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A NOTE ON γ TRACES FOR THE WILSON ACTION
IN THE CONTINUUM LIMIT

B. Berg^{*)}, A. Billoire and D. Foerster
CERN -- Geneva

ABSTRACT

In $d = 4$ and $d = 2$ dimensions we calculate averages of certain products of γ matrices $\langle \pi(1 \pm \gamma_\mu) \rangle_L$ with respect to closed lattice paths of length L . The approach to the asymptotic behaviour for large L is considered and found to be quite different in $d = 4$ and $d = 2$ dimensions.

^{*)} On leave from II. Institut für Theoretische Physik der Universität Hamburg, F.R.G.

Lattice colour gauge theory with fermions (quarks) has been formulated and investigated by Wilson^{1),2)}. We consider an oriented, infinite, d dimensional hypercubic lattice. The bilinear fermions can be integrated out and one is led to an effective action that gives rise to certain graphical rules described in Ref. 2). Let us arbitrarily fix a link ℓ_0 on the lattice. The contribution of the fermion part of the effective action at the link ℓ_0 is

$$S_{\ell_0}(\{u\}) = - \sum_{L=2}^{\infty} \frac{K^{2L}}{L} \sum_{P_{2L}(\ell_0)} \text{Tr} \left(\prod_{\ell \in P_{2L}} (1 + \gamma_{\ell}) u_{\ell} \right) \quad (1)$$

The second sum goes over all positively oriented paths $P_{2L}(\ell_0)$ of length $2L$ through the given link ℓ_0 . Euclidean γ matrices $\{\gamma_{\mu}, \gamma_{\nu}\} = \delta_{\mu\nu}$ and elements U_{ℓ} of the $SU(n)$ gauge group are attached to the link ℓ of the lattice. Further, the notation $\gamma_{-\ell} = -\gamma_{\ell}$, $U_{-\ell} = U_{\ell}^*$ is used and only dimension $d = 2, 4$ is considered. Paths with spikes do not contribute because $(1 + \gamma_{\mu})(1 - \gamma_{\mu}) = 0$.

For the free theory $U_{\ell} = 1$ for all ℓ and Eq. (1) becomes

$$S_{\ell_0} = S_{\ell_0}(\{1\}) = - \sum_{L=2}^{\infty} \frac{K^{2L}}{L} \langle \text{Tr} \rangle_{2L} N_{2L} \quad (2)$$

Here N_L ($L = 4, 6, \dots$, now) is the number of paths $P_L(\ell_0)$ without spikes (fermionic paths) and

$$\langle \text{Tr} \rangle_L = \langle \text{Tr} (1 + \gamma_{\mu}) \rangle_L = \frac{1}{N_L} \sum_{P_L(\ell_0)} \text{Tr} \left(\prod_{\ell \in P_L} (1 + \gamma_{\ell}) \right) \quad (3)$$

In the following we consider the free case. The hopping parameter K is related to the fermion mass m by Ref. 2)

$$m = \frac{1 - 2dK}{2Ka} \quad (4)$$

The continuum limit is obtained for lattice spacing $a \rightarrow 0$ and m fixed, hence $K \rightarrow 1/2d$ in the continuum limit. In the range $0 \leq K \leq 1/2d$ the series (2) is convergent.

In view of recent very active work³⁾ on Monte Carlo calculations within the approach²⁾, it might be of some interest to consider the continuum limit $K \rightarrow 1/2d$ in some more detail. From a simple entropy argument one recognizes $N_L \propto (2d-1)^L$ for $L \rightarrow \infty$. A closed formula for N_L is⁴⁾

$$N_L = \frac{1}{2d} \int \frac{d^d p}{(2\pi)^d} \left[(E_+)^L + (E_-)^L + 2(d-1) \right] \quad (5)$$

with $0 \leq p_\mu \leq 2\pi$

$$E_{\pm} = \sum_{\mu} \cos p_{\mu} \pm \sqrt{\left(\sum_{\mu} \cos p_{\mu} \right)^2 - 2d+1}$$

For large L the asymptotic behaviour is

$$N_L^{as} = \frac{7^L}{L^2} \left(\frac{3}{4\pi} \right)^2 \left(1 + O\left(\frac{1}{L}\right) \right) \quad \text{in } d=4 \quad (6a)$$

and

$$N_L^{as} = \frac{3^L}{L} \frac{1}{4\pi} \left(1 + O\left(\frac{1}{L}\right) \right) \quad \text{in } d=2 \quad (6b)$$

We have checked that N_L^{as} is a good approximation for reasonable values of L , for instance $L > 20$. To give a value: $N_{20}^{as}/N_{20} = 1.14$ in $d=4$ and $N_{20}^{as}/N_{20} = 1.10$ in $d=2$ dimensions.

As the continuum limit is approached for $K \rightarrow 1/2d$, the series (2) is divergent for $K > 1/2d$. In the region $1/2d < K \leq 1/2d-1$ the divergence must be due to the average over the γ traces and we may deduce the leading behaviour $\langle \text{Tr} \rangle_L \propto (2d/2d-1)^L$. More precisely, we have derived the formula

$$\langle \overline{\text{Tr}} \rangle_L = \frac{2^L}{4 N_L} \int \frac{d^d p}{(2\pi)^d} \left[(A_+)^L + (A_-)^L \right] \quad (7)$$

$$0 \leq p_\mu \leq 2\pi$$

with

$$A_\pm = 2 \left(\sum_\mu \cos p_\mu \pm i \sqrt{\sum_\mu \sin^2 p_\mu} \right)$$

For large L Eq. (7) has the asymptotic behaviour

$$\langle \overline{\text{Tr}} \rangle_L^{\text{as}} = \left(\frac{8}{7} \right)^L \frac{1024}{3 L^2} \left(1 + O\left(\frac{1}{L}\right) \right) \quad \text{in } d = 4 \quad (8a)$$

$$\langle \overline{\text{Tr}} \rangle_L^{\text{as}} = - \left(\frac{4}{3} \right)^L \frac{8}{L} \left(1 + O\left(\frac{1}{L}\right) \right) \quad \text{in } d = 2 \quad (8b)$$

Note the different signs in (8a) and (8b). Quite general in $d = 2n$ ($n = 1, 2, \dots$) dimensions the asymptotic sign is "+" for n even and "-" for n odd.

A remarkable point is the slow approach to the asymptotic behaviour. We have calculated $\langle \text{Tr} \rangle_L$ up to $L = 50$ ($d=4$) and $L = 44$ ($d=2$). The values are given in Table 1. Additionally we give the number N_L of fermionic paths up to $L = 20$ in Table 2. To guide the eye we have also graphically presented part of the results for $\langle \text{Tr} \rangle_L$ in a figure. We note that in four dimensions $\langle \text{Tr} \rangle_L$ changes sign for paths of length $L \geq 24$. In Ref. 5) $\langle \text{Tr} \rangle_L$ was considered up to $L = 12$ ($d=4$) and the sign change has not been noticed. The approach to the asymptotic behaviour (8) is slow but steady in two dimensions. In four dimensions the asymptotic behaviour is not attained even for paths of length as large as $L = 50$, only the derivative $d/dL \langle \text{Tr} \rangle_L^{\text{as}}$ reaches its asymptotic behaviour there. In view of the large number of paths (cf. e.g., N_{20}) this seems rather surprising.

Finally we remark that in the four-dimensional full theory¹⁾ long paths might become important if one has (as due to renormalization in the case of a massless pion) to approach the value $K = 1/8$ from above.

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Table 1

$\langle \text{Tr} \rangle_L$ in $d = 4$ and $d = 2$ up to $L = 50$

L	$\langle \text{Tr} \rangle_L, d = 4$	$\langle \text{Tr} \rangle_L, d = 2$
4	- 8.	- 4.
6	- 18.526 316	- 16.
8	- 22.660 033	- 18.370 370
10	- 25.159 705	- 26.947 368
12	- 25.775 427	- 36.280 795
14	- 24.742 890	- 51.714 117
16	- 22.177 103	- 75.134 144
18	- 18.200 307	- 112.452 446
20	- 12.907 348	- 171.979 346
22	- 6.360 958	- 268.076 476
24	1.411 375	- 424.242 295
26	10.418 896	- 679.785 967
28	20.709 069	- 1 100.346 288
30	32.369 771	- 1 796.073 260
32	45.532 646	- 2 952.259 113
34	60.378 008	- 4 881.489 882
36	77.141 600	- 8 112.361 140
38	96.123 604	-13 540.776 904
40	117.700 332	-22 688.316 418
42	142.339 155	-38 144.121 215
44	170.617 355	-64 321.672 245
46	203.245 803	-
48	241.098 548	-
50	285.249 590	-

Table 2

Number N_L of fermionic paths up to
 $L = 20$ in $d = 4$ and $d = 2$ dimensions

L	$N_L, d = 4$	$N_L, d = 2$
2	0	0
4	6	2
6	114	6
8	3 618	54
10	122 100	380
12	4 369 140	3 020
14	162 818 586	23 702
16	6 265 666 554	190 526
18	247 355 380 068	1 544 244
20	9 969 925 727 106	12 649 062

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FIGURE CAPTION

$\langle \text{Tr} \rangle_L$ versus L in $d = 4$ (upper line) and $d = 2$ dimensions (lower line).
The asymptotic behaviour is also given: dashed lines.

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Graph of $\log_{10} \frac{W}{W_0}$ vs. $\log_{10} \frac{L}{L_0}$ for $W_0 = 100$ and $L_0 = 10$

where W is weight and L is length. The curves are for different values of n and m .

The curves are for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and $m = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. The curves are for different values of n and m .