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UNITARY REALIZATIONS OF THE NON-COMPACT
SYMMETRY GROUPS OF SUPERGRAVITY

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INTRODUCTION

In this talk I will be reporting on some work done in collaboration with Cihan Saclioğlu on the oscillator-like unitary representations of the non-compact groups of extended supergravity theories^{1,2} (ESGT). Our construction uses boson annihilation and creation operators whose transformation properties are the same as the vector fields in the corresponding supergravity theories.

In the first part of my talk I will briefly summarize how non-compact groups emerge in ESGTs^{3,4} and motivate the study of their unitary representations following mainly the ideas of Ellis, Gaillard and Zumino^{5,6}. After this motivation I will give the bosonic construction of the Lie algebras of these non-compact groups and the corresponding unitary representations. In the last part of my talk I will discuss the relevance of these representations to the attempts to extract a realistic grand unified theory (GUT) from $N = 8$ ESGT^{5,6} and stress the point that with an additional assumption one may be able to obtain a realistic GUT based on $SU(5)$ in $N = 5$ ESGT with a generation group $U(1)$. I conclude with some comments on the infinite dimensional Lie superalgebras that contain the generators of supersymmetry and of the non-compact symmetry groups.

NON-COMPACT GROUPS OF EXTENDED SUPERGRAVITY THEORIES

In their important work on $N = 8$ ESGT, Cremmer and Julia have shown that in ESGTs for $N = 5, 6, 8$ the natural $SO(N)$ invariance that can be extended to $SU(N)$ via chiral dual transformations can

be further enlarged to a non-compact invariance group on-shell³. The first non-compact invariance group of this type was found for the $N = 4$ theory by Cremmer, Ferrara and Scherk⁷.

Under the action of the non-compact invariance group G the vector field strengths get transformed into their duals and together form a linear representation of G , whereas the scalar fields transform non-linearly as the coset space G/H where H is the maximal compact subgroup of G . The largest invariance group of these theories on-shell has the form $G_{\text{global}} \otimes H_{\text{local}}$ where the local invariance group H_{local} is isomorphic to (but not identical with) the maximal compact subgroup of G_{global} . The Fermi fields ($s = 1/2$ or $s = 3/2$) are all singlets under G_{global} and transform as some non-trivial linear representation of H_{local} . The graviton is a singlet of both G_{global} and H_{local} . The content of the fundamental fields that enter ESGTs ($N = 4-8$) in four dimensions and the corresponding groups G_{global} and H_{local} are listed below⁴:

Table 1

N	8	7	6	5	4
Spin 2	1	1	1	1	1
Spin 3/2	8	7+1	6	5	4
Spin 1	28	21+7	15+1	10	6
Spin 1/2	56	35+21	20+6	10+1	4
Spin 0	70	35+35	15+15	5+5	1+1
G_{global} Rank	$E_{7(+7)}$ 7	$E_{7(+7)}$ 7	$SO^*(12)$ 6	$SU(5,1)$ 5	$SU(4) \times SU(1,1)$ 4
H_{local} Rank	$SU(8)$ 7	$SU(8)$ 7	$U(6)$ 6	$U(5)$ 5	$U(4)$ 4

Cremmer and Julia constructed the $N = 8$ ESGT in four dimensions by dimensional reduction from the simple supergravity in eleven dimensions combined with duality transformations. The emergence of the exceptional group $E_{7(7)}$ as a symmetry of this theory is rather surprising. In fact $N = 8$ ESGTs in d space-time dimensions obtained from the eleven dimensional simple supergravity all have non-compact global invariance groups belonging to the E-series⁸. The Lie groups E_n are all finite dimensional for $n \leq 8$. The Lie algebra of E_9 is infinite dimensional and corresponds to a Kac-Moody extension of the Lie algebra of E_8 ⁹. The invariance groups of $N = 8$ ESGTs in various space-time dimensions d are listed below⁸:

Table 2

d	G_{global}	H_{local}
9	$GL(2, R)$	$SO(2)$
8	$E_{3(+3)} \equiv SL(3, R) \times SL(2, R)$	$SO(3) \times SO(2)$
7	$E_{4(+4)} \equiv SL(5, R)$	$SO(5)$
6	$E_{5(+5)} \equiv SO(5, 5)$	$SO(5) \times SO(5)$
5	$E_{6(+6)}$	$USp(8)$
4	$E_{7(+7)}$	$SU(8)$
3	$E_{8(+8)}$	$SO(16)$

Note that the rank of H_{local} is in general not equal to the rank of G_{global} .

THE QUESTION OF BOUND STATES IN EXTENDED SUPERGRAVITY THEORIES

The non-compact global invariance groups in four dimensional ESGTs make their appearance only in those theories that contain fundamental scalar fields. These scalar fields sit on the coset space G/H and the "gauge fields" associated with H_{local} are all composites of the scalar fields as in the two dimensional CP^N models¹⁰. The Lagrangian for the scalar fields has the general form

$$\mathcal{L} = \text{Tr} (V^{-1} D_{\mu} V)^2 \quad (1)$$

where

$$D_{\mu} V \equiv \partial_{\mu} V - V A_{\mu} \quad \text{and} \quad A_{\mu} = V^{-1} \partial_{\mu} V$$

with $V(x)$ denoting a matrix function of the scalar fields parametrizing the coset space G/H . The potential problem with ghosts due to the non-compactness of G is avoided by the gauging of its maximal compact subgroup¹¹.

Now as was pointed out by Gell-Mann the fundamental fields that enter the largest ESGT ($N=8$) in four dimensions do not have a rich enough structure to accommodate the basic fields of a realistic gauge theory of strong, weak and electromagnetic interactions¹². Thus it was thought that some of the fields entering such a theory might have to be made composites of the fundamental fields of $N = 8$ ESGT in order to make contact with elementary particle physics^{13,14}. The first important development in this direction came from the suggestion of Cremmer and Julia that the classically non-propagating composite gauge fields of $SU(8)_{\text{local}}$ in $N = 8$ ESGT may become dynamical on the quantum level³. Their suggestion was motivated by the analogy with the two-dimensional CP^N models whose study at large N limit shows that the composite gauge field develops a pole at $p^2 = 0$ in its propagator and hence becomes dynamical on the quantum level¹⁰. Nissimov and Pacheva have extended this analysis to the three-dimensional generalized non-linear sigma models and shown that in the large N limit these theories have a phase in which the composite gauge fields develop poles at $p^2 = 0$ in their propagators and become propagating^{15,16}.

The major attempt at making contact with the elementary particle physics came from Ellis, Gaillard and Zumino (originally together with Maiani) who postulated that in $N = 8$ ESGT in addition to massless gauge fields other massless bound states (fermionic and bosonic) form whose low energy effective theory is a grand unified theory (GUT) based on $SU(5)$ with three generations of quarks and leptons^{5,6}. All the "elementary" particles entering this GUT (quarks, leptons, gauge fields, Higgs fields) are all composites of the fundamental "preon" fields of $N = 8$ ESGT with a binding scale of order m_{planck} . The only truly elementary particle at low energies is the graviton which is a singlet of the $E_{7(7)}_{\text{global}} \otimes SU(8)_{\text{local}}$ symmetry of the theory.

Ellis, Gaillard and Zumino chose the supercurrent multiplet of bound states from which to construct a realistic GUT. In this multiplet the vector bound states transform like the adjoint representation of $SU(8)_{\text{local}}$. In addition to the particles needed for a realistic GUT this supercurrent multiplet contains many unwanted helicity states. These unwanted helicity states cannot be

made supermassive in an $SU(5)$ or $SU(3) \times SU(2) \times U(1)$ invariant fashion without introducing an infinite set of supermultiplets of bound states^{6,17,18}. In fact, an infinite set of supermultiplets seems to be necessary for a vector-like embedding of $SU(3)$ colour in these theories¹⁷⁻¹⁹. Both of these are semi-phenomenological motivations for looking for an infinite set of bound states in the $N = 8$ ESGT.

On the other hand there are strong theoretical arguments indicating that in ESGTs with non-compact invariance groups there may be infinite set of bound states formed. The strongest of these arguments is the analogy with two- and three-dimensional generalized σ models. We have already mentioned that in the two-dimensional generalized CP^N models with $G_{\text{global}} \times H_{\text{local}}$ invariance the composite gauge fields associated with H_{local} become dynamical on the quantum level. Haber et al. have shown that, in the large N limit, these gauge fields get confined and act as the binding force between the scalar fields, and the spectrum of theory consists of scalar bound states that transform like a linear representation of the parent global symmetry group G_{global} ²⁰. Similarly, Nissimov and Pacheva have shown that, in the three-dimensional supersymmetric generalized σ -models, the phase in which the composite gauge fields become dynamical massless "gluons" on the quantum level has a bound state spectrum consisting of equal mass $s = 0$ bosons and $s = 1/2$ fermions transforming like a linear representation of the global invariance group G_{global} ¹⁶.

Now, as was first mentioned by Zumino¹⁸ and later elaborated by Ellis, Gaillard and Zumino⁶, if the $N = 8$ supergravity theory has a phase in which the composite gauge fields of $SU(8)_{\text{local}}$ become dynamical and act as the binding force between the other fields, in such a way that the bound state spectrum corresponds to a linear representation of $E_{7(7)}$, then one must have an infinite set of bound states since all the non-trivial unitary representations of non-compact groups are infinite dimensional.

Moreover, as was reported in this workshop²¹, Grisaru and Schnitzer have shown that under certain reasonable assumptions the scattering amplitudes in extended supergravity theories seem to reggeize, indicating again the possibility of an infinite set of bound states²².

Further clues in this direction come again from the study of two-dimensional models. Makhankov and Pashaev²³, in their study of the non-linear Schrödinger equation with a non-compact $U(1,1)$ invariance, find that the spectrum of soliton solutions is far richer than the compact case and suggest that this be understood in the language of unitary realizations of the non-compact invariance group. Similar results were obtained by Rabinovici in the case of σ models with a non-compact global invariance group²⁴.

All these theoretical arguments provide sufficient motivation for a study of the unitary representations of the non-compact groups of ESGTs.

THE BOSONIC CONSTRUCTION OF THE LIE ALGEBRAS OF THE NON-COMPACT GROUPS OF EXTENDED SUPERGRAVITY THEORIES

The construction of the Lie algebras of non-compact groups using boson annihilation and creation operators is well known in the physics literature²⁵. Consider, for example, a set of boson annihilations and creation operators a_i and $a_i^\dagger$ ($i=1,\dots,n$) satisfying the commutation relations

$$\begin{aligned} [a_i, a_j^\dagger] &\equiv [a_i, a_j] = \delta_i^j & i, j = 1, \dots, n \\ [a_i, a_j] &= 0 = [a_i^\dagger, a_j^\dagger] & a_i^\dagger \equiv a^i \end{aligned} \quad (2)$$

where the creation operators are denoted with upper indices. Then the bilinear operators $a^i a_j \equiv a_i^\dagger a_j$ generate the Lie algebra of $U(n)$ under commutation. The diboson annihilation and creation operators $a_i a_j$ and $a^i a^j$ close into the set $a^i a_j$ under commutation and all together they generate the Lie algebra of the non-compact group $Sp(2n, \mathbb{R})$. If one uses two sets of boson operators $a_i (a^i)$ and $b_i (b^i)$ $i = 1, \dots, n$ then the generators $I_j^i = a^j a_i + b_i b^j$ of $U(n)$ can be extended to the Lie algebra of $SP(2n, \mathbb{R})$ by using symmetric diboson annihilation and creation operators:

$$S_{ij} = a_i b_j + a_j b_i, \quad S^{ij} = a^i b^j + a^j b^i \quad (3)$$

or to the Lie algebra of $SO(2n)^*$ by using antisymmetric diboson annihilation and creation operators^{1,2}:

$$A_{ij} = a_i b_j - a_j b_i, \quad A^{ij} = a^i b^j - a^j b^i \quad (4)$$

Note that in either extension the rank of the Lie algebra does not change. The Lie algebras of $SU(m, n)$ and $SO(m, n)$ can be similarly constructed².

In Ref. 1 this standard construction has been extended to the case when the boson operators transform like the antisymmetric tensor representation of $SU(n)$. Under the restriction that the Lie algebra of the non-compact group have the same rank as its maximal compact subgroup [$U(n)$ or $SU(n)$ as the case may be] one finds that such an extension yields a finite dimensional Lie algebra only for $n \leq 8$. For $n < 4$ this extended construction corresponds to the standard construction outlined above. For $n = 4, 5, \dots, 8$ one obtains exactly the Lie algebras of the non-compact

symmetry groups of ESGTs in four dimensions for $N = 4, 5, \dots, 8$. Let us now summarize this construction using an arbitrary number of pairs of boson annihilation and creation operators as was done in Ref. 2. The use of more than one pair of boson operators corresponds to taking direct sums of copies of the resulting Lie algebra. However, this trivial extension on the Lie algebra level makes it possible to construct much larger classes of unitary irreducible representations as will be explained in the next section.

Consider the antisymmetric boson operators

$$a_{ij}(K) = -a_{ji}(K) \quad \text{and} \quad b_{ij}(K) = -b_{ji}(K) \\ i, j = 1, \dots, n \quad K = 1, \dots, N$$

that satisfy the commutation relations

$$[a_{ij}(K), a^{kl}(L)] = \delta^{KL} (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) = [b_{ij}(K), b^{kl}(L)] \quad (5) \\ [a_{ij}(K), a_{kl}(L)] = 0 = [b_{ij}(K), b_{kl}(L)]$$

where the creation operators are denoted by upper SU(n) indices. Then the bilinear operators

$$I_j^i \equiv \vec{a}^{im} \cdot \vec{a}_{jm} + \vec{b}_{jm} \cdot \vec{b}^{im} \equiv \sum_{K=1}^N \sum_{m=1}^n a^{im}(K) a_{jm}(K) + b_{jm}(K) b^{im}(K) \quad (6)$$

generate the Lie algebra of U(n). Separating out the trace part Q generating the U(1) subgroup we can write it as:

$$I_j^i = T_j^i + \frac{1}{n} \delta_j^i Q, \quad Q = \vec{a}^{kl} \cdot \vec{a}_{kl} + \vec{b}_{kl} \cdot \vec{b}^{kl} \quad (7)$$

where T_j^i generate the SU(n) Lie algebra.

$n = 4$: In this case the Lie algebra of U(4) can be extended to the Lie algebra of $SU(4) \times SU(1,1)$ by the diboson annihilation and creation operators Q^- and Q^+ defined as

$$Q^+ = \frac{1}{4} \epsilon_{ijkl} \vec{a}^{ij} \cdot \vec{b}^{kl} \quad Q^- = \frac{1}{4} \epsilon^{ijkl} \vec{a}_{ij} \cdot \vec{b}_{kl} \quad (8)$$

$$[Q, Q^+] = \mp Q^\mp \quad (9) \\ [Q^-, Q^+] = 2Q$$

n = 5: Again using the SU(5) invariant totally antisymmetric ϵ tensor we can define diboson annihilation and creation operators

$$\begin{aligned} A_i &= \frac{\sqrt{2}}{4} \epsilon_{ijklm} \vec{a}^{jk} \cdot \vec{b}^{lm} \\ A^i &= \frac{\sqrt{2}}{4} \epsilon^{ijklm} \vec{a}_{jk} \cdot \vec{b}_{lm} \end{aligned} \quad (10)$$

which close into the I_j^i and together form the Lie algebra of SU(5,1):

$$\begin{aligned} [A_i, A^j] &= T_i^j - \frac{3}{10} \delta_i^j Q \\ [T_j^i, A_k] &= -\delta_k^i A_j + \frac{1}{5} \delta_j^i A_k \\ [Q, A_i] &= 4A_i \end{aligned} \quad (11)$$

n = 6: In this case the diboson operators constructed using the ϵ tensor do not close into the U(6) subalgebra. To close the algebra one has to introduce pairs of boson annihilation and creation operators $\vec{v}(\vec{v}^\dagger)$ and $\vec{w}(\vec{w}^\dagger)$ that are singlet under SU(6). Then the diboson operators

$$\begin{aligned} A_{ij} &= \frac{1}{4} \epsilon_{ijklmp} \vec{a}^{kl} \cdot \vec{b}^{mp} + \frac{1}{\sqrt{2}} (\vec{a}_{ij} \cdot \vec{v} + \vec{b}_{ij} \cdot \vec{w}) \\ A^{ij} &= \frac{1}{4} \epsilon^{ijklmp} \vec{a}_{kl} \cdot \vec{b}_{mp} + \frac{1}{\sqrt{2}} (\vec{a}^{ij} \cdot \vec{v}^\dagger + \vec{b}^{ij} \cdot \vec{w}^\dagger) \end{aligned} \quad (12)$$

together with SU(6) and U(1) generators

$$\begin{aligned} T_j^i &= \vec{a}^{im} \cdot \vec{a}_{jm} + \vec{b}^{im} \cdot \vec{b}_{jm} - \frac{1}{6} \delta_j^i (\vec{a}^{kl} \cdot \vec{a}_{kl} + \vec{b}^{kl} \cdot \vec{b}_{kl}) \\ Q &= \vec{a}^{kl} \cdot \vec{a}_{kl} + \vec{b}^{kl} \cdot \vec{b}_{kl} - 6 \vec{v}^\dagger \cdot \vec{v} - 6 \vec{w} \cdot \vec{w}^\dagger \end{aligned} \quad (13)$$

form the Lie algebra of SO(12)*:

$$\begin{aligned} [A_{ij}, A^{kl}] &= \frac{1}{2} (\delta_i^k T_j^\ell + \delta_j^\ell T_i^k - \delta_i^\ell T_j^k - \delta_j^k T_i^\ell) - \frac{1}{12} (\delta_i^k \delta_j^\ell - \delta_j^k \delta_i^\ell) Q \\ [T_j^i, A_{kl}] &= -\delta_k^i A_{jl} - \delta_l^i A_{kj} + \frac{1}{3} \delta_j^i A_{kl} \\ [Q, A^{kl}] &= 4A^{kl} \end{aligned} \quad (14)$$

n = 7: In this case as well the diboson operators constructed from the antisymmetric tensor boson operators do not close into the U(7) generators. To close the algebra one needs to introduce pairs of boson operators $\vec{v}_i(\vec{v}^i)$, $\vec{w}_i(\vec{w}^i)$ transforming like the fundamental representation of SU(7). Doing this however makes the construction in this case equivalent to the case for n = 8. Referring the reader for details to Refs. 1 and 2 we now give the n = 8 construction.

n = 8: In this case the diboson annihilation and creation operators constructed using the ϵ tensor transform like the totally antisymmetric tensor representation of rank four under SU(8). Since this corresponds to a real representation 70 of SU(8) one has to take "self-conjugate" combinations of the diboson operators. Thus one finds that the diboson operators

$$V_{ijkl} = \vec{a}_{[ij} \cdot \vec{b}_{kl]} + \frac{1}{4} \epsilon_{ijklrstu} \vec{a}^{rs} \cdot \vec{b}^{tu} \quad (15)$$

where the bracket $[ijkl]$ denotes antisymmetrization of all four indices, close into the Lie algebra of SU(8):

$$\begin{aligned} [V_{ijkl}, V_{mnpq}] &= -\frac{1}{144} \epsilon_{ijklrstu} \epsilon^{vstuabcd} \epsilon_{abcdmnpq} T_v^r \\ [T_j^i, V_{mnpq}] &= \frac{1}{6} \epsilon_{mnpqjrst} V^{irst} - \frac{1}{2} \delta_j^i V_{mnpq} \end{aligned} \quad (16)$$

and together form the Lie algebra of $E_{7(7)}$. The self-conjugacy of the 70 operators V_{ijkl} is reflected in the identity

$$V_{ijkl}^\dagger \equiv V^{ijkl} = \frac{1}{4!} \epsilon^{ijklrstu} V_{rstu}$$

To summarize, we have the following representation content for the boson operators for various n and the resulting Lie algebras L:

			<u>L</u>
n = 4:	$a_{ij}(b_{ij})$	<u>6</u>	of $SU(4) \rightarrow SU(4) \times SU(1,1)$
n = 5:	$a_{ij}(b_{ij})$	<u>10</u>	of $SU(5) \rightarrow SU(5,1)$
n = 6:	$a_{ij}(b_{ij}) \oplus v(w)$	<u>15+1</u>	of $SU(6) \rightarrow SO(12)^*$
n = 7:	$a_{ij}(b_{ij}) \oplus v_i(w_i)$	<u>21+7</u>	of $SU(7) \rightarrow E_{7(7)}$
n = 8:	$a_{ij}(b_{ij})$	<u>28</u>	of $SU(8) \rightarrow E_{7(7)}$

The boson operators have exactly the same representation content as the vector fields in extended supergravity theories and the Lie algebras (L) are those of the non-compact groups (G) of the respective theories as well. Moreover, under the action of the non-compact group generated by L, the boson operators $a_{ij}(b_{ij})$ get mixed with $b_{ij}(a_{ij})$ and together form a linear representation of G. The vector field strengths in ESGTs and their duals transform into each other under the action of the non-compact invariance group and form a linear representation in exactly the same fashion.

OSCILLATOR-LIKE UNITARY REPRESENTATIONS OF NON-COMPACT GROUPS OF SUPERGRAVITY

With the exception of $E_{7(7)}$ all the Lie algebras constructed above have a Jordan structure with respect to their subalgebras generating a maximal compact subgroup, i.e., they decompose as a vector space direct sum

$$L = L^- \oplus L^0 \oplus L^+ \quad (17)$$

where

$$(L^+)^{\dagger} \cong L^- \quad (L^0)^{\dagger} \cong L^0$$

and L^0 contains the generator Q of a U(1) factor group which gives the grading, i.e.,

$$\begin{aligned} [Q, L^+] &= L^+ \\ [Q, L^-] &= -L^- \\ [Q, L^0] &= 0 \end{aligned} \quad (18)$$

The simple non-compact Lie groups which have a Jordan structure with respect to their maximal compact subgroups have been classified²⁶. They are the groups $SU(m, n)$, $SO(n, 2)$, $Sp(2n, \mathbb{R})$, $SO(2n)^*$, $E_6(-14)$ and $E_7(-25)$, where $E_6(-14)$ and $E_7(-25)$ have maximal compact subgroups $SO(10) \times SO(2)$ and $E_6 \otimes SO(2)$, respectively. The Lie algebras of all these groups can be constructed in terms of boson annihilation and creation operators and one can give a general construction of a class of unitary representations of these groups in the Fock space of the boson operators^{1,2}.

Consider the Fock space constructed from the tensor product of Fock spaces of individual boson operators. The vacuum state $|0\rangle$ will be a tensor product of the individual vacua $|0\rangle$

$$|0\rangle \equiv |0\rangle|0\rangle\ldots|0\rangle$$

which is annihilated by all the annihilation operators.

$$a_i(K) |0\rangle = 0 = b_i(K) |0\rangle \quad K = 1, \dots, N$$

Now choose a set of states $|\psi_A\rangle$ in our Fock space which is annihilated by all the diboson operators in the L^- space and transform as a certain representation of the maximal compact subgroup generated by L^0 :

$$L^- |\psi_A\rangle = 0 \quad (19)$$

Then the infinite set of states generated by applying the operators L^+ on $|\psi_A\rangle$ form the basis of a unitary representation of the non-compact group G :

$$|\psi_A\rangle, L^+ |\psi_A\rangle, L^+ L^+ |\psi_A\rangle, \dots \quad (20)$$

If the states $|\psi_A\rangle$ are chosen such that they transform like an irreducible representation of the maximal compact subgroup then the corresponding representation of the non-compact group is also irreducible. The proof of this theorem is very simple and uses only the condition (19) and the Jordan structure of the Lie algebra L^1 .

We should note that the Jordan structure does not guarantee the existence of states $|\psi_A\rangle$ annihilated by L^- space that transform irreducibly under the compact subgroup. For example, in the construction of the Lie algebra of $SO(12)^*$ in terms of antisymmetric tensor and singlet boson operators of $SU(6)$ [see Eq. (12)], such states do not exist in the Fock space, whereas in the construction of $SO(12)^*$ in terms of boson operators transforming like the fundamental representation of $SU(6)$ there exists an infinite set of such states. In cases when there does not exist any such set of states $|\psi_A\rangle$ annihilated by the L^- space the application of our method leads to reducible unitary representations.

To illustrate our construction we now give a detailed study of the corresponding representations of $SU(5,1)$ using its Lie algebra given by Eq. (11). Now any of the states of the form

$$[a^{ij}(1)]^{n_1} [a^{kl}(2)]^{n_2} \dots [a^{pq}(N)]^{n_N} |0\rangle$$

or of the form

$$i, j, \dots = 1, \dots, 5$$

$$[b^{ij}(1)]^{n_1} [b^{kl}(2)]^{n_2} \dots [b^{pq}(N)]^{n_N} |0\rangle$$

are annihilated by the operators $A^i = \epsilon^{ijk\ell m} a_{jk} b_{\ell m}$ belonging to the L^- space. Any irreducible subset $|\psi_A\rangle$ of these states can be used as the starting representation of $U(5)$ for the construction of a unitary irreducible representation (UIR) of $SU(5,1)$.

Restricting ourselves to the case of only one pair of boson operators a and b and assuming that the annihilation operators transform as the representation $\underline{10}$ of $SU(5)$ we find the following transformation properties of various states:

$$\begin{aligned} a^{jk}|0\rangle &\rightarrow |(1,1,1,0,0)\rangle \\ a^{jk}a^{\ell m}|0\rangle &\rightarrow |(2,1,1,1,1)\rangle + |(2,2,2,0,0)\rangle \\ b^{jk}|0\rangle &\rightarrow |(1,1,1,0,0)\rangle \\ \dots \end{aligned} \quad (21)$$

where $(m_1, \dots, m_5)$ denotes a representation of $U(5)$ whose Young Tableaux contain m_i boxes in the i th row. It is easy to see that any irreducible representation of $SU(5)$ with a definite $U(1)$ charge can be constructed by repeated application of a^{jk} followed by a suitable projection operator. Thus we can take any irreducible representation of $SU(5)$ as the starting states for the construction of an irreducible representation of $SU(5,1)$.

Starting from states $|\psi_A\rangle$ transforming like a representation $(m_1, m_2, m_3, m_4, m_5)$ of $U(5)$ that are annihilated by the operators A_i^1 of the L^- space we can construct the states

$$(A_i)^k |\psi_A\rangle \quad k = 1, 2, \dots$$

by repeated application of the operators A_i of the L^+ space. Under $U(5)$ they transform like

$$\begin{aligned} (A_i)^k |\psi_A\rangle &\xrightarrow{\sim} (2k, k, k, k, k) \otimes (m_1, m_2, m_3, m_4, m_5) \\ k &= 0, 1, 2, \dots \end{aligned} \quad (22)$$

and form the basis of an irreducible unitary representation of $SU(5,1)$.

In the case of the Lie algebra of $E_{7(7)}$ we do not have a Jordan structure with respect to the Lie algebra of its maximal compact subgroup $SU(8)$. It decomposes as

$$L = T_j^i \oplus V_{ijkl} = 63 \oplus 70 \quad (23)$$

where T_j^i are the generators of $SU(8)$ and V_{ijkl} the non-compact generators. On a set of states $|\psi_A\rangle$ that are constructed by acting on the vacuum state with the creation operators a^{ij} ($i, j = 1, \dots, 8$) and which transform irreducibly under $SU(8)$ we can apply the non-compact generators to generate new states

$$|\psi_A\rangle, V_{ijkl}|\psi_A\rangle, VV|\psi\rangle, \dots \quad (24)$$

Clearly these states form the basis of a unitary representation of $E_{7(7)}$. However, they are infinitely reducible. This can be seen as follows: the product of two copies of the V 's contains an $SU(8)$ singlet, i.e.,

$$\begin{aligned} V_{ijkl} V_{mnpq} &= \frac{1}{2} \{V_{ijkl}, V_{mnpq}\} + \frac{1}{2} [V_{ijkl}, V_{mnpq}] = \\ &= (1 + 720 + 1764)_{\text{sym}} + 63_{\text{antisym}} \end{aligned} \quad (25)$$

This means that every irreducible representation of $SU(8)$ that occurs in the infinite set of states (24) will reappear again after two applications of the V 's. Thus the multiplicity of an $SU(8)$ representation occurring in the set (24) is infinite. This implies that the unitary representation we have is infinitely reducible since the multiplicity of an irreducible representation of the maximal compact subgroup inside a UIR of a non-compact group is always less than or equal to its dimension²⁷.

There is a method due to Gell-Mann for constructing a class of UIRs of some non-compact groups on certain coset spaces of their maximal compact subgroups^{28,29}. His method is applicable to $E_{7(7)}$ and is quite simple for determining the multiplicities of representations of $SU(8)$. For example, one possible coset space on which to realize the UIRs of $E_{7(7)}$ is $SU(8)/Sp(8)$. In this case the multiplicities of representations of $SU(8)$ inside a UIR of $E_{7(7)}$ are determined by the number of $Sp(8)$ singlets they contain. Unfortunately, this method cannot be applied to our construction simply because the boson operators we use transform linearly under $SU(8)$ rather than non-linearly as a certain coset space of $SU(8)$ satisfying Gell-Mann's criteria²⁹. However, though reducible, our representations of $E_{7(7)}$ may still be of relevance for physical applications since, as explained in the next section, the compatibility of supersymmetry with non-compact invariance groups do in general imply reducible unitary representations.

GRAND UNIFIED THEORIES AND NON-COMPACT SYMMETRY GROUPS OF SUPERGRAVITY

In their attempt to extract an effective "low energy" GUT from the $N = 8$ ESGT, Ellis, Gaillard and Zumino have considered only the supercurrent multiplet of bound states from which to choose their "elementary" particles (quarks, leptons, etc.). The choice of this multiplet was dictated by the requirement that it contain the composite gauge fields of $SU(8)_{\text{local}}$. In addition to the "low energy" "elementary" particles this composite multiplet contains many unwanted helicity states. Since there seems to be no experimental evidence for the existence of these additional particles at low energies it was suggested that they be made

superheavy at the order of Planck mass³⁰. The same problem also arises in the scheme of Derendinger, Ferrara and Savoy who do not restrict themselves to the current supermultiplet alone³¹. To make these unwanted states massive in an $SU(5)$ or even $SU(3) \times SU(2) \times U(1)$ invariant fashion so as to be left with three families of chiral massless fermions, one needs to introduce an infinite set of composite supermultiplets^{6,17}. The fact that one needs to introduce an infinite set of supermultiplets to get rid of the unwanted helicity states leads one to consider an alternative possibility: instead of taking both the gauge fields and the "matter fields" (quarks and leptons) for an effective low energy GUT from the same supermultiplet one takes the gauge fields from one supermultiplet and the matter fields from other composite supermultiplets. If such a scenario is adopted one need not go to the largest ESGT to obtain a realistic GUT as an effective low energy theory. Already the $N = 5$ ESGT with $SU(5,1)_{\text{global}} \times U(5)_{\text{local}}$ invariance may in principle be large enough to accommodate such a theory. If the gauge fields were taken from the supercurrent multiplet then in analogy with the CP^N models we would expect them to be singlets under the global invariance group G_{global} ³². Thus if one goes outside the supercurrent multiplet, to be consistent one must then choose both the matter multiplets and the gauge fields from among the infinite set of bound states forming representations of G_{global} . Such a scenario however requires a resolution of an apparent conflict between supersymmetry and unitary realizations of G_{global} . As mentioned earlier the basic Fermi fields of ESGTs are all inert under G_{global} . Thus one would naively expect only the bosonic bound states with integer helicity to form unitary representations of G_{global} . But if supersymmetry is valid at any level then it would imply that these infinite towers of bosonic bound states have fermionic partners and together form infinite towers of supermultiplets. This in turn means that the fermionic bound states must also form unitary representations of the non-compact G_{global} . One may ask whether there is a super invariance group of these theories that contains the non-compact symmetry group as well as the supersymmetry transformations and transforms the bosonic and fermionic bound states into each other. The fact that there is no simple finite dimensional supergroup whose even subgroup contains $E_{7(7)}$ as a factor group makes one suspect that such a group may be infinite dimensional. In fact, when one takes the commutator of non-compact symmetry generators and supersymmetry generators one obtains new generalized supersymmetry generators⁶. Thus starting from the Lie algebra of the non-compact group and supersymmetry generators one can generate an infinite dimensional superalgebra with generalized momenta and generalized supercharges^{6,33}. The existence of this infinite dimensional superalgebra does not automatically ensure the compatibility of unitary realization of G_{global} on the bound states and supersymmetry. One has to show that one can realize this infinite dimensional algebra unitarily³³. For the class of UIRs given in the previous section

this compatibility can be shown easily³³ as follows: in the infinite set of generalized supersymmetry generators thus obtained, there are some that commute with the non-compact generators belonging to the L^- space. These operators F_α carry half integer helicity $|\lambda| = 1/2$. Thus by applying on the initial set of states $|\psi_A\rangle$ annihilated by the L^- space with these generalized supergenerators F_α we can create half-integer helicity states annihilated by the L^- space³⁴:

$$L^- F_\alpha |\psi_A\rangle = F_\alpha L^- |\psi_A\rangle = 0$$

By repeated application of the F_α 's we can create half-integer and integer spin helicity states that form a supermultiplet. All of these states are annihilated by the operators in the L^- space and hence can be used as the starting states for constructing oscillator-like unitary representations. Diagrammatically we have

At each helicity λ we have an infinite tower of states forming a unitary representation of G_{global} . This shows that the unitary realizations of the non-compact symmetries of ESGTs on the bound states can be made compatible with supersymmetry. However, the resulting unitary representations will in general not be irreducible. This is because even if the initial set of states $|\psi_A\rangle$ transform irreducibly under the maximal compact subgroup H of G_{global} , the states $F_\alpha|\psi_A\rangle$, $F_\alpha F_\alpha|\psi_A\rangle$, ... etc., are in general reducible and consequently the unitary representations of G_{global} obtained by repeated application of L^+ operators on these states will also be reducible. The infinite dimensional group that contains G_{global} and the supersymmetry transformations will transform all these states into each other. It would be interesting to know if one such unitary supermultiplet will be rich enough to accommodate all the elementary particles of a realistic GUT.

Before concluding let us remind ourselves once more that the crucial problem in the attempts to extract a realistic GUT from ESGTs is to show that there are indeed bound states formed that

correspond to the "elementary" particles observed so far. If they are formed and fall into representations of $G_{\text{global}} \times H_{\text{local}}$ invariance of these theories one must then show that one can break these larger symmetries as well as the supersymmetries so as to be left with the observed low energy symmetries⁶ and particles. One would also like to know what this breaking implies for the infinite dimensional superalgebra incorporating supersymmetry and the global invariance group. On the other hand as was reported in this meeting in the largest ESGT (N=8) one has the option of gauging the SO(8) invariance at the price of losing $E_{7(7)}$ global invariance while still preserving supersymmetry^{35,36}. If the bound states do indeed form representations of $E_{7(7)}$ what would happen to them when one turns on the SO(8) gauge couplings? Do they become unbound or do the non-singlets of SO(8) get confined? Will there be any trace of $E_{7(7)}$ left? Hopefully by the time of the next meeting we will have answers to some of these problems.

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