ON SEVERAL PARAMETER RENORMALIZATION GROUP EQUATIONS
AND RENORMALIZATION SCHEME SPECIFICATIONS

A. Peterman
CERN - Geneva

A B S T R A C T

An example of the explicit realization of the full renormalization group for physical quantities is investigated. It introduces two-indice beta functions $\beta_{ij}(g_j)$:

- i) the first index specifying the renormalization scheme chosen (including the scale μ);
- ii) the second index specifying the coupling of which the appropriate beta function is the differential transformation.

The treated example is presented in a 4- ϵ dimensional $O(n)$ massless $(\vec{\varphi} \cdot \vec{\varphi})^2$ theory with emphasis on the calculation of the fixed point.

Recently Stevenson ¹⁾ gave for the first time an explicit complete realization of the full renormalization group ²⁾ and not just the subgroup for scale transformations of the subtraction point ^{2),3)}. This realization is restricted to quantities of physical significance (i.e., to measurable quantities) in the massless case with one single coupling constant ³⁾. Its aim is to specify the renormalization scheme (RS) by this realization, which he shows to be effectively so in the restricted cases he has considered. However, the generalization to more complicated cases should not present insurmountable difficulties. Multiplicative renormalizable quantities are now being studied and, with more unphysical parameters, seem to account for such quantities as, for example, the Green's functions of a given theory. The realization ¹⁾ is not specific to QCD, but embodies any Lagrangian field theories for which perturbative expansions are valid. Stevenson's ¹⁾ aim is to determine the optimum value of a perturbative result truncated after the i th term by searching for those values of the parameters which characterize the RS, the symbolic expression $\partial R^{(i)} / \partial(RS) = 0$. Here $R^{(i)}$ is a physical quantity truncated at the i th term for both the coefficients and for the coupling constant. The latter is indeed defined as the solution of the equation

$$\frac{\partial a^{(i)}}{\partial \tau} = b a^{(i)2} (1 + c_1 a^{(i)} + c_2 a^{(i)2} + \dots + c_{i-1} [a^{(i)}]^{i-1}) \quad (1)$$

whereas $R^{(i)}$ is given by

$$R^{(i)} = a^{(i)N} (1 + r_1 a^{(i)} + \dots + r_{i-1} [a^{(i)}]^{i-1}) \quad (2)$$

N = any real number, even non-integer.

Stevenson exploits the fact that the full renormalization group ²⁾, in addition to describing scale transformations, also expresses the property that, summed to its solution, the perturbative expansion becomes independent of the RS [the full renormalization group (RG) accounts for both scale transformations and RS changes. This is proved ¹⁾ in particular for physical quantities in the massless case with a single coupling constant. But the full RG is large enough to embody other cases too.] However, the truncated expressions are RS dependent and differ by a term of order a^{i+1} if the expansion has been truncated at the i th term. Moreover, in such a case, the set of equations is finite, and only i of them are non-trivial. More explicitly, for the physical quantity R , one has the set of RG equations which guarantees the invariance of R under general RG transformations :

$$\frac{\partial R}{\partial \tau} = \left\{ \frac{\partial}{\partial \tau} \Big|_{a \text{ const.}} + \beta(a) \frac{\partial}{\partial a} \right\} R = 0 \quad (3)$$

for $j=1$ with some limiting procedure specified and

$$\frac{\partial R}{\partial c_j} = \left\{ \frac{\partial}{\partial c_j} \Big|_{a \text{ const.}} + \beta_j(a) \frac{\partial}{\partial a} \right\} R = 0, \quad (j=2,3,\dots) \quad (4)$$

As was said before, only $i-1$ of these equations are non-trivial in i th order perturbation theory. Equations (3) and (4) are formally satisfied in i th order and in any RS due to cancellations between the coefficients r and the coupling parameter a in the expansion of R [see Eq. (2)]. This is only a formal result since there is a remainder, but this remainder is of the same order as other terms already neglected. However, this remainder can be large if the series does not converge. Furthermore, it also indicates the likely behaviour of the higher order terms since these must be such as to satisfy (3) and (4), namely that the derivatives with respect to the renormalization scheme parameters must finally yield zero. This criterion determines the optimum RS according to Stevenson's analysis ¹⁾. [See, however, Refs. 4) and 5).] *)

---

*) Pennington ⁴⁾ in his analysis introduces also the two indices $h_{\alpha i}(\alpha)$ under the form $h_{\alpha i} = (\partial \alpha(R)) / (\partial \rho_i)$ where $\alpha(R)$ is the coupling in some RS, say R , related to a scheme of reference $\alpha(0)$ by

$$\alpha(R) \equiv \alpha^{(n)}(R) = \alpha^{(n)}(0) [1 + \rho_1 \alpha^{(n)}(0) + \dots + \rho_n (\alpha^{(n)}(0))^n]$$

[in $\alpha^{(n)}$, n indicates the order of truncation]. His realization, instead of taking the coefficients of the usual b function: $b_i(R)$, as in Ref. 1), takes as independent variables the ρ_i appearing in the expression of $\alpha(R)$ in terms of $\alpha(0)$, i.e., the set $\{\rho_i\}$. He gives explicitly $h_{1\alpha}$, $h_{2\alpha}$ and $h_{3\alpha}$ [respectively in his notation $(\partial \alpha(R)) / (\partial \rho_1)$, $(\partial \alpha(R)) / (\partial \rho_2)$, ... Eqs. (38), (35) and (32) of Ref. 4] together with equations of the type

$$\frac{\partial \pi}{\partial \rho_i} = \left\{ \frac{\partial}{\partial \rho_i} \Big|_{\alpha(R)} + h_{i\alpha} \frac{\partial}{\partial \alpha} \right\} \pi = 0$$

analogous to Eq. (4) above [Ref. 1)], π being a physical quantity in scheme R [notice his formula for $\pi^{(4)}(R)$, Eq. (29).

$$\pi^{(4)}(R) = \pi_0 + \alpha(R) \pi_1 + \alpha^2(R) \pi_2(R) + \alpha^3(R) \pi_3(R) + \alpha^4(R) \pi_4(R)] .$$

For physical observables, Stevenson's choice of renormalization parameters is a complete one since any quantity which depends on a and τ only through $a(\tau, c_1)$ satisfies (3) as does R and depends on the same renormalization parameters as a . In other words, such a quantity is a function of the coefficients of the β function $c_2, c_3, \dots$ [Eq. (1)] characterizing a in a particular RS.

We now report on an example in solid state theory; the $O(n)\chi(\vec{\phi}, \vec{\phi})^2$ massless theory in $4-\epsilon$ dimensions, where one sees explicitly the cancellations which make the n th order formally satisfy Eq. (4). Moreover, these cancellations are used to find the position of the fixed point without having to resort directly to the β_1 function (the usual RG β function for scale transformations). However, no attempt is made to optimize the result following the more sophisticated method of Stevenson¹⁾ (see, however, the additional remarks). Indeed, this would require a still more "improved" approximation method over the already "improved perturbation theory" of the commonly used subgroup of the full RG. (This new approximation method is no longer a polynomial in the coupling a .)

We therefore consider renormalization groups with more than the single parameter μ , the renormalization mass, and allow two additional parameters σ and ρ . While an over-all change in μ , $\mu = \mu \exp(-c_1)$ is still possible, one can, for instance, renormalize the wave function, not at $p^2 = \mu^2$ but at $p^2 = \mu^2 e^{2\sigma}$, i.e., ($\sigma \equiv c'_2$)

$$\Gamma^{(2)}(p, \lambda) \Big|_{p^2=0} = 0 \quad (\text{massless})$$

$$\frac{\partial}{\partial p^2} \Gamma^{(2)}(p, \lambda) \Big|_{p^2 = \mu^2 e^{2\sigma}} = 1 \quad (\text{wave function renormalization})$$

The third parameter ρ is connected with the definition of the coupling constant or, more precisely, with the angle between the two incoming momenta p_1, p_2 .

We write

$$\Gamma^{(4)}(p_1, p_2, p_3, \lambda) \Big|_{RP} = \lambda$$

where RP means that the p 's which define λ are such that

$$(p_1 + p_2)^2 = (2 - \rho) \mu^2, \quad p_1^2 = p_2^2 = (1 - \rho/4) \mu^2$$

so that $p_1 p_2 = -(\rho/4)\mu^2$ or, ($\rho \equiv c_2''$)

$$p_i p_j = \frac{1}{4} (4\delta_{ij} - \rho) \mu^2$$

(i, j run over 1, 2.) $|\rho| < 4$. The same is true of course of i, j running over 3, 4. For convenience, our parameter ρ is defined by

$$H = \int_0^1 \frac{dx}{x} \int_0^1 dy L_n \left[1 - x + (1 - \frac{\rho}{4}) \frac{x}{y} (1 - x(1-y)) \right]$$

(with $\rho=1$ one recovers the usual symmetry point for $p_1 p_2 p_3 p_4 = H \rightarrow J$ of Brézin et al. 6)).

It is well known that this model provides an attractive way of handling critical phenomena and especially of computing various critical exponents 6). The quantities computed with this theory and within our renormalization scheme will therefore depend on μ, σ and ρ (or, equivalently, on c_1, c_2' and c_2''), explicitly and/or implicitly through λ . Of course, any quantity of physical significance (an observable) will be independent of these parameters in the limit explained above. The exponent η and the combination $(2-\eta)(1/\gamma-1)$ are defined respectively in terms of two quantities $A(\lambda, \rho, \sigma, \epsilon)$ and $B(\lambda, \rho, \sigma, \epsilon)$ taken at the fixed point $\lambda = \lambda^*$ which this theory exhibits for $\epsilon < 4$. Explicitly, we have

$$A(\lambda^*, \rho, \sigma, \epsilon) = \eta(\lambda^*, \rho, \sigma, \epsilon)$$

$$B(\lambda^*, \rho, \sigma, \epsilon) = (2-\eta)(\frac{1}{\gamma}-1) \equiv \hat{\eta}(\lambda^*, \rho, \sigma, \epsilon)$$

These exponents are measurable quantities and therefore independent of the parameters ρ, σ . One can thus write

$$\left(\frac{\partial}{\partial \rho} + \beta_\rho(\lambda) \frac{\partial}{\partial \lambda} \right) A(\lambda, \rho, \sigma, \epsilon) \Big|_{\lambda=\lambda^*} = 0 \quad (\text{or } A \rightleftharpoons B)$$

$$\left(\frac{\partial}{\partial \sigma} + \beta_\sigma(\lambda) \frac{\partial}{\partial \lambda} \right) B(\lambda, \rho, \sigma, \epsilon) \Big|_{\lambda=\lambda^*} = 0$$

with

$$\beta_\rho(\lambda) = \frac{\partial \lambda}{\partial \rho} ; \quad \beta_\sigma(\lambda) = \frac{\partial \lambda}{\partial \sigma} ; \quad (\lambda \equiv \lambda(\mu e^{-\epsilon}, \rho, \sigma))$$

the differential transformations of the coupling λ . In the first non-vanishing approximation one can compute

$$-4\beta_p(\lambda) = -\frac{\lambda^3}{36}(5n+22)\frac{dH}{d\alpha} + \dots \equiv h_p\lambda; \quad (i=p \text{ and } \alpha \equiv (1-\frac{p}{4}) \text{ here})$$

$$\beta_\sigma(\lambda) = \frac{\lambda^3}{72}(n+2) + \dots \equiv h_\sigma\lambda; \quad (i=\sigma \text{ here and } \alpha \text{ as above})^{*)}$$

As well as optimizing the fixed point beyond order λ^3 [i.e., contributions due to $O(\lambda^4, \lambda^3\epsilon)$, etc.], one can, in a straightforward way, compute A and B as either equation will determine the position of λ^* . This is an alternative way of finding the fixed point without calculating $\beta(\lambda) \equiv \mu(\partial/\partial\mu)\lambda$. It is not a simpler way but it shows that other RG equations can be used to calculate λ^* . To make the bridge with Stevenson's realization of the full RG, one should then consider $c_2 = D + (c'_2 + c''_2) \cdot K$ and write down a single $O(\lambda^3)$ β_i function, namely $\beta_2 = \partial\lambda/\partial c_2$ where c_2 is the coefficient of $\mu(\partial/\partial\mu)\lambda = \beta(\lambda)$ of order λ^3 which contains the unphysical parameters $c'_2 \equiv \rho$, $c''_2 \equiv \sigma$. However, in our example, it is better to enlarge the full RG by splitting c_2 into parts depending separately on c'_2 and c''_2 . However, it should be emphasized that they both occur in $\beta(\lambda)$ at the same order in λ as c_2 in a well-defined combination $(c'_2 + c''_2) \cdot K + D$.

The problem of optimization, which Stevenson attempted to solve ¹⁾, has not been addressed here. Our aim was to give a realization of the general renormalization group which governs both scale transformations and the RS specification. The index i in the two-indice differential transformations - e.g., the function h_{ie} in the summary of Ref. 2) - has been often questioned concerning its real significance. We hope that the examples given in this paper will shed some light on this question, even for the dullest minds:

- 1) the index e specifies of which coupling h_{ie} is the differential transformation;
- 2) the first index in the set $\{i\}$ concerns scale transformations (τ in Stevenson's realization);
- 3) all other indices of the set specify the renormalization scheme ⁷⁾.

We end with some additional remarks concerning the nature of series expansions. Indeed the general RG equations, although valid order-by-order, govern the non-perturbative physical strengths and it is interesting to see how optimization procedures act when the perturbative expansions are not guaranteed to sum up to the non-perturbative physical quantity and have presumably a zero radius of convergence in field theory.

---

*) Geometrical factors $[(2\pi)^{-d/2}/\Gamma(d/2)]$ per loop are included in the definition of the coupling λ , ($d=4-\epsilon$).

ADDITIONAL REMARKS

Another popular procedure for getting the "optimum value" is the so-called fastest apparent convergence method (FAC). Grosso modo the optimum is found there by finding the intersection ⁴⁾ of the two last approximations, if available. (In the next-to-leading order, it is the intersection of this next-to-leading curve with the leading one.) For more details see Refs. 1) of 4), for instance. Pennington ⁴⁾ has shown that the optima, as determined by FAC or PMS [principle of minimum sensitivity ¹⁾], do not differ substantially. However, PMS imposes more constraints, being based on the mathematical requirements of the full RG, information neglected by FAC.

Concerning this requirement, which affects true physical quantities (perturbation expansions summed up), there has been some criticism about the convergence of the expansions of the β function and of the physical quantities themselves. These criticisms cast doubt on the nature of the convergence which might well be very poor, perhaps not even Borel summable, and therefore question the additional information about the full RG used in PMS if the convergence is so bad. However, this is a problem that one should consider at a deeper level. For if the β function and the physical quantities' expansion are really poor, then not only is the PMS procedure somewhat questionable but indeed the whole of perturbative QCD. Partial resummations (leading logs for instance) do indeed lose mathematical legitimacy for these kinds of expansions: thus we have no approximate theory and therefore nothing to optimize !

It seems more reasonable to take the other attitude when both the β function and the physical quantities are reasonably convergent, in which case perturbative QCD might be relevant. In the absence of such an a priori hypothesis, the whole structure collapses - which might very well be the case - (but this is not a constructive hypothesis). If the former view ^{*)} is erroneous, conflicting results with the experimental data will warn us soon enough. But up to now this does not seem to be the case.

---

^{*)} The a priori hypothesis.

A P P E N D I X

$H(z)$ is a real monotonically increasing function in the interval $0 \leq z \leq 1$. It increases from $-\text{Li}_2(1) = -(\pi^2/6)$ for $z=0$ up to $\text{QLi}(-1,+1) = 1.17195362\dots$ for $z=1$ *). In the case of the symmetric renormalization point, $z=3/4$ and $H(3/4) = 0.74955(89)$ (in contradiction with values quoted in the literature), a value obtained by the computer adaptative integration subroutine RIWIAD, with 1000 subvolumes and 10 iterations and checked by analytical upper and lower bounds.

---

*) A function closely related to the "inverse tangent integral".

R E F E R E N C E S

- 1) P.M. Stevenson, Phys.Rev. D23, 2916 (1981).
- 2) E.C.G. Stueckleberg and A. Petermann, Helv.Phys.Acta 26, 499 (1953);
- 2) N.N. Bogoliubov and D.V. Shirkov, Dokladi Nauk USSR 103, 203 (1955).
- 3) M. Gell-Mann and F. Low, Phys.Rev. 95, 1300 (1954).
- 4) M.R. Pennington, Durham preprint (May, 1981). See also :
W. Furmanski and R. Petronzio, CERN preprint TH.3046 (1981) - to be
published in Zeit.f.Phys.
- 5) R. Roberts, in preparation.
- 6) E. Brézin, J.C. Le Guillou and J. Zinn-Justin, Phys.Rev. D9, 1121
(1974).
- 7) See also : R.J. Crewther, Cargèse Summer Institute lectures (1975),
Plenum Press (1976) - Eds. Lévy et al. - for a five-parameter
realization of the full RG in the $\lambda\phi^4$ theory.