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AND AMBIGUITIES OF QUANTIZATION:
A PARTICLE IN A MAGNETIC FIELD

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Abstract.

New Hamiltonians yielding classical equations of motion for a charged particle in a uniform magnetic field are found. It is shown explicitly that after the canonical quantization the new Hamiltonians describe different quantum systems which have no in common with the usual quantum particle in a magnetic field, because the spectra and measurability properties of the same conserved observables (kinetic energy, generalized angular momentum, etc.) for each system are quite different. The solutions and propagators of the Schrödinger equation with the new Hamiltonians are obtained.

1. Introduction.

It is well known nowadays (see, e.g., [1-5] and references therein) that a given system of the second order classical equations of motion (Newton's equations) can be derived either from an infinite number of different equivalent Lagrangians (or Hamiltonians) dependent on coordinates, velocities, and time, or from no one. In the case of the systems of the first type (nonessentially nonselfadjoint systems [4]) the differences between equivalent Lagrangians by no means can be exhausted by trivial additions of total time derivatives. Therefore it is quite natural to pose the following question: whether quantum systems arising after a quantization (canonical or Feynman's, for example) of different but classically equivalent Hamiltonians or Lagrangians are physically equivalent or not? This question was discussed somehow both in the mentioned papers and in some others [6-13]. All the authors arrive at the conclusion that after the quantization one would obtain physically different systems; however, there are only a few papers in which this problem was investigated sufficiently completely and convincingly. Concretely, only the following systems were considered in this connection till now: a free particle [9], a one-dimensional motion with a linear friction [11-12], and a two-dimensional isotropic oscillator [10]. The first and the second systems were investigated also in refs. [5, 13], in which it was shown explicitly that different classically equivalent Lagrangians (or Hamiltonians) lead to essentially different (thus nonequivalent) evolution operators (propagators of the Schrödinger equation) in the cases of a free particle,

a motion with a linear friction, and a particle in a magnetic field.

In this paper we analyze in detail the ambiguity of quantization arising due to the existence of different classically equivalent Hamiltonians in the case of a charged particle moving in a uniform magnetic field. This case has certain advantages over that mentioned above. Firstly, the system is not too simple, because it is essentially two-dimensional, while the systems considered in refs. [9-12] were in fact one-dimensional, since the equations of motion for different degrees of freedom (coordinates) were uncoupled. But it is well-known [1,3-6] that the problem of finding Lagrangians leading to the given equations of motion in the multidimensional case differs in principle from the same problem in the one-dimensional case. Secondly, the model discussed in this paper relates to the well-studied real physical system, so that the predictions of the different theories can be, in principle, compared with the experimental data. Besides, the equivalent Lagrangians studied in the paper are nonsingular, unlike, e.g., the case of the motion with friction [5,11-13]; thus our analysis is quite correct from the mathematical point of view.

We will obtain new Hamiltonians yielding the usual classical equations of motion of a particle in a uniform magnetic field, but the reader will see by his own eyes that after the canonical quantization these Hamiltonians will describe quantum systems which have no in common with the usual quantum particle in a magnetic field. Therefore we think that this is one of the most convincing examples illustrating the nonequ-

ivalence of quantum systems corresponding to different but classically equivalent Lagrangians or Hamiltonians.

2. Classically equivalent Hamiltonians.

The motion of a charged particle in the plane perpendicular to the uniform constant magnetic field is described in the classical case by the following equations:

$$(1) \quad \ddot{x} = \omega \dot{y}; \quad \ddot{y} = -\omega \dot{x}; \quad \omega = \text{const}$$

The standard Lagrangian yielding these equations is well-known:

$$(2) \quad \mathcal{L}_0 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\omega}{2}(x\dot{y} - \dot{x}y)$$

To find another Lagrangian yielding the same equations of motion let us consider a generalization of eqs.(1) to the case when the friction force proportional to the velocity is present:

$$(3) \quad \begin{aligned} \ddot{x} + \gamma \dot{x} - \omega \dot{y} &= 0 \\ \ddot{y} + \gamma \dot{y} + \omega \dot{x} &= 0 \end{aligned}$$

Eqs.(3) in the form which they are written in cannot be derived from any Lagrangian, since the Helmholtz conditions [1,4] for them are not fulfilled (see Appendix). However, if one multiplies these equations by appropriate integrating factors, then new equations (equivalent to (3)) can be obtained from a Lagrangian. To find it, let us rewrite the system of real equations (3) as follows,

$$(4) \quad \ddot{z} + \bar{\gamma} \dot{z} = 0; \quad z = x + iy; \quad \bar{\gamma} = \gamma + i\omega$$

Equivalent Lagrangians for equation (4) were studied in many papers. The most simple of them is the Bateman-Caldirola Lagrangian [14,15]

$$(5) \quad \tilde{\mathcal{L}} = \frac{1}{2} e^{\gamma t} \dot{z}^2$$

Taking real and imaginary parts of $\tilde{\mathcal{L}}$, one obtains two real Lagrangians yielding equations equivalent to (3):

$$(6) \quad \tilde{\mathcal{L}}_1 = e^{\gamma t} \left[\dot{x} \dot{y} \cos \omega t + \frac{1}{2} (\dot{x}^2 - \dot{y}^2) \sin \omega t \right]$$

$$(7) \quad \tilde{\mathcal{L}}_2 = e^{\gamma t} \left[\frac{1}{2} (\dot{x}^2 - \dot{y}^2) \cos \omega t - \dot{x} \dot{y} \sin \omega t \right]$$

Lagrangian (7) was found for the first time in ref. [16]. It can be transformed to (6) by means of the rotation of the coordinate axes:

$$(8) \quad \begin{aligned} \tilde{\mathcal{L}}_2(x, y, t) &= \tilde{\mathcal{L}}_1(x', y', t) \\ x &= \frac{1}{\sqrt{2}} (x' + y'); \quad y = \frac{1}{\sqrt{2}} (y' - x') \end{aligned}$$

Therefore we consider further Lagrangian $\tilde{\mathcal{L}}_1$ only. Hamiltonian corresponding to it is as follows,

$$(9) \quad \tilde{\mathcal{H}}_1 = e^{-\gamma t} \left[p_x p_y \cos \omega t + \frac{1}{2} (p_x^2 - p_y^2) \sin \omega t \right]$$

In ref. [17] the following Hamiltonian was proposed to describe the motion with friction in a magnetic field:

$$(10) \quad \mathcal{H}' = \frac{1}{2} e^{-\gamma t} \left\{ \left(p_x + \frac{\omega}{2} y e^{\gamma t} \right)^2 + \left(p_y - \frac{\omega}{2} x e^{\gamma t} \right)^2 \right\}$$

However, one can easily check that this Hamiltonian yields not equations (3) but the following ones:

$$(11) \quad \begin{aligned} \ddot{x} + \gamma \left(\dot{x} - \frac{\omega}{2} y \right) - \omega y &= 0 \\ \ddot{y} + \gamma \left(\dot{y} + \frac{\omega}{2} x \right) + \omega x &= 0 \end{aligned}$$

Although some solutions of eqs.(3) and (11) may be similar in the case of $\gamma \ll \omega$, none the less these equations describe different physical systems. Moreover, it can be shown that any Lagrangian leading to eqs.(3) multiplied by coordinates- and velocities-independent integrating factors can be expressed as a linear combination of $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ with constant coefficients (see Appendix A).

Equation (4) can be derived also from Sona's Hamiltonian [18]

$$(12) \quad \tilde{\mathcal{H}} = e^{\rho} + \tilde{\gamma} z$$

(We choose dimensionless variables). Writing $\rho = \rho_x + i\rho_y$ and calculating the real part of $\tilde{\mathcal{H}}$ one obtains the new Hamiltonian

$$(13) \quad \tilde{\mathcal{H}}_3 = e^{\rho_x} \cos \rho_y + \gamma x - \omega y$$

One can easily verify that it leads to eqs.(3). (The imaginary part of $\tilde{\mathcal{H}}$ yields the Hamiltonian for equations being obtained from (3) by the replacement $\gamma \rightarrow \omega$, $\omega \rightarrow -\gamma$).

The Lagrangian corresponding to $\tilde{\mathcal{H}}_3$ has the form

$$(14) \quad \tilde{\mathcal{L}}_3 = \frac{1}{2} \dot{x} \ln(\dot{x}^2 + \dot{y}^2) - \dot{y} \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) - \dot{x} + \omega y - \gamma x$$

It yields the equations of motion

$$(15) \quad \begin{aligned} (\dot{x}^2 + \dot{y}^2)^{-1} \left[\dot{y} (\ddot{x} + \gamma \dot{x} - \omega \dot{y}) - \dot{x} (\ddot{y} + \gamma \dot{y} + \omega \dot{x}) \right] &= 0 \\ (\dot{x}^2 + \dot{y}^2)^{-1} \left[\dot{x} (\ddot{x} + \gamma \dot{x} - \omega \dot{y}) + \dot{y} (\ddot{y} + \gamma \dot{y} + \omega \dot{x}) \right] &= 0 \end{aligned}$$

which are equivalent to eqs.(3) provided $\dot{x}^2 + \dot{y}^2 \neq 0$.

We do not intend to look for all equivalent Lagrangians or Hamiltonians for eqs.(3) (their number is infinite). We shall confine ourselves to Hamiltonians (9) and (13) and show that after canonical quantization (i.e. after the replacement of p_x and p_y by the operators $-i\hbar \partial/\partial x$ and $-i\hbar \partial/\partial y$) we shall obtain the systems with quite different physical properties. To compare these two systems with the conventional model of a quantum particle in the magnetic field described by the Hamiltonian

$$(16) \quad \mathcal{H}_0 = \frac{1}{2} \left(p_x + \frac{\omega}{2} y \right)^2 + \frac{1}{2} \left(p_y - \frac{\omega}{2} x \right)^2$$

we assume hereafter $\gamma = 0$, i.e. we shall consider besides Hamiltonian (16) the following Hamiltonians:

$$(17) \quad \mathcal{H}_1 = p_x p_y \cos \omega t + \frac{1}{2} (p_x^2 - p_y^2) \sin \omega t$$

and

$$(18) \quad \mathcal{H}_3 = e^{p_x} \cos p_y - \omega y$$

3. Quantization and spectra of conserved quantities.

Different Hamiltonians lead to different relations between generalized momenta and velocities:

$$(19a) \quad \begin{aligned} \dot{x} &= p_x + \frac{\omega}{2} y \\ \dot{y} &= p_y - \frac{\omega}{2} x \end{aligned} \quad \text{for } \mathcal{H}_0$$

$$(19b) \quad \begin{aligned} \dot{x} &= p_y \cos \omega t + p_x \sin \omega t \\ \dot{y} &= -p_y \sin \omega t + p_x \cos \omega t \end{aligned} \quad \text{for } \mathcal{H}_1$$

$$(19c) \quad \begin{aligned} \dot{x} &= e^{p_x} \cos p_y \\ \dot{y} &= -e^{p_x} \sin p_y \end{aligned} \quad \text{for } \mathcal{H}_3$$

Differentiating these equations with respect to time and taking into account the second set of Hamilton's equations

$$(20a) \quad \begin{aligned} \dot{p}_x &= \frac{\omega}{2} (p_y - \frac{\omega}{2} x) \\ \dot{p}_y &= -\frac{\omega}{2} (p_x + \frac{\omega}{2} y) \end{aligned} \quad \text{for } \mathcal{H}_0$$

$$(20b) \quad \dot{p}_x = \dot{p}_y = 0 \quad \text{for } \mathcal{H}_1$$

$$(20c) \quad \begin{aligned} \dot{p}_x &= 0 \\ \dot{p}_y &= \omega \end{aligned} \quad \text{for } \mathcal{H}_3$$

one obtains in all three cases the same equations (1) (without integrating factors). Since in the classical case the complete information on the system can be obtained from its equations

of motion only, without any appeal to Hamiltonians or Lagrangians, different Hamiltonians (16)-(18) are indeed equivalent from the point of view of classical mechanics (see also the next section).

Note that if one considers equations (1) as a system of ordinary differential equations only and forgets their physical meaning as a very special case of more general equations of motion of a charged particle in arbitrary nonuniform electromagnetic fields, then one has no quite convincing arguments to prefer, e.g., Hamiltonian (16) to Hamiltonian (17). The evident disadvantage of Hamiltonian (17) is that it is time-dependent and nonpositive definite. But it is invariant with respect to space translations, as well as the original system of equations (1), while the conventional Hamiltonian (16) is explicitly space inhomogeneous.

Now let us investigate the properties of quantum systems described by Hamiltonian operators obtained from (16)-(18) by means of the replacements $p_x \rightarrow \hat{p}_x = -i\hbar \partial/\partial x$; $p_y \rightarrow \hat{p}_y = -i\hbar \partial/\partial y$. Their physical differences manifest themselves to the most extent in different measurability properties of various physical observables and in the characters of their spectra.

For example, in the conventional case the velocity operators (19a) do not commute, so that two orthogonal projections of the velocity cannot be measured exactly simultaneously, while in the case of Hamiltonians (17) and (18) such measurements are quite possible, because the operators $\hat{x}$ and $\hat{y}$ given by eqs.(19b) or (19c) commute. (See Appendix B, where

the physical aspect of this item is discussed).

The consequence of equations of motion (1) is the conservation of the kinetic energy $E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$. In the conventional case the operator $\hat{E}$ coincides with the Hamiltonian operator $\hat{H}_0$ and has the (infinitely degenerate) discrete spectrum: $E_n = \hbar\omega(n + \frac{1}{2})$, $n = 0, 1, 2 \dots$. Just this discrete character of the energy spectrum is responsible for the existence of the nonzero magnetic susceptibility in the state of the thermodynamical equilibrium. However, in the case of Hamiltonians (17) and (18) the kinetic energy operator has the form, respectively, $\hat{E}_1 = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2)$ and $\hat{E}_3 = \frac{1}{2}\exp(2\hat{p}_x)$. Therefore in both unusual cases the kinetic energy spectra are continuous. Consequently, if one defined the equilibrium density matrix, as well as in the conventional case, by the relation $\hat{\rho} = \exp(-\hat{E}/kT)$, then it would coincide with the free particle equilibrium density matrix. This means that the equilibrium magnetic susceptibility for systems with Hamiltonians (17) and (18) equals zero. It was supposed in ref.[8] that nonconventional Hamiltonians or Lagrangians may be significant for the description of matter at the quarks or subquarks level. Our example shows that if it were true, such systems would possess very unusual physical properties.

Equations of motion (1) possess well-known linear integrals of motion

$$(21) \quad \xi = x + \dot{y}/\omega; \quad \eta = y - \dot{x}/\omega$$

These conserved quantities determine the coordinates of the center of the circle along which the particle rotates in

the magnetic field. In the usual case, according to eq.(19a), the corresponding operators are as follows,

$$(22a) \quad \begin{aligned} \hat{\xi}_0 &= \frac{1}{2} \hat{x} + \hat{p}_y / \omega \\ \hat{\eta}_0 &= \frac{1}{2} \hat{y} - \hat{p}_x / \omega \end{aligned}$$

They do not commute; therefore the position of the center of the circle cannot be known exactly, in this case. However, in the cases (17) and (18) one obtains commuting operators:

$$(22b) \quad \begin{aligned} \hat{\xi}_1 &= \hat{x} + \frac{1}{\omega} (\hat{p}_x \cos \omega t - \hat{p}_y \sin \omega t) \\ \hat{\eta}_1 &= \hat{y} - \frac{1}{\omega} (\hat{p}_x \sin \omega t + \hat{p}_y \cos \omega t) \end{aligned}$$

$$(22c) \quad \begin{aligned} \hat{\xi}_3 &= \hat{x} - \frac{1}{\omega} \sin \hat{p}_y e^{\hat{p}_x} \\ \hat{\eta}_3 &= \hat{y} - \frac{1}{\omega} \cos \hat{p}_y e^{\hat{p}_x} \end{aligned}$$

so that in these cases the position of the center can be determined precisely. Moreover, while in the usual case the kinetic energy operator $\hat{E}_0 = \hat{H}_0$ commutes both with $\hat{\xi}_0$ and $\hat{\eta}_0$, in the case b) both commutators $[\hat{E}_1, \hat{\xi}_1]$ and $[\hat{E}_1, \hat{\eta}_1]$ are nonzero, and in the case c) one has $[\hat{E}_3, \hat{\xi}_3] \neq 0$ but $[\hat{E}_3, \hat{\eta}_3] = 0$.

The consequence of equations of motion (1) is the conservation of the analogue of angular momentum - the quantity

$$(23) \quad M = x\dot{y} - \dot{x}y + \frac{\omega}{2}(x^2 + y^2) \equiv \frac{\omega}{2}(\xi^2 + \eta^2) - \frac{1}{\omega} E$$

In the usual case the operator $\hat{M}_0$ commutes with the energy operator $\hat{E}_0 = \hat{H}_0$, and it has the standard form of the

operator of the generalized angular momentum:

$$(24a) \quad \hat{M}_0 = \hat{x} \hat{p}_y - \hat{y} \hat{p}_x$$

Therefore in the usual case the spectrum of the observable M is discrete, as well as the energy spectrum. Quite different situations take place in cases b) and c). Firstly, in these both cases the operators $\hat{M}$ and $\hat{E}$ do not commute - thus the kinetic energy and the analogue of angular momentum cannot be measured simultaneously. Secondly, the spectra of $\hat{M}_1$ and $\hat{M}_3$ are continuous. This can be easily understood, if one writes these operators as follows:

$$(24b) \quad \hat{M}_1 = \frac{\omega}{2} (\hat{\xi}_1^2 + \hat{\eta}_1^2) - \frac{1}{2\omega} (\hat{p}_x^2 + \hat{p}_y^2)$$

$$(24c) \quad \hat{M}_3 = \frac{\omega}{2} (\hat{\xi}_3^2 + \hat{\eta}_3^2) - \frac{1}{2\omega} e^{2\hat{p}_x}$$

Indeed, one can verify that three sets of operators: $(\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y)$; $(\hat{\xi}_1, \hat{\eta}_1, \hat{p}_x, \hat{p}_y)$; $(\hat{\xi}_3, \hat{\eta}_3, \hat{p}_x, \hat{p}_y)$ - obey the same commutation relations:

$$(25) \quad \begin{aligned} [\hat{x}, \hat{y}] &= [\hat{p}_x, \hat{p}_y] = [\hat{\xi}_1, \hat{\eta}_1] = [\hat{\xi}_3, \hat{\eta}_3] = [\hat{\xi}_1, \hat{p}_y] = \\ &= [\hat{\xi}_3, \hat{p}_y] = [\hat{\eta}_1, \hat{p}_x] = [\hat{\eta}_3, \hat{p}_x] = 0; \\ [\hat{\xi}_1, \hat{p}_x] &= [\hat{\eta}_1, \hat{p}_y] = [\hat{\xi}_3, \hat{p}_x] = [\hat{\eta}_3, \hat{p}_y] = i\hbar \end{aligned}$$

Consequently, each set can be transformed to any other one by means of a unitary (canonical) transformation due to Stone-von Neumann's theorem. The matrix elements of these transformations are in fact the eigenfunctions of the commu-

ting operators $\hat{f}$ and $\hat{g}$ (22b)-(22c) satisfying the corresponding Schrödinger equations. In the case (22b) they are as follows,

$$\begin{aligned}
 \langle x, y, t | \hat{f}, \hat{g} \rangle &= \frac{\omega}{2\pi\hbar} \exp \left\{ \frac{i\omega}{\hbar} \left[\frac{1}{2}(y^2 - x^2) \cos \omega t \right. \right. \\
 (26b) \quad &+ xy \sin \omega t + \hat{f}(x \cos \omega t - y \sin \omega t) - \hat{g}(x \sin \omega t + y \cos \omega t) \\
 &+ \hat{f}\hat{g} \sin \omega t + \frac{1}{2}(\hat{g}^2 - \hat{f}^2) \cos \omega t \left. \right] \Big\}; \\
 (\alpha \hat{f}_1 + \beta \hat{g}_1) | \hat{f}, \hat{g} \rangle &= (\alpha \hat{f} + \beta \hat{g}) | \hat{f}, \hat{g} \rangle; \\
 i\hbar \frac{\partial}{\partial t} | \hat{f}, \hat{g} \rangle &= \hat{H}_1 | \hat{f}, \hat{g} \rangle; \quad \langle \hat{f}', \hat{g}' | \hat{f}, \hat{g} \rangle = \delta(\hat{f} - \hat{f}') \delta(\hat{g} - \hat{g}')
 \end{aligned}$$

In the case c) the explicit expression can be easily obtained in the momentum representation:

$$\begin{aligned}
 \langle p_x, p_y, t | \hat{f}, \hat{g} \rangle &= \frac{1}{2\pi\hbar} \exp \left\{ \frac{1}{i\hbar} \left[\hat{f} p_x + \hat{g} p_y - \omega \hat{g} t \right. \right. \\
 (26c) \quad &+ \frac{1}{\omega} \sin p_y e^{p_x} \left. \right] \Big\}; \\
 (\alpha \hat{f}_3 + \beta \hat{g}_3) | \hat{f}, \hat{g} \rangle &= (\alpha \hat{f} + \beta \hat{g}) | \hat{f}, \hat{g} \rangle; \\
 i\hbar \frac{\partial}{\partial t} | \hat{f}, \hat{g} \rangle &= \hat{H}_3 | \hat{f}, \hat{g} \rangle; \quad \langle \hat{f}', \hat{g}' | \hat{f}, \hat{g} \rangle = \delta(\hat{f} - \hat{f}') \delta(\hat{g} - \hat{g}')
 \end{aligned}$$

Since unitary transformations do not change spectra, the spectrum of the operator $\hat{M}_1$ is the same as the energy spectrum of the two-dimensional "upside-down oscillator", i.e. the system with the potential energy $V(\rho) = -\frac{\omega^2}{2} \rho^2$. The motion in such a potential is infinite; therefore the spectrum is continuous: $-\infty < M_1 < \infty$.

Similarly, the operator $\hat{M}_3$ in the momentum representation looks like the Hamiltonian of the two-dimensional system with the potential $V(x) = -\frac{1}{2\omega^2} \exp(2x)$. The motion in this potential is infinite, and the spectrum is continuous again.

Note that the operators $\hat{M}_0$ and $\hat{M}_1$ do not commute with

the pairs $\hat{\mathcal{F}}_0, \hat{\mathcal{Q}}_0$ and $\hat{\mathcal{F}}_1, \hat{\mathcal{Q}}_1$, respectively, while $\hat{M}_3$ commutes with $\hat{\mathcal{Q}}_3$ and does not commute with $\hat{\mathcal{F}}_3$.

It is obvious from eq.(24b) that the operator $\hat{M}_1$ commutes with the operator

$$(27b) \quad \hat{J}_1 = \hat{\mathcal{F}}_1 \hat{p}_y - \hat{\mathcal{Q}}_1 \hat{p}_x = (\hat{x} \hat{p}_y - \hat{y} \hat{p}_x) + \frac{2}{\omega} \hat{H}_1$$

Evidently, the spectrum of $\hat{J}_1$ is discrete: $J_1 = \hbar j$, $j = 0, \pm 1, \pm 2, \dots$.

The corresponding classical conserved quantity is

$$(28) \quad J = \cos \omega t (x \dot{x} - y \dot{y} + \frac{2}{\omega} \dot{x} \dot{y}) + \sin \omega t (\frac{1}{\omega} (\dot{x}^2 - \dot{y}^2) - x \dot{y} - \dot{x} y)$$

In the standard case its operator is

$$(27a) \quad \hat{J}_0 = \frac{2}{\omega} \cos \omega t (\hat{p}_x \hat{p}_y + \frac{\omega^2}{4} \hat{x} \hat{y}) + \frac{1}{\omega} \sin \omega t [\hat{p}_x^2 - \hat{p}_y^2 + \frac{\omega^2}{4} (\hat{x}^2 - \hat{y}^2)]$$

while in the case c) one obtains the following expression:

$$(27c) \quad \hat{J}_3 = \{ \hat{x}, e^{\hat{p}_x} \} \cos(\hat{p}_y - \omega t) + e^{\hat{p}_x} \{ \hat{y}, \sin(\hat{p}_y - \omega t) \} - \frac{1}{\omega} e^{2\hat{p}_x} \sin(2\hat{p}_y - \omega t)$$

where $\{\hat{a}, \hat{b}\}$ means the anticommutator of the operators $\hat{a}$ and $\hat{b}$. One can check that $[\hat{J}_0, \hat{M}_0] \neq 0$.

If we chose another gauge of the vector potential in the standard Hamiltonian, e.g., if we considered the Hamiltonian

$$\mathcal{H}_0' = \frac{1}{2}(\hat{p}_x + \omega \hat{y})^2 + \frac{1}{2}\hat{p}_y^2$$

then the form of the operators-integrals of motion would be also changed:

$$\begin{aligned}\hat{\mathcal{X}}_0' &= \hat{x} + \hat{p}_y/\omega; & \hat{\mathcal{Y}}_0' &= -\hat{p}_x/\omega; \\ \hat{M}_0' &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x + \frac{\omega}{2}(\hat{x}^2 - \hat{y}^2)\end{aligned}$$

None the less the spectra of these operators and commutation relations between them would be the same:

$$\begin{aligned}[\hat{\mathcal{X}}_0', \hat{\mathcal{E}}_0'] &= [\hat{\mathcal{Y}}_0', \hat{\mathcal{E}}_0'] = [\hat{M}_0', \hat{\mathcal{E}}_0'] = 0; \\ \hat{\mathcal{E}}_0' &= \hat{H}_0'; & [\hat{\mathcal{X}}_0', \hat{\mathcal{Y}}_0'] &= -i\hbar/\omega\end{aligned}$$

Since the operator $\hat{M}_0'$ can be expressed as follows,

$$\begin{aligned}\hat{M}_0' &= \hat{x}\hat{\pi}_y - \hat{y}\hat{\pi}_x; \\ \hat{\pi}_x &= \hat{p}_x + \frac{\omega}{2}\hat{y}; & \hat{\pi}_y &= \hat{p}_y + \frac{\omega}{2}\hat{x}; & [\hat{\pi}_x, \hat{\pi}_y] &= 0\end{aligned}$$

its spectrum remains discrete: $M_0' = \hbar j$, $j=0, \pm 1, \pm 2, \dots$.

Consequently, the differences of spectra and measurability properties of the same conserved observables like the kinetic energy, generalized angular momentum, etc., shown above, cannot be related to any gauge transformation. These differences are the convincing evidence in favour of the statement that quantum Hamiltonians arising by means of the canonical procedure from different classically equivalent Hamiltonians describe, generally speaking, quite different quantum systems obeying none the less in the classical limit the same equations of mo-

tion of the second order.

4. Propagators and canonical transformation .

Let us compare also the propagators of the Schrödinger equations with Hamiltonians (16)-(18). In the standard case we have the well-known expression [19]

$$\begin{aligned} G_0(x_2, y_2, t; x_1, y_1, 0) &= \frac{\omega}{4\pi i \hbar \sin(\frac{1}{2}\omega t)} \\ (29a) \quad &\times \exp\left\{ \frac{i\omega}{4\hbar} \cot\left(\frac{\omega t}{2}\right) [(x_2 - x_1)^2 + (y_2 - y_1)^2] \right. \\ &\left. + \frac{i\omega}{2\hbar} (x_1 y_2 - x_2 y_1) \right\} \end{aligned}$$

In the case of Hamiltonian $\hat{H}_1$ one can calculate the Green function (propagator) either integrating over the complete set of solutions (26b) or using the general formulae for Green functions of the Schrödinger equations with quadratic Hamiltonians given in ref. [20] :

$$\begin{aligned} G_1(x_2, y_2, t; x_1, y_1, 0) &= \frac{\omega}{4\pi \hbar \sin(\frac{1}{2}\omega t)} \\ (29b) \quad &\times \exp\left\{ \frac{i\omega}{4\hbar} [(y_2 - y_1)^2 - (x_2 - x_1)^2] \right. \\ &\left. - 2 \cot\left(\frac{\omega t}{2}\right) (x_2 - x_1)(y_2 - y_1) \right\} \end{aligned}$$

In the third case a simple expression can be obtained in the momentum representation after integrating over the set of solutions (26c):

$$\begin{aligned} G_3(p_x^{(2)}, p_y^{(2)}, t; p_x^{(1)}, p_y^{(1)}, 0) &= \delta(p_x^{(2)} - p_x^{(1)}) \delta(p_y^{(2)} - p_y^{(1)} - \omega t) \\ (29c) \quad &\times \exp\left[-\frac{2i}{\hbar \omega} \exp(p_x^{(1)}) \sin\left(\frac{\omega t}{2}\right) \cos\left(p_y^{(2)} - \frac{\omega t}{2}\right) \right] \end{aligned}$$

Evidently, these three propagators are quite different and cannot be transformed to each other by gauge transformations. However, the question remains whether exist other unitary (canonical) transformations relating the solutions of the Schrödinger equations with different Hamiltonians.

The answer is affirmative (and trivial) provided time-dependent transformations are admitted. Indeed, let us consider a system with a Hamiltonian $\hat{H}$. The solutions of the Schrödinger equations with this Hamiltonian can be represented as follows, $\Psi(t) = \hat{U}(t) \Psi(0)$, $\hat{U}(t)$ being the evolution operator (the propagator is its matrix element in the representation chosen), satisfying the equation

$$(30) \quad i\hbar \frac{d\hat{U}}{dt} = \hat{H} \hat{U}, \quad t > 0; \quad \hat{U}(0) = \text{unit operator}$$

Similarly, for a system with a Hamiltonian $\hat{H}'$ one has, $\Psi'(t) = \hat{U}'(t) \Psi'(0)$. If the evolution operators and their inverse ones exist, then the following relations hold:

$$(31) \quad \begin{aligned} \Psi'(t) &= \hat{S}(t) \Psi(t); \\ \hat{S}(t) &= \hat{U}'(t) \hat{S}_0 [\hat{U}(t)]^{-1} \end{aligned}$$

where $\hat{S}_0$ is an arbitrary time-independent unitary operator. Evidently, the operator $\hat{S}(t)$ is unitary due to the unitarity of the operators $\hat{U}(t)$, $\hat{U}'(t)$, $\hat{S}_0$. Consequently, using time-dependent canonical transformation one can reduce any given quantum system possessing a unitary evolution operator to an arbitrary other quantum system. To understand better the mē-

ture of such transformations, let us consider the most simple example: the free motion described by two different classically equivalent Hamiltonians: $H_0^{(0)} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2)$ and $H_1^{(0)} = \hat{p}_x \hat{p}_y$.

The propagators in the coordinate representation are given by eqs. (29a-b) with $\omega = 0$. Since the Hamiltonians do not depend on time, in both cases $\hat{U}(t) = \exp(-it\hat{H}/\hbar)$. Choosing $\hat{S}_0 = 1$ one finds the following transformation operator:

$$(32) \quad \hat{S}(t) = \exp \left\{ \frac{it}{\hbar} \left[\frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2) - \hat{p}_x \hat{p}_y \right] \right\}$$

One can easily check the following equation:

$$(33) \quad \hat{H}_1^{(0)} = \hat{S} \hat{H}_0^{(0)} \hat{S}^+ + i\hbar \hat{S} \hat{S}^+$$

Other operators are transformed according to the relation $\hat{A}_1 = \hat{S} \hat{A}_0 \hat{S}^+$. Denoting the operators corresponding to the system with the Hamiltonian $\hat{H}_1^{(0)}$ with a prime, one obtains:

$$(34) \quad \begin{aligned} \hat{p}_x' &= \hat{p}_x; & \hat{p}_y' &= \hat{p}_y; \\ \hat{x}' &= \hat{x} + (\hat{p}_x - \hat{p}_y)t; \\ \hat{y}' &= \hat{y} + (\hat{p}_y - \hat{p}_x)t \end{aligned}$$

This transformation is time-dependent; moreover, it mixes up generalized coordinates with generalized momenta (although preserves commutation relations). This means that the physical significance of the operators of coordinates $\hat{x}$, $\hat{y}$ in the case of Hamiltonian $\hat{H}_0^{(0)}$ and $\hat{x}'$, $\hat{y}'$ in the case of Hamiltonian $\hat{H}_1^{(0)}$ is quite different, although equations of motion are the same in both cases: $\ddot{x} = \ddot{y} = 0$ and $\ddot{x}' = \ddot{y}' = 0$.

Consequently, it is just one more confirmation of the statement that different classically equivalent Hamiltonians describe after the canonical quantization different physical systems.

One could think that different Hamiltonians describe different physical systems even at the classical level. It seems to us that this is not true, since in the classical case there is no physical quantity which could not be calculated on the basis of Newton's equations of motion: if we know $x(t)$ and $\dot{x}(t)$, we can calculate the energy, kinetic and angular momentum, and so on; but to find $x(t)$ and $\dot{x}(t)$ one needs equations of motion only, while the knowledge of such things as Hamiltonian, Lagrangian, action, etc., is not necessary, in principle. Hamiltonians, Lagrangians, actions, generalized momenta and so on in classical mechanics play only auxiliary roles and have no direct physical meaning, since they cannot be measured independently and directly in the classical case. These quantities acquire the physical meaning only in the quantum case, in which coordinates and generalized momenta become truly independent variables due to the uncertainty relations, and each physical observable requires, generally speaking, its own independent measuring procedure.

5. Solutions of the Schrödinger equation.

Various complete systems of solutions of the Schrödinger equations in the case of the standard Hamiltonian for a particle in a magnetic field were found in many papers, e.g., in refs.

[21-22]. Here we obtain some solutions in the case of unusual Hamiltonians. Eigenfunctions of the operators of the centre of the orbit $\hat{\xi}$ and $\hat{\eta}$ are given by eqs.(26). Eigenstates of the kinetic energy operators $\hat{E}_1$ and $\hat{E}_2$ in the coordinate representation, obviously, have the form of plane waves. Therefore let us calculate the eigenfunctions of the generalized angular momentum operator $\hat{M}_1$ (24b), which can be expressed in terms of coordinates and momenta operators as follows,

$$(35) \quad \begin{aligned} \hat{M}_1 = & (\hat{x} \cos \omega t - \hat{y} \sin \omega t) \hat{p}_x \\ & - (\hat{x} \sin \omega t + \hat{y} \cos \omega t) \hat{p}_y + \frac{\omega}{2} (\hat{x}^2 + \hat{y}^2) \end{aligned}$$

We will find the mutual eigenfunctions of the commuting operators $\hat{M}_1$ and $\hat{J}_1$ (27b), satisfying the Schrödinger equation. Thus we have to solve the following system of equations:

$$\begin{aligned} \hat{M}_1 \varphi &= \hbar \mu \varphi; \\ \hat{J}_1 \varphi &= \hbar j \varphi; \\ i \hbar \frac{\partial \varphi}{\partial t} &= \hat{H}_1 \varphi, \end{aligned}$$

where μ is an arbitrary real number, and j is an arbitrary integer, as was shown earlier.

Making the orthogonal coordinate transformation

$$(36) \quad \begin{aligned} \tilde{x} &= x \cos(\frac{\omega}{2} t) - y \sin(\frac{\omega}{2} t); \\ \tilde{y} &= x \sin(\frac{\omega}{2} t) + y \cos(\frac{\omega}{2} t) \end{aligned}$$

one can write these equations as follows,

$$(37) \quad \tilde{y} \frac{\partial \varphi}{\partial \tilde{y}} - \tilde{x} \frac{\partial \varphi}{\partial \tilde{x}} = i \left[\frac{\omega}{2 \hbar} (\tilde{x}^2 + \tilde{y}^2) - \mu \right] \varphi$$

$$(38) \quad \tilde{y} \frac{\partial \varphi}{\partial \tilde{x}} - \tilde{x} \frac{\partial \varphi}{\partial \tilde{y}} + \frac{2i\hbar}{\omega} \frac{\partial^2 \varphi}{\partial \tilde{x} \partial \tilde{y}} = -i\mu \varphi$$

$$(39) \quad \frac{\partial \varphi}{\partial t} = \frac{\omega}{2} \left(\tilde{y} \frac{\partial \varphi}{\partial \tilde{x}} - \tilde{x} \frac{\partial \varphi}{\partial \tilde{y}} \right) + i\hbar \frac{\partial^2 \varphi}{\partial \tilde{x} \partial \tilde{y}}$$

From eqs.(38)-(39) one obtains $\varphi(\tilde{x}, \tilde{y}, t) = \exp(-\frac{i}{2}\omega j t)$
 $\times \tilde{\varphi}(\tilde{x}, \tilde{y})$. Eq.(37) is of the first order, so that one can easily find its general solution:

$$(40) \quad \tilde{\varphi}(\tilde{x}, \tilde{y}) = \tilde{y}^{-i\mu} \exp\left[\frac{\omega}{4i\hbar}(\tilde{x}^2 - \tilde{y}^2)\right] f(\tilde{x}\tilde{y})$$

f being an arbitrary function. Substituting expression (40) in-to eq.(38) one arrives at the following equation for determining function $f(z)$:

$$(41) \quad z f'' + (1 - i\mu) f' + \left(\frac{j\omega}{2\hbar} - \frac{\omega^2}{4\hbar^2} z \right) f(z) = 0$$

$$z \equiv \tilde{x}\tilde{y}$$

This equation can be reduced to the equation for the confluent hypergeometric function. Note that eq.(41) is invariant with respect to the transformations $z \rightarrow -z$, $j \rightarrow -j$; therefore it is sufficiently ^{to} solve it only for $z > 0$.

Using the results and designations from §6.2 of ref. [23], one obtains the following expression:

$$(42) \quad \varphi_{j,\mu}(\tilde{x}, \tilde{y}, t) = N \tilde{y}^{-i\mu} \exp\left[\frac{\omega}{4i\hbar}(\tilde{x}^2 - \tilde{y}^2) - \frac{\omega}{2\hbar} \tilde{x} \tilde{y} - \frac{i}{2} \omega j t\right]$$

$$\times \Psi\left(\frac{1}{2} - \frac{j}{2} - \frac{i\mu}{2}, 1 - i\mu; \frac{\omega}{\hbar} \tilde{x} \tilde{y}\right);$$

$$\omega \tilde{x} \tilde{y} > 0; \quad -\infty < \mu < \infty; \quad j = 0, \pm 1, \pm 2,$$

(N - normalization constant). For large values of the product $\tilde{x} \tilde{y}$ this function has the following asymptotics:

$$\varphi_{j,\mu}(\tilde{x}, \tilde{y}, t) \sim \left|\frac{\tilde{y}}{\tilde{x}}\right|^{-\frac{i\mu}{2}} |\tilde{x} \tilde{y}|^{\frac{1}{2}(j-1)}$$

$$\times \exp\left[-\frac{\omega}{2\hbar} |\tilde{x} \tilde{y}| + \frac{\omega}{4i\hbar}(\tilde{x}^2 - \tilde{y}^2)\right]$$

If we chose the function Φ instead Ψ as the solution of the confluent hypergeometric equation, then we would obtain the inappropriate asymptotics for large values of $|\tilde{x} \tilde{y}|$:

$\varphi \sim \exp(+\frac{\omega}{2\hbar} |\tilde{x} \tilde{y}|)$. To make function (42) single-valued, we define the function $\tilde{y}^{-i\mu}$ by the relation

$$\tilde{y}^{-i\mu} = \exp\left(-\frac{i\mu}{2} \ln \tilde{y}^2\right)$$

In the special case of $j=0$ the function $\varphi_{0,\mu}$ can be expressed in terms of the McDonald function (see eq.6.9(13) from [23]):

$$(43) \quad \varphi_{0,\mu}(\tilde{x}, \tilde{y}, t) = \left|\cosh\left(\frac{\pi}{2}\mu\right) \frac{\omega}{4\pi^3\hbar}\right|^{\frac{1}{2}} \left(\frac{\tilde{x}}{\tilde{y}}\right)^{\frac{i\mu}{2}}$$

$$\times \exp\left[\frac{\omega}{4i\hbar}(\tilde{x}^2 - \tilde{y}^2)\right] K_{\frac{i\mu}{2}}\left(\left|\frac{\omega \tilde{x} \tilde{y}}{2\hbar}\right|\right);$$

$$(44) \quad \int_{-\infty}^{\infty} \varphi_{0,\mu}(x, y) \varphi_{0,\mu'}^*(x, y) dx dy = \delta(\mu - \mu')$$

To calculate the normalization constant one should make the change of variables $\tilde{x} \tilde{y} = u$, $\tilde{x}/\tilde{y} = v$ in integral (44)

and take into account formula 7.14(36) from [23] for the integral of the product of McDonald's functions:

$$\int_0^{\infty} K_{\nu}(t) K_{\nu}(t) dt = \frac{1}{4} \Gamma\left(\frac{1+\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2}\right) \Gamma\left(\frac{1+\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2}\right),$$

the relation

$$(45) \quad \int_0^{\infty} t^{i\alpha-1} dt = \int_{-\infty}^{\infty} e^{i\alpha x} dx = 2\pi \delta(\alpha)$$

and the well-known properties of the gamma-function.

It is interesting to consider the special case of a free particle $\omega = 0$. In this case the angular momentum operator $\hat{M}_1^{(0)}$ (it is an integral of motion) has the unusual form:

$$(46) \quad \hat{M}_1^{(0)} = \hat{x} \hat{y} - \hat{y} \hat{x} = \hat{x} \hat{p}_x - \hat{y} \hat{p}_y$$

It commutes with the operator

$$K_1^{(0)} = \lim_{\omega \rightarrow 0} (\omega \hat{J}_1) = 2 \hat{H}_1^{(0)} = 2 \hat{p}_x \hat{p}_y$$

The mutual eigenfunctions of the operators $\hat{M}_1^{(0)}$ and $\hat{K}_1^{(0)}$ have the following form:

$$(47) \quad \begin{aligned} \varphi_{\mu, k}(x, y, t) &= \frac{1}{4(\pi \hbar)^{1/2}} \left(\frac{x}{y}\right)^{\frac{i\mu}{2}} \exp\left(-\frac{i}{2} k t - \frac{\pi}{4} \mu\right) \\ &\quad \times H_{i\mu}^{(1)}\left(\left[\frac{2}{\hbar} k x y\right]^{\frac{1}{2}}\right); \\ \hat{M}_1^{(0)} \varphi_{\mu, k} &= \hbar \mu \varphi_{\mu, k}; \quad \hat{K}_1^{(0)} \varphi_{\mu, k} = \hbar k \varphi_{\mu, k}; \\ \langle \varphi_{\mu', k'} | \varphi_{\mu, k} \rangle &= \delta(\mu - \mu') \delta(k - k') \end{aligned}$$

$H_{i\mu}^{(1)}$ is Hankel's function [23] (we believe $\sqrt{z} = +i\sqrt{|z|}$ for $z < 0$).

The normalization constant can be calculated in the same manner as in the previous case if one replaces the Hankel function by its asymptotical formula

$$H_{i\mu}^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp\left[i\left(z - \frac{\sqrt{\pi}}{2} - \frac{\pi}{4}\right)\right] \\ |z| \rightarrow \infty$$

and uses the relation $\delta(\sqrt{x} - \sqrt{x_0}) = 2\sqrt{x_0} \delta(x - x_0)$.

Besides conserved quantities with continuous spectra (like E_1 or M_1), the system with Hamiltonian $\hat{H}_1$, as well as an arbitrary other quantum system, possesses conserved quantities with discrete and even equidistant spectra. For example, any operator of the type $\hat{a}^+ \hat{a}$, $\hat{a}$ and $\hat{a}^+$ being the operators - integrals of motion satisfying the boson - like annihilation and creation operators commutation relation $[\hat{a}, \hat{a}^+] = 1$, belongs to this class [24]. One can choose, e.g., the following operators - integrals of motion as the boson - like annihilation operators:

$$(48) \quad \hat{\alpha} = (2\hbar\omega)^{-\frac{1}{2}} (\omega \hat{\xi}_1 + i\hat{p}_x); \\ \hat{\beta} = (2\hbar\omega)^{-\frac{1}{2}} (\omega \hat{\eta}_1 + i\hat{p}_y); \quad [\hat{\alpha}, \hat{\beta}] = [\hat{\alpha}, \hat{\beta}^+] = 0$$

However, it is more convenient to consider certain linear combinations of these operators, namely,

$$(49) \quad \hat{A} = \frac{1}{\sqrt{2}} (\hat{\alpha} + i\hat{\beta}) = \frac{1}{2} (\hbar\omega)^{-\frac{1}{2}} [\omega (\hat{\xi}_1 + i\hat{\eta}_1) - \hat{p}_y + i\hat{p}_x]; \\ \hat{B} = \frac{1}{\sqrt{2}} (\hat{\alpha} - i\hat{\beta}) = \frac{1}{2} (\hbar\omega)^{-\frac{1}{2}} [\omega (\hat{\xi}_1 - i\hat{\eta}_1) + \hat{p}_y + i\hat{p}_x];$$

$$[\hat{A}, \hat{A}^+] = [\hat{B}, \hat{B}^+] = 1;$$

$$[\hat{A}, \hat{B}] = [\hat{A}, \hat{B}^+] = 0$$

One can easily check the following relations:

$$\begin{aligned} \hat{D}_+ &= \hat{A}^+ \hat{A} + \hat{B}^+ \hat{B} = \frac{1}{\hbar} \left(\hat{M}_1 + \frac{2}{\omega} \hat{E}_1 \right) - 1; \\ \hat{D}_- &= \hat{B}^+ \hat{B} - \hat{A}^+ \hat{A} = \frac{1}{\hbar} \hat{J}_1 \end{aligned} \quad (50)$$

Thus the discrete nature of the spectrum of the operator $\hat{J}_1$ becomes self-evident. The most simple and direct way to find the eigenstates of the operators (50) is to construct first the coherent states [25] $|\alpha, \beta\rangle$ satisfying the equations (α, β are arbitrary complex numbers)

$$\begin{aligned} \hat{A} |\alpha, \beta\rangle &= \alpha |\alpha, \beta\rangle; \\ \hat{B} |\alpha, \beta\rangle &= \beta |\alpha, \beta\rangle \end{aligned} \quad (51)$$

and the Schrödinger equation with the Hamiltonian $\hat{H}_1$. Using the general formulae for the coherent states of arbitrary multidimensional nonstationary quantum systems with quadratic Hamiltonians found in ref. [26] we obtain the following expression for the normalized coherent states:

$$\begin{aligned} \langle x, y, t | \alpha, \beta \rangle &= \left(\frac{\omega}{2\pi\hbar} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\omega}{4\hbar} (x^2 + y^2) - \frac{i\omega}{4\hbar} (\tilde{x}^2 - \tilde{y}^2) \right. \\ &+ \left(\frac{\omega}{4\hbar} \right)^{\frac{1}{2}} \left[\tilde{\alpha} (1-i)(\tilde{x} - \tilde{y}) + \tilde{\beta} (1+i)(\tilde{x} + \tilde{y}) \right] \\ &\left. - \frac{i}{2} (\tilde{\alpha}^2 + \tilde{\beta}^2) - \frac{1}{2} (|\alpha|^2 + |\beta|^2) \right\} \end{aligned} \quad (52)$$

where $\tilde{\alpha} = \alpha e^{\frac{i\omega t}{2}}$, $\tilde{\beta} = \beta e^{-\frac{i\omega t}{2}}$, and the values $\tilde{x}$ and $\tilde{y}$

are defined by eqs.(36). In the case of the standard Hamiltonian coherent states for a particle moving in a uniform magnetic field were constructed in ref.[22].

Expanding the coherent state $|\alpha, \beta\rangle$ in a power series with respect to the parameters α and β one obtains the eigenfunctions of the operators $\hat{A}^+ \hat{A}$ and $\hat{B}^+ \hat{B}$ [22,25,26]:

$$\begin{aligned}
 |n, m\rangle &= \left(\frac{\omega}{2\pi\hbar} \frac{i^{n+m}}{2^{n+m} n! m!} \right)^{\frac{1}{2}} \exp \left\{ \frac{i}{2} \omega t (n-m) \right. \\
 &\quad \left. - \frac{\omega}{4\hbar} (x^2 + y^2) - \frac{i\omega}{4\hbar} (\tilde{x}^2 - \tilde{y}^2) \right\} H_n \left(\frac{i}{2} \left(\frac{\omega}{\hbar} \right)^{\frac{1}{2}} (\tilde{y} - \tilde{x}) \right) \\
 &\quad \times H_m \left(\frac{1}{2} \left(\frac{\omega}{\hbar} \right)^{\frac{1}{2}} (\tilde{y} + \tilde{x}) \right); \quad \langle m', n' | m, n \rangle = \delta_{nn'} \delta_{mm'}, \\
 &\quad \hat{A}^+ \hat{A} |n, m\rangle = n |n, m\rangle; \quad \hat{B}^+ \hat{B} |n, m\rangle = m |n, m\rangle
 \end{aligned}
 \tag{53}$$

$H_n(z)$ is the Hermite polynomial [23]. The variables $\tilde{x}$ and $\tilde{y}$ are given by eqs.(36). Since the function $H_n(z)$ is either even or odd, the function $H_n(iz)$ is real up to the factor i^n .

The solutions obtained can be easily generalized to the case of Hamiltonian (9). However, it must be clear from this paper that they have no relation to the problem of the motion of a real quantum charged particle under the influence of the Lorentz force and the force of friction; therefore we do not give these formally more general solutions here.

Appendix A.

Let us replace nonlagrangian equations of motion (3) by the equivalent system of equations

$$\begin{aligned} G_1 &\equiv f_{11}(t)(\ddot{x} + \delta\dot{x} - \omega\dot{y}) + f_{12}(t)(\ddot{y} + \delta\dot{y} + \omega\dot{x}) = 0 \\ (A1) \quad G_2 &\equiv f_{21}(t)(\ddot{x} + \delta\dot{x} - \omega\dot{y}) + f_{22}(t)(\ddot{y} + \delta\dot{y} + \omega\dot{x}) = 0 \\ f_{11}f_{22} - f_{12}f_{21} &\neq 0 \end{aligned}$$

and try to choose the functions $f_{ik}(t)$ in such a manner that functions $G_i(\ddot{q}, \dot{q}, q, t)$ (where $q \equiv (x, y)$) would satisfy the necessary and sufficient conditions of the existence of a Lagrangian yielding equations (A1), i.e.

$$(A2) \quad G_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}; \quad i=1, 2$$

These conditions (known in literature as Helmholtz' conditions) are as follows (see, e.g., [1,4]),

$$(A3) \quad \frac{\partial G_i}{\partial \ddot{q}_K} = \frac{\partial G_K}{\partial \ddot{q}_i}$$

$$(A4) \quad \frac{\partial G_i}{\partial \dot{q}_K} + \frac{\partial G_K}{\partial \dot{q}_i} = \frac{d}{dt} \left(\frac{\partial G_i}{\partial \ddot{q}_K} + \frac{\partial G_K}{\partial \ddot{q}_i} \right)$$

$$(A5) \quad \frac{\partial G_i}{\partial q_K} - \frac{\partial G_K}{\partial q_i} = \frac{1}{2} \frac{d}{dt} \left(\frac{\partial G_i}{\partial \dot{q}_K} - \frac{\partial G_K}{\partial \dot{q}_i} \right)$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial \dot{q}_i} + \ddot{q}_i \frac{\partial}{\partial \ddot{q}_i}$. In the case of functions G_i of the type (1) eqs.(A3)-(A5) lead to the following system of equations for the functions $f_{ik}(t)$:

$$(A6) \quad f_{12} = f_{21}$$

$$(A7) \quad \omega \frac{d}{dt} (f_{11} + f_{22}) = 0$$

$$(A8) \quad \dot{f}_{12} = \gamma f_{12} + \frac{\omega}{2} (f_{22} - f_{11})$$

$$(A9) \quad \dot{f}_{11} = \gamma f_{11} + \omega f_{12}$$

$$(A10) \quad \dot{f}_{22} = \gamma f_{22} - \omega f_{12}$$

If $\gamma \neq 0$, $\omega \neq 0$ the general solutions of this system are

$$(A11) \quad \begin{aligned} f_{11} = -f_{22} &= e^{\gamma t} (\alpha \cos \omega t + \beta \sin \omega t) \\ f_{12} &= e^{\gamma t} (\beta \cos \omega t - \alpha \sin \omega t) \end{aligned}$$

α , β being arbitrary constant parameters. Thus one obtains that any equivalent Lagrangian for system (3), which is quadratic with respect to the velocities, has the following form (note that $f_{ik} = \partial^2 \mathcal{L} / \partial \dot{q}_i \partial \dot{q}_k$):

$$(A12) \quad \mathcal{L} = \beta \tilde{\mathcal{L}}_1 + \alpha \tilde{\mathcal{L}}_2 + \frac{d}{dt} \varphi(q, t)$$

$\alpha, \beta = \text{const}$, and φ being an arbitrary function; $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ are given by formulas (6)-(7) of the paper. If $\gamma=0$, then an arbitrary quadratic with respect to the velocities equivalent Lagrangian for system (1) has the form

$$(A13) \quad \mathcal{L} = \beta \mathcal{L}_1 + \alpha \mathcal{L}_2 + c \mathcal{L}_0 + \frac{d\varphi}{dt}$$

$c, \alpha, \beta = \text{const}$

where $\mathcal{L}_0$ is an arbitrary standard Lagrangian for this system (e.g., given by eq.(2) of the paper).

Note that in this case the addition of a total derivative $d\varphi/dt$ is equivalent to a gauge transformation of the vector potential in the standard Lagrangian.

Appendix B.

Here we will show the physical nature of the well-known uncertainties relation

$$(B1) \quad \Delta v_x \Delta v_y \geq \frac{e\hbar c}{2m^2 c}$$

for the projections of the velocity of a charge particle with mass m and charge e moving in a uniform magnetic field $\mathcal{H}$ directed along the z - axis. This relation follows from the standard Hamiltonian, but it does not hold for unusual Hamiltonians like $\hat{H}_1$ or $\hat{H}_3$.

Let us consider the procedure of measuring the v_x -projection of the velocity based on using Compton's effect. The value of v_x can be calculated by the formula (provided $v \ll c$)

$$\delta_x \approx \frac{c}{2} \frac{\omega' - \omega}{\omega}$$

where ω and ω' are the frequencies of photons before and after the scattering, respectively (for the sake of simplicity we consider only the backscattering of photons moving along the X - axis). Note that the scheme considered is similar to that considered in ref.[27] in connection with the relation

$\Delta x \Delta p \geq \hbar/2$. To guarantee the accuracy of the measurement $\Delta \omega'$ one should measure the frequency ω' with the accuracy $\Delta \omega' \leq 2\omega \Delta \delta_x / c$. Therefore the duration of the measurement τ must satisfy the relation $\tau \geq 2\pi / \Delta \omega' = \pi c / (\omega \Delta \delta_x)$.

During this period of time the X - component of the velocity suddenly changes from δ_x to $\delta_x - \delta \delta_x$, where

$$\delta \delta_x \approx \frac{2\hbar\omega}{mc}$$

because of the interaction with a photon. It is essential that the moment of time when this change happens is unknown. But the change of the value δ_x leads to the change of the value δ_y due to the equation of motion $\dot{\delta}_y = -\frac{e\mathcal{H}}{mc} \delta_x$ (i.e. due to the properties of Lorentz' force).

Consequently, the measurement of the value δ_x inevitably disturbs the value δ_y , so that after the measurement it can not be predicted with the accuracy better than

$$\begin{aligned} \Delta \delta_y &\approx \frac{e\mathcal{H}}{mc} \delta \delta_x \tau \geq \frac{e\mathcal{H}}{mc} \frac{2\hbar\omega}{mc} \frac{\pi c}{\omega \Delta \delta_x} \\ &= \frac{e\mathcal{H}}{m^2 c} \frac{2\pi\hbar}{\Delta \delta_x} \end{aligned}$$

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