)history of the line of research reported here.

Mirror symmetry rose to prominence around 1990 after a computation by Candelas, de la Ossa, Green and Parkes [1]. These authors used the conjectural equivalence of \( \mathcal{N} = 2 \) superconformal field theories attached to mirror pairs \((X,Y)\) of Calabi-Yau manifolds (as well as some other information about the use of these manifolds for compactifying string theory) to make a prediction about the number of rational curves on a generic quintic threefold. That prediction was expressed in terms of the periods of an assumed mirror manifold previously constructed by Greene and Plesser [2].

The mathematical theory relevant to the computations on the mirror manifold ("B-model") was rapidly understood to be related to classical Hodge theory [3]. Based on the development of Gromov-Witten theory, the enumerative predictions ("A-model") were verified over subsequent years, culminating in the proof of the now classical "mirror theorems" [4–6],

\[
\text{Gromov-Witten theory of } X \xleftarrow{\text{solved by}} \text{Hodge theory of } Y \quad \quad (1.1)
\]

Of importance for the subject of this workshop is the parallel development of the so-called Calabi-Yau/Landau-Ginzburg correspondence. Originally proposed in [7], it was fully developed in [8] into an equivalence (in the sense that there exists a continuous
interpolation) between definitions of physical theories involving non-trivial geometries and non-trivial (critical) interactions. In the simplest case of hypersurfaces in projective space (where \( V \) is a homogeneous polynomial and \( \Gamma \) an appropriate discrete group)

non-linear sigma-model on \( X = \{V = 0\} \subset \mathbb{P}^4 \) related to Landau-Ginzburg orbifold \( V/\Gamma \)

(1.2)

In 1994, M. Kontsevich proposed to understand the mathematical origin of mirror symmetry from an underlying equivalence of triangulated categories [9],

\[
\text{Fuk}(X) \cong D^b(Y)
\]

(1.3)

In 2000, K. Hori and C. Vafa [10] (see [11] for earlier work) gave a physical proof of mirror symmetry as an equivalence

\[
\text{Sigma-model on } X \xleftrightarrow{\text{dual to by}} \text{Landau-Ginzburg orbifold } W/\Gamma
\]

(1.4)

by exploiting the linear sigma model underlying (1.2) (Note that this correspondence is different from (1.2), which is operating on the same side of the mirror symmetry. The discrete group in (1.4) is in general different from the one in (1.2).)

In 2002, M. Kontsevich made a proposal for the D-brane category that would underly the correspondence (1.2) (in the B-model) and could also be used on one side of homological mirror symmetry: The category of matrix factorizations [12]. Essentially

\[
D^b(Y) \cong \text{MF}(W/\Gamma)
\]

(1.5)

This proposal was picked up by D. Orlov [13], and by two groups of physicists [14,15].

I became interested in matrix factorizations around that time. Together with K. Hori, I wrote a few papers [16–18] exploring the possibility of using (1.5) for concrete computations in mirror symmetry with D-branes. (The physicists' intuition being that computations in Landau-Ginzburg models are much simpler than those in the derived category. But there is another side to that coin.) In particular, we proposed in [17] a mirror conjecture relating the set of 625 real quintics (as objects in the Fukaya category) to a set of 625 matrix factorizations of the mirror Landau-Ginzburg superpotential. This will be reviewed below.

The paper [19], which is of especial relevance for this talk, contains three main results. (i) The detailed definition of the right hand side of (1.5), and especially a definition of the important \( \mathbb{Z} \)-grading. (ii) An index theorem for matrix factorizations
in the framework of (1.5), including a formula for the Chern character. (iii) A discussion of the notion of stability for the category of matrix factorizations, including a formula for the central charge (an additive complex function whose argument is the slope).

The equivalence (1.5) was proven by D. Orlov [20]. An index theorem for matrix factorizations was also proven by D. van Straten [21], and more recently in [22]. The formula for the central charge plays a rôle in [23] and subsequent work. But the definition of a stability condition is not yet complete. For a relevant discussion of geometric invariant theory involving non-reductive group actions, see [24].

The main evidence for the conjecture of [17] appeared in 2006 [25]. My interest in the problem was revived after J. Solomon computed the number of real lines on the quintic [26]. Using a heuristic route analogous to that orginally employed in [1], I was then able to predict the number of real rational curves in all degrees. We verified the enumerative predictions in [27], and explained the B-model computation in [28], so that the result of [25] can now be stated as a real mirror correspondence, generalizing (1.1) ¹

real Gromov-Witten theory of \((X, L)\) normal function attached to \((Y, C)\)

(1.6)

In the rest of this talk, I will describe the various ingredient in the open string mirror correspondence (1.6). In the final section, I will describe a few consequences that one might draw for the (possible) rôles of matrix factorizations in mirror symmetry.

Remark. As in the review [29], I will here concentrate on progress made on open mirror symmetry on compact Calabi-Yau threefolds. For a review of the progress on non-compact manifolds, see [30]. Landau-Ginzburg models and matrix factorizations also play a rôle in mirror symmetry for non-Calabi-Yau manifolds, some of which was discussed at other talks in this conference.

2 A-model

Our interest is concentrated on the quintic Calabi-Yau \(X = \{ V = 0 \} \subset \mathbb{P}^4\), defined as the vanishing locus of a homogeneous degree 5 polynomial \(V\) in 5 complex variables \(x_1, \ldots, x_5\). We assume that \(X\) is defined over the reals, which means that all coefficients

¹Here, \(L\) is the real quintic, and \(C \subset Y\) is an algebraic cycle in the mirror family, see below for the details.
of $V$ are real (possibly up to some common phase). The real locus $\{x_i = \bar{x}_i\} \subset X$ is then a Lagrangian submanifold with respect to the standard symplectic structure, and after choosing a flat $U(1)$ connection, will define an object in the (derived) Fukaya category $Fuk(X)$.

Both the topological type and the homology class in $H_3(X;\mathbb{Z})$ of the real locus depend on the complex structure of $X$ (the choice of (real) polynomial $V$). On the other hand, the Fukaya category is independent of the choice of $V$ (real or not). The object in $Fuk(X)$ that we shall refer to as the real quintic is defined from the real locus $L$ of $X$ when $V$ is the Fermat quintic $V = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$. It is not hard to see that topologically, $L \cong \mathbb{R}P^3$. There are therefore two choices of flat bundles on $L$, and we will denote the corresponding objects of $Fuk(X)$ by $L_+$ and $L_-$, respectively. More precisely, since $Fuk(X)$ depends on the choice of a complexified Kahler structure on $X$, we define $L_\pm$ for some choice of Kahler parameter $t$ close to large volume $\text{Im}(t) \to \infty$, and then continue it under Kahler deformations. In fact, the rigorous definition of the Fukaya category is at present only known infinitesimally close to this large volume point [31]. However, $Fuk(X)$ does exist over the entire stringy Kahler moduli space of $X$, and at least some of the structure varies holomorphically. Our interest here is in the variation of the categorical structure associated with $L_\pm$ over the entire stringy Kahler moduli space of $X$, identified via mirror symmetry with the complex structure moduli space of the mirror quintic, $Y$.

The Fermat quintic is invariant under more than one anti-holomorphic involution. If $\mathbb{Z}_5$ denotes the multiplicative group of fifth roots of unity, we define for $\chi = (\chi_1, \ldots, \chi_5) \in (\mathbb{Z}_5)^5$ an anti-holomorphic involution $\sigma_\chi$ of $\mathbb{P}^4$ by its action on homogeneous coordinates

$$\sigma_\chi : x_i \mapsto \chi_i \bar{x}_i. \quad (2.1)$$

The Fermat quintic is invariant under any $\sigma_\chi$. The involution and the fixed point locus only depend on the class of $\chi$ in $(\mathbb{Z}_5)^5/\mathbb{Z}_5 \cong (\mathbb{Z}_5)^4$, and we obtain in this way $5^4 = 625$ (pairs of) objects $L_\pm^{[\chi]}$ in $Fuk(X)$.

We emphasize again that although we have defined the Lagrangians $L_\pm^{[\chi]}$ as fixed point sets of anti-holomorphic involutions of the Fermat quintic, we can think of the corresponding objects of $Fuk(X)$ without reference to the complex structure.

5
3 B-model

Let $V \in \mathbb{C}[x_1, x_2, \ldots, x_5]$ be a polynomial. A *matrix factorization* of $V$ is a $\mathbb{Z}_2$-graded free $\mathbb{C}[x_1, \ldots, x_5]$-module $M$ equipped with an odd endomorphism $Q : M \to M$ of square $V$,

$$Q^2 = V \cdot \text{id}_M \quad (3.1)$$

The category $\text{MF}(V)$ is the triangulated category of matrix factorizations with morphisms given by $Q$-closed morphisms of free modules, modulo $Q$-exact morphisms. Matrix factorizations are well-known objects since the mid '80's, see in particular [12], and it was proposed by Kontsevich that $\text{MF}(V)$ should be a good description of B-type D-branes in a Landau-Ginzburg model based on the worldsheet superpotential $V$ [13–15]. To apply this to the case of interest, we need a little bit of extra structure.

When $V$ is of degree 5, the so-called homological Calabi-Yau/Landau-Ginzburg correspondence [20] states that the derived category of coherent sheaves of the projective hypersurface $X = \{V = 0\} \subset \mathbb{P}^4$ is equivalent to the graded, equivariant category of matrix factorizations of the corresponding Landau-Ginzburg superpotential,

$$D^b(X) \cong \text{MF}(V/\mathbb{Z}_5) \quad (3.2)$$

where $\mathbb{Z}_5$ is the group of 5-th roots of unity acting diagonally on $x_1, \ldots, x_5$. To pass to the mirror quintic $Y$ by the standard Greene-Plesser construction, we replace $V$ with the one-parameter family of potentials $W$ given by

$$W = \frac{1}{5}(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5) - \psi x_1 x_2 x_3 x_4 x_5 \quad (3.3)$$

and enlarge the orbifold group to obtain (1.5)

$$D^b(Y) \cong \text{MF}(W/(\mathbb{Z}_5)^4) \quad (3.4)$$

where $(\mathbb{Z}_5)^4 = \text{Ker}(\mathbb{Z}_5^5 \to \mathbb{Z}_5)$ is the subgroup of phase symmetries of $W$ whose product is equal to 1.

To describe an object mirror to the real quintic, we begin with finding a matrix factorization of the one-parameter family of polynomials (3.3). If $S \cong \mathbb{C}^5$ is a 5-dimensional vector space, we can associate to its exterior algebra a $\mathbb{C}[x_1, \ldots, x_5]$-module $M = \wedge^* S \otimes \mathbb{C}[x_1, \ldots, x_5]$. It naturally comes with the decomposition

$$M = M_0 + M_1 + M_2 + M_3 + M_4 + M_5, \quad \text{where} \ M_5 = \wedge^5 S \otimes \mathbb{C}[x_1, \ldots, x_5], \quad (3.5)$$
and the $\mathbb{Z}_2$-grading $(-1)^*$. Let $\eta_i$ ($i = 1, \ldots, 5$) be a basis of $S$ and $\bar{\eta}_i$ the dual basis of $S^*$, both embedded in $\text{End}(M)$. We then define two families of matrix factorizations $(M, Q_{\pm})$ of $W$ by

$$Q_{\pm} = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \left( x_i^2 \eta_i + x_i^3 \bar{\eta}_i \right) \pm \sqrt{5} \prod_{i=1}^{5} (\eta_i - x_i \bar{\eta}_i)$$

(3.6)

To check that $Q_{\pm} = W \cdot \text{id}_M$, one uses that $\eta_i, \bar{\eta}_i$ satisfy the Clifford algebra

$$\{ \eta_i, \eta_j \} = \delta_{ij}$$

(3.7)

as well as the ensuing relations

$$\{(x_i^2 \eta_i + x_i^3 \bar{\eta}_i), (\eta_i - x_i \bar{\eta}_i)\} = 0 \quad \text{and} \quad (\eta_i - x_i \bar{\eta}_i)^2 = -x_i$$

(3.8)

The matrix factorization (3.6) is quasi-homogeneous ($\mathbb{C}^*$-gradable), but we will not need this data explicitly.

Now to specify objects in $\text{MF}(W/\Gamma)$, where $\Gamma = \mathbb{Z}_5$ or $\mathbb{Z}_5^4$ for the quintic and mirror quintic, respectively, we have to equip $M$ with a representation of $\Gamma$ such that $Q_{\pm}$ is equivariant with respect to the action of $\Gamma$ on the $x_i$. Since $Q_{\pm}$ is irreducible, this representation of $\Gamma$ on $M$ is determined up to a character of $\Gamma$ by a representation on $S$, i.e., an action on the $\eta_i$. For $\gamma \in \Gamma$, we have $\gamma(x_i) = \gamma_i x_i$ for some fifth root of unity $\gamma_i$. We then set $\gamma(\eta_i) = \gamma_i^{-2} \eta_i$, making $Q_{\pm}$ equivariant. As noted, this representation is unique up to an action on $M_0$, i.e., a character of $\Gamma$.

For the mirror quintic, $\Gamma = \text{Ker}(\mathbb{Z}_5) \rightarrow \mathbb{Z}_5$, so $\Gamma^* = (\mathbb{Z}_5)^5 / \mathbb{Z}_5$, and we label the characters of $\Gamma$ as $[\chi]$. The corresponding objects of $\text{MF}(W/\Gamma)$ constructed out of $Q_{\pm}$ (3.6) are classified as $Q_{\pm}^{[\chi]} = (M, Q_{\pm}, \rho_{[\chi]})$, where $\rho_{[\chi]}$ is the representation on $M$ we just described.

By using the explicit algorithm of [32], one may obtain representatives of the matrix factorizations $Q_{\pm}^{[\chi]}$ in $D^b(Y)$, which it would be interesting to analyze them further. A particularly nice one is the bundle

$$\text{Ker} \left( \begin{array}{c} M_0(2) \\ \oplus M_2(1) \\ \oplus M_4(0) \end{array} \right) \xrightarrow{Q_{\pm}} \left( \begin{array}{c} M_1(4) \\ \oplus M_3(3) \\ \oplus M_5(2) \end{array} \right)$$

(3.9)

where $M_\chi(k) = \Lambda^\chi S \otimes \mathcal{O}_M(k)$. See [28] for details.
4 Correspondence

Conjecture [17]. There is an equivalence of categories $\text{Fuk}(X) \cong \text{MF}(W/(\mathbb{Z}_5)^4)$ which identifies the 625 pairs of objects $L_{\pm}^{[x]}$ with the 625 pairs of equivariant matrix factorizations $Q_{\pm}^{[x]}$.

Since (as follows from the calculation of the intersection indices below), the images of the 625 objects in K-theory generate the lattices of topological charges of the respective categories, it is natural to propose that, in an appropriate sense, the $L_{\pm}^{[x]}$ generate $\text{Fuk}(X)$ and the $Q_{\pm}^{[x]}$ generate $\text{MF}(W/(\mathbb{Z}_5)^4)$. In combination with the Conjecture (and the equivalence (1.5)), this would establish homological mirror symmetry for the pair (quintic, mirror quintic). Similar statements should also hold for other Calabi-Yau hypersurfaces or complete intersections, although the details would be different (possibly substantially so).

5 Evidence 1

The first main evidence for the Conjecture is the identity of intersection indices, originally due to [33].

Let us start with the geometric intersection index between 2 $L^{[x]}$ and $L^{[x]}$. Because of the projective equivalence, we have to look at the intersection of the fixed point loci of $\sigma_x$ and $\sigma_{x'}$ from (2.1) where $\omega$ runs over the 5 fifth roots of unity. It is not hard to see that topologically

$$\text{Fix}(\sigma_x) \cap \text{Fix}(\sigma_{x'}) \cap X \cong \mathbb{R}^{d-2}, \quad \text{where } d = \# \{ x_i = \omega x_i \}. \quad (5.1)$$

After making the intersection transverse by a small deformation in the normal direction, we obtain a vanishing contribution for $d = 0, 1, 3, 5,$ and $\pm 1$ for $d = 2, 4$, where the sign depends on the non-trivial phase differences $x_i^* \omega x_i^*$. Explicitly, one finds

$$L^{[x]} \cap L^{[x']} = \sum_{\omega \in \mathbb{Z}_5} f_i(\chi^* \omega \chi), \quad (5.2)$$

where

$$f_i(\chi) = \begin{cases} \prod_{i=1}^{5} \text{sgn}(\text{Im}(\chi_i)), & \text{if } \# \{ i, \chi_i = 1 \} = 2, 4 \\ 0, & \text{else.} \end{cases} \quad (5.3)$$

---

$^2$The intersection index, being topological, does not depend on the Wilson lines on the A-branes. For the B-branes, it is correspondingly independent of the sign of the square root in (3.6).
To compute the intersection index between the matrix factorizations, we use the index theorem of [19]. It says in general

$$\chi_{\text{Hom}}((M, Q, \rho), (M', Q', \rho')) := \sum_i (-1)^i \dim \text{Hom}^i((M, Q, \rho), (M', Q', \rho')) =$$

$$= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Str}_{M', \rho'}(\gamma)^* \frac{1}{\prod_{i=1}^5 (1 - \gamma_i)} \text{Str}_{M, \rho}(\gamma), \quad (5.4)$$

where $\gamma_i$ are the eigenvalues of $\gamma \in \Gamma$ acting on the $x_i$, and $\rho, \rho'$ are the representations of $\Gamma$ on $M$. For $M = M', Q = Q'$ and $\rho = \rho_{|x}; \rho' = \rho_{|x'}$ described above, this evaluates to

$$-\frac{1}{5^4} \sum_{\gamma \in (\mathbb{Z}_5)^4} \chi(\gamma)^* \chi(\gamma) \prod_{i=1}^5 (\gamma_i + \gamma_i^2 - \gamma_i^3 - \gamma_i^4) = -\sum_{\omega \in \mathbb{Z}_5} f_2(\chi^* \omega \chi), \quad (5.5)$$

where

$$f_2(\chi) = \begin{cases} \prod_{i=1}^5 \text{sgn}(\text{Im}(\chi_i)), & \text{if } \#\{i, \chi_i = 1\} = 0 \\ 0 & \text{else.} \end{cases} \quad (5.6)$$

We do not know any generally valid result from the representation theory of finite cyclic group which shows that (5.2) and (5.5) coincide. It is however not hard to check by hand or computer that for all $\chi$,

$$\sum_{\omega \in \mathbb{Z}_5} (f_1 + f_2)(\omega \chi) = 0. \quad (5.7)$$

Hence

$$L^{[\chi]} \cap L^{[\chi']} = \chi_{\text{Hom}}(Q^{[\chi]}, Q^{[\chi']}) \quad (5.8)$$

as claimed. A further computation shows that the rank of the $625 \times 625$ dimensional intersection matrix (5.2) is 204, which is equal to the rank of $H_3(X; \mathbb{Z})$, and the determinant is one. So the classes of the $L^{[\chi]}$ generate the homology, as claimed above.

### 6 Evidence 2

Consider the endomorphism algebra $\text{Hom}^*(Q, Q)$ of the matrix factorization $Q = Q_+$, as objects in $\text{MF}(W/(\mathbb{Z}_5)^4)$. This algebra is $\mathbb{Z}$-graded thanks to the homogeneity of $W$ [19]. We also have $\text{Hom}^0(Q, Q) \cong \mathbb{C}$ since $Q$ is irreducible, and this implies $\text{Hom}^3(Q, Q) \cong \mathbb{C}$ by Serre duality. Finally, it is shown in [17] that

$$\text{Hom}^1(Q, Q) = \text{Hom}^2(Q, Q) = \begin{cases} 0 & \psi \neq 0 \\ \mathbb{C} & \psi = 0 \end{cases} \quad (6.1)$$
The appearance of an additional cohomology element in $\text{Hom}^1(Q, Q)$ is another reflection of some results of [33] and was the initial motivation to investigate mirror symmetry for the real quintic.

To interpret (6.1) in the A-model, we recall that the morphism algebra of objects in the Fukaya category is defined using Lagrangian intersection Floer homology [31]. For the endomorphism algebra of a single Lagrangian, Floer homology is essentially a deformation of ordinary Morse homology by holomorphic disks.

For example, consider the real quintic $L \cong \mathbb{R}P^3$. Think of $\mathbb{R}P^3$ as $\mathbb{C}P^3 / \mathbb{Z}_2$, and embed the $S^3$ in $\mathbb{R}^4$ as $(y_0, \ldots, y_3)$ as $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$. A standard Morse function for $\mathbb{R}P^3$ in this presentation is given by $f = y_0^2 + 2y_3^2$ restricted to the $S^3$. This Morse function is self-indexing and has one critical point in each degree $i = 0, 1, 2, 3$. The Morse complex takes the form

$$C^0 \xrightarrow{0} C^1 \xrightarrow{\delta} C^2 \xrightarrow{0} C^3$$

(6.2)

Working with integer coefficients, $C^i \cong \mathbb{Z}$ for all $i$, we have $\delta = 2$, and the complex (6.2) computes the well-known integral cohomology of $\mathbb{R}P^3$.

To compute Floer homology of the real quintic, we have to deform (6.2) by holomorphic disks, i.e., $\delta = 2 + O(e^{-\epsilon/2})$. In the standard treatments, such as [31], this requires taking coefficients from a certain formal (Novikov) ring with uncertain convergence properties. In other words, Floer homology is at present only defined in an infinitesimal neighborhood of the large volume point in moduli space (which leads to the often heard remark that $HF^*(L, L)$ is isomorphic to $H^*(L)$). It is however natural to expect that we may in fact analytically continue (6.2) to the opposite end of moduli space, $\psi = 0$, where it is (conjecturally) identified with the deformation complex of $Q$. In particular, we conjecture $\delta(\psi = 0) = 0$ so as to reproduce (6.1).

### 7 “Proof”

The original construction of mirror symmetry by Greene and Plesser [2] exploited the fact that at $\psi = 0$, we may reduce the equivalence between the Fermat quintic $\{\sum x_i^5 = 0\}$ and the mirror quintic at $\psi = 0$, the LG orbifold $\sum x_i^5 / (\mathbb{Z}_5)^4$, to the equivalence between minimal models

$$\text{Landau-Ginzburg model } x^5 \xrightarrow{\text{mirror to}} \text{Landau-Ginzburg orbifold } x^5 / \mathbb{Z}_5$$

(7.1)
This equivalence has been explained many times in the literature (ultimately, it is a special case of Arnold's strange duality). At the level of D-branes, the correspondence identifies the building blocks of (3.6) (at $\psi = 0$), namely the factorizations

$$x^5 = x^2 \cdot x^3$$

(7.2)

with the vanishing cycles of opening angle $2\pi/5$ sketched in the figure. As can be seen, these vanishing cycles are the closest to the real slice as they can be, and the topological charges agree. (This is indicated by the shaded region in the figure. See e.g., [34] for details.)

An orbifold of the fifth tensor product of this correspondence leads to the identification between (3.6) and the real quintic. To make these observations into an actual proof, one would need to explain precisely the deviation between the real slice and the vanishing cycle. This should, again, be a consequence of holomorphic disks, or rather their Landau-Ginzburg analogue, perhaps in the framework of Fan-Jarvis-Ruan theory [35].

8 Main evidence

The main evidence for the above conjecture is the enumeration of holomorphic disks using mirror symmetry [25,27,28].

We introduce the generating function of open Gromov-Witten invariants for the pair $(X, L) = (\text{quintic}, \text{real quintic})$ defined in [26].

$$T_A(q) = \log q + \frac{1}{4} + \sum_{d \text{ odd}} \tilde{n}_d q^{d/2}$$

(8.1)

where $d$ indexes the degree of the holomorphic disk. One may compute the $\tilde{n}_d$ similarly to the ordinary ($g = 0$) Gromov-Witten invariants by localization on the moduli space of maps from the disk to $\mathbb{P}^4$. We then pull back $T_A(q)$ under the standard mirror map

$$q = q(z) = \exp(\omega_1(z)/\omega_0(z))$$

(8.2)

where $\omega_0$ and $\omega_1$ are the power series and first logarithmic around $z = 0$ solutions of the Picard-Fuchs differential operator

$$\mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4), \quad \theta = \frac{d}{dz}, \quad z = (5\psi)^{-5}$$

(8.3)
Namely, we define

$$T_B(z) = \omega_0(z)T_A(q(z))$$  \hspace{1cm} (8.4)$$

Then the main result of [27] is the inhomogeneous Picard-Fuchs equation

$$\mathcal{L}T_B(z) = \frac{15}{8}\sqrt{z}$$  \hspace{1cm} (8.5)$$

As shown in [28], one can identify $T_B(z)$ as a particular truncated normal function\(^3\) associated with the algebraic cycle $C_+ - C_-$, where

$$C_\pm = \{x_1 + x_2 = 0, x_3 + x_4 = 0, x_5^2 \pm \sqrt{5}\psi x_1 x_2 = 0\} \subset \{W = 0\}$$  \hspace{1cm} (8.6)$$

are two families of curves in the mirror quintic. Namely, we have

$$T_B = T_B(z) = \int_\Gamma \hat{\Omega}$$  \hspace{1cm} (8.7)$$

where $\Gamma$ is a particular three-chain bounding $C_+ - C_-$, and $\hat{\Omega}$ is a particular choice of holomorphic three-form on $Y$, defined as Poincaré residue by the formula

$$\hat{\Omega} = \left(\frac{5}{2\pi i}\right)^3 \psi \text{Res}_{W=0} \frac{\alpha}{W}$$  \hspace{1cm} (8.8)$$

where $\alpha$ is the four-form on projective space

$$\alpha = \sum_i (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_5$$  \hspace{1cm} (8.9)$$

The choice of holomorphic three-form in (8.8) is precisely the one for which the Picard-Fuchs equation of the mirror quintic takes the form (8.3).

So finally, the equivalence (8.4) is evidence for the Conjecture because the cycles $C_\pm$ provide representatives of the second algebraic Chern class of our matrix factorizations $Q_\pm$. In general, Grothendieck’s theory of Chern classes provides a map

$$c_2^{\text{alg}} : D^b(Y) \to \text{CH}^1(Y)$$  \hspace{1cm} (8.10)$$

from the derived category of coherent sheaves to the Chow groups of algebraic cycles modulo rational equivalence. Composing this map on one side with the equivalence $D^b(Y) \cong \text{MF}(W/(\mathbb{Z}_\delta)^4)$, and on the other with the Abel-Jacobi map, we obtain the construction of a truncated normal function starting from a virtual matrix factorization of zero topological charge. In particular [28]

$$c_2^{\text{alg}}(Q_+) - c_2^{\text{alg}}(Q_-) = [C_+ - C_-] \in \text{CH}^2(Y)$$  \hspace{1cm} (8.11)$$

\(^3\)We recall the definition below.
9 Consequences, Infinitesimal invariant

What are the lessons of all of this for the geometry and physics of Landau-Ginzburg models?

First of all, there are a few lose ends to tie up in what is known already. For example, what is the proper definition of the Chow group of algebraic cycles that we mentioned at the end of the previous section? (Such a definition might already exist in general abstraction, unbeknownst to me.)

Secondly, it seems quite unfortunate that one has to go through a fairly long chain of correspondences, and a somewhat delicate geometric computation to arrive at the rather simple inhomogeneity in (8.5). In [36], I show that in fact there does exist a shortcut, along the following lines.

Recall that the variation of Hodge structure associated with the family of mirror quintics \( Y \rightarrow B \) is concerned with the variation of the Hodge decomposition

\[
H^3(Y; \mathbb{C}) = H^{3,0}(Y) \oplus H^{2,1}(Y) \oplus H^{1,2}(Y) \oplus H^{0,3}(Y)
\]

of the third cohomology group \( H^3(Y; \mathbb{C}) \cong H^3(Y; \mathbb{Z}) \otimes \mathbb{C} \) with \( z = (5\psi)^{-5} \in B \). After forming the Hodge filtration

\[
F^p H^3(Y) = \bigoplus_{p' \geq p} H^{p',3-p'}(Y)
\]

we may write the important condition of Griffiths transversality of the VHS as

\[
\nabla F^p H^3(Y) \subset F^{p-1} H^3(Y) \otimes \Omega_B
\]

where \( \nabla \) is the flat ("Gauss-Manin") connection originating from the local triviality of \( H^3(Y; \mathbb{Z}) \) over the moduli space \( B \).

Moreover, we have the Griffiths intermediate Jacobian fibration which is the fibration \( J^3(Y) \rightarrow B \) of complex tori

\[
J^3(Y) = \frac{H^3(Y)}{F^2 H^3(Y) \oplus H^3(Y; \mathbb{Z})} \cong (F^2 H^3(Y))^*/H_3(Y; \mathbb{Z})
\]

Then, a Poincaré normal function of the variation of Hodge structure is a holomorphic section \( \nu \) of \( J^3(Y) \) satisfying Griffiths transversality for normal functions

\[
\nabla \nu \in F^1 H^3(Y) \otimes \Omega_B
\]
where \( \tilde{\nu} \) is an arbitrary lift of \( \nu \) from \( J^3(Y) \) to \( H^3(Y) \) (the condition (9.5) does not depend on the lift).

Finally, we recall the notion of the infinitesimal invariant of a normal function (see [37] for details). In our context, this invariant is nothing but the open string version of the well-known Yukawa coupling, \( \kappa \), which is a section of \( (F^3H^3)^{-2} \otimes \text{Sym}^3\Omega_B \to B \) arising from the third iterate of the differential period mapping,

\[
H^1(TY) \to \bigoplus \text{Hom}(H^{p,q}(Y), H^{p-1,q+1}(Y)).
\]  

(9.6)

Namely, by choosing a lift \( \tilde{\nu} \) of \( \nu \) form \( J^3(Y) \) to \( H^3(Y) \), we may contract \( \nabla \tilde{\nu} \) with \( F^2H^3(Y) \otimes \Omega_B \), and use (9.5) to obtain a section \( \delta \) of \( (F^3H^3)^{-1} \otimes \text{Sym}^2\Omega_B \to B \). Note that in contrast to (9.5), this definition does depend on the lift. [The usual definition of the infinitesimal invariant takes the class of \( \delta \) in \( H^1 \) of the Koszul complex]

\[
F^2H^3 \to F^1H^3 \otimes \Omega_B \to F^0H^3 \otimes \Omega_B^2,
\]  

(9.7)

which is independent of the lift.]

We are now nearing the punchline of the shortcut to (8.5). By contraction with your choice \( \Omega \) of section of \( F^3H^3 \), the Yukawa coupling is usually written as

\[
\kappa = \int_Y \Omega \wedge \nabla^3\Omega
\]  

(9.8)

which has a "Landau-Ginzburg" equivalent (written here up to normalization)

\[
\kappa_{ijk} = \text{Res} \frac{\partial_i W \partial_j W \partial_k W}{(dW)^5}.
\]  

(9.9)

To write a similar formula for the infinitesimal invariant, we have to specify a lift of the normal function. I have studied two distinguished lifts. The first is the so-called real lift [38], which exploits the (non-holomorphic) splitting (9.1) of the Hodge filtration, and in which the infinitesimal invariant satisfies the interesting equation

\[
\bar{\delta}_k \delta_{ij} = -\kappa^{\text{L}}_{ijk} \bar{\delta}_{ik}
\]  

(9.10)

The other possibility is what I want to call the "Landau-Ginzburg lift", in which we have a formula similar to (9.9)

\[
\delta_{ij} = \text{Res} \frac{\text{Str}(\partial_i W \partial_j Q(dQ)^5)}{(dW)^5}
\]  

(9.11)
in terms of the residue for matrix factorizations introduced in [39].

To finish up, I think it is fair to say that the Landau-Ginzburg formulation of D-brane categories has served very well to develop intuition, but that several quantitative details are still better understood in the geometric description. This is true in A- and B-model, and among physicists and mathematicians. All can benefit from filling in the details of the correspondence.

References
