Topological content of dimensional reduction

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We consider supersymmetric Yang-Mills theories which have nontrivial topological features and show that in this case the dimensional-reduction technique has a topological content. We find that a central charge of topological origin can be obtained, by dimensional reduction, from a topological charge in a higher-dimensional supersymmetry algebra (but not from an extra component of P_μ). We interpret this fact in terms of commutative diagrams of spheres.

I. INTRODUCTION

As was shown by Witten and Olive,¹ a supersymmetry algebra can have a topological charge as a central charge. In particular, they considered the $N=2$ supersymmetric $SU(2)$ Yang-Mills theory in $3+1$ dimensions. They found that the supersymmetry algebra, in the phase of broken gauge symmetry, has two central charges: an electric charge and the magnetic charge of the 't Hooft-Polyakov monopoles (electric-magnetic duality).

In a recent paper² the author considered the $USp(2)$ supersymmetric $SU(2)$ Yang-Mills theory in $4+1$ dimensions. It was shown that an electric charge appears in the supersymmetry algebra in the phase of broken gauge symmetry, while the topological charge of the solitons (instantons in the four-dimensional Euclidean space) appears in the phase of unbroken gauge symmetry (no electric-magnetic duality).

A consequence of the electric-magnetic duality in $3+1$ dimensions is that the soliton supermultiplet has not only the same number of states as the massive supermultiplet of elementary particles, but also an identical spin structure.³ Instead, the soliton supermultiplet in $4+1$ dimensions⁴ has the same number of states as the massive supermultiplet of elementary particles, but a different spin structure because of the absence of the electric-magnetic duality.

Olive⁵ showed that the electric charge and the magnetic charge, that appear in the $(3+1)$ -dimensional supersymmetry algebra as central charges, can be obtained by dimensional reduction from the fifth and the sixth components of P_μ in the $(5+1)$ -dimensional supersymmetry algebra, respectively. In this paper we suggest a new interpretation of the dimensional reduction in the topological sec-

tor. We show that a topological charge that modifies a supersymmetry algebra as a central charge can be obtained, by dimensional reduction, from another topological charge (and not from an extra component of P_μ) in a higher-dimensional supersymmetry algebra. The results that we get in the dimensional reduction of a supersymmetry algebra with a topological charge as a central charge (from $5+1$ to $4+1$ dimensions, from $4+1$ to $3+1$ dimensions, and then directly from $5+1$ to $3+1$ dimensions), show that the dimensional reduction can have a topological content. Thus we interpret the dimensional-reduction technique (in the nontrivial topological sector) in terms of commutative diagrams of spheres (Secs. IV, VI, and VIII).

In Sec. II, we give a short review of the $USp(2)$ supersymmetric $SU(2)$ Yang-Mills theory in $5+1$ dimensions on-shell.^{6,2} Then we explicitly calculate the supersymmetry algebra by means of the canonical formalism and find that there is a surface integral Z_μ ($\mu=0,1,2,3,5,6$) that we do not include in P_μ , but interpret as a central charge of topological origin.

In Sec. III, we perform the dimensional reduction of the $(5+1)$ -dimensional supersymmetry algebra to $4+1$ dimensions. We recover the $(4+1)$ -dimensional supersymmetry algebra² in both phases. In particular, we find that the topological charge of the $(4+1)$ -dimensional solitons can be obtained, by dimensional reduction, from the sixth component of Z_μ .

In Sec. V, we show that we can get the results of Witten and Olive in the $(3+1)$ -dimensional supersymmetry algebra by dimensional reduction from the $(4+1)$ -dimensional supersymmetry algebra only if it is considered in the phase of unbroken gauge symmetry, that is the phase in which the $(4+1)$ -dimensional theory has solitons. Thus we can recov-

er the electric-magnetic duality in the $(3+1)$ -dimensional supersymmetry algebra by dimensional reduction from the $(4+1)$ -dimensional supersymmetry algebra where there is no duality.

In Sec. VII we perform the dimensional reduction of the $(5+1)$ -dimensional supersymmetry algebra directly to $3+1$ dimensions, and again we recover the results of Witten and Olive. In particular, we find that the two magnetic charges in the $(3+1)$ -dimensional supersymmetry algebra can be obtained by dimensional reduction from Z_μ , for $\mu=5$ and $\mu=6$. Sec. IX is devoted to some additional remarks.

II. THE SUPERSYMMETRY ALGEBRA IN $5+1$ DIMENSIONS

Let us consider the $\text{USp}(2)$ -supersymmetric $\text{SU}(2)$ Yang-Mills theory in $5+1$ dimensions on-shell.^{6,2} The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - i\bar{\Lambda}^{ia}\Gamma_\mu D^\mu \Lambda_i^a, \quad (2.1)$$

where

$$\begin{aligned} D_\mu \Lambda_i^a &= \partial_\mu \Lambda_i^a + g\epsilon_{abc}A_\mu^b \Lambda_i^c, \\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc}A_\mu^b A_\nu^c \\ (\mu, \nu &= 0, 1, 2, 3, 5, 6, \quad i = 1, 2, \quad a, b, c = 1, 2, 3) \end{aligned} \quad (2.2)$$

the Γ_μ are 8×8 Dirac matrices whose explicit representation is given in Ref. 2, all the fields are in the adjoint representation of the gauge group $\text{SU}(2)$, and the Λ_i^a are two eight-component complex spinors constrained by the Weyl condition and by the $\text{USp}(2)$ -covariant Majorana condition,

$$(1 - i\Gamma_7)\Lambda_i^a = 0, \quad (2.3)$$

$$\Lambda_i^a = C_6 \Omega_{ij} \bar{\Lambda}^{jaT}, \quad (2.4)$$

where

$$\Gamma_7 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_5 \Gamma_6,$$

C_6 is the charge-conjugation operator in six dimensions, and Ω_{ij} is the real symplectic metric used to raise and lower indices. The supersymmetry transformations are

$$\begin{aligned} \delta A_\mu^a &= -i\bar{\alpha}^i \Gamma_\mu \Lambda_i^a, \\ \delta \Lambda_i^a &= \frac{1}{2} \Sigma_{\mu\nu} F^{\mu\nu a} \alpha_i, \end{aligned} \quad (2.5)$$

where

$$\Sigma_{\mu\nu} = \frac{1}{4} [\Gamma_\mu, \Gamma_\nu],$$

and the parameters α_i are two constant Weyl spinors constrained by the $\text{USp}(2)$ -covariant Majorana con-

dition. The supersymmetry transformations (2.5) close on-shell,

$$[\delta(\alpha), \delta(\beta)] = \bar{\alpha}^i \partial \beta_i + \text{gauge transformation} + \text{equation of motion}. \quad (2.6)$$

The conserved supersymmetry currents are

$$J_i^\mu = \frac{i}{2} F^{\rho\sigma a} \Gamma_\rho \Gamma_\sigma \Gamma^\mu \Lambda_i^a. \quad (2.7)$$

And the supersymmetry charges are

$$Q_i = \int d^5 \vec{x} J_i^0. \quad (2.8)$$

The supersymmetry charges, written in terms of the fields and their conjugate momenta, are

$$Q_j = \int d^5 \vec{x} \left[i\pi_{A_l}^b \Gamma_l \Lambda_j^b + \frac{i}{2} F^{lmnb} \Gamma_l \Gamma_m \Gamma_0 \Lambda_j^b \right], \quad (2.9)$$

$$\bar{Q}^i = \int d^5 \vec{x} \left[i\pi_{A_p}^a \Lambda^{ia\dagger} \Gamma_p \Gamma_0 + \frac{i}{2} F^{pqaa} \Lambda^{ia\dagger} \Gamma_p \Gamma_q \right],$$

where

$$\begin{aligned} \pi_{A_l}^a &= -F^{0la}, \\ l, m, p, q &= 1, 2, 3, 5, 6, \quad i, j = 1, 2, \quad a, b = 1, 2, 3 \\ \pi_{\Lambda_i}^a &= i\Lambda^{ia\dagger}. \end{aligned}$$

The equal-time commutators and anticommutators are

$$\begin{aligned} [A_\mu^a(\vec{x}), \pi_{A_\nu}^b(\vec{y})] &= i\delta_{\mu\nu} \delta^{ab} \delta^5(\vec{x} - \vec{y}), \\ \{\Lambda_\alpha^{ia\dagger}(\vec{x}), \Lambda_{j\beta}^b(\vec{y})\} &= \frac{1}{2} \delta_j^i \delta_{\alpha\beta} \delta^{ab} \delta^5(\vec{x} - \vec{y}) \\ (\mu, \nu &= 0, 1, 2, 3, 5, 6, \quad \alpha, \beta = 1, \dots, 8, \\ i, j &= 1, 2, \quad a, b = 1, 2, 3). \end{aligned} \quad (2.10)$$

We explicitly calculate the supersymmetry algebra. Paying attention to boundary terms, we get

$$\begin{aligned} \{\bar{Q}_\alpha^i, Q_\beta^j\} &= \Omega^{ij}(\Gamma^\mu)_{\beta\alpha} P_\mu + \Omega^{ij}(\Gamma_\mu \Gamma_7)_{\beta\alpha} Z_\mu \\ (\mu &= 0, 1, 2, 3, 5, 6, \quad \alpha, \beta = 1, \dots, 8, \quad i, j = 1, 2), \end{aligned} \quad (2.11)$$

where

$$Z_\mu = \int d^5 \vec{x} I_{0\mu} \quad (2.12)$$

and $I_{0\mu}$ is the time component of the antisymmetric tensor

$$I_{\nu\mu} = \frac{1}{8} \epsilon_{\nu\mu\rho\sigma\tau\eta} F^{\rho\sigma a} F^{\tau\eta a} \quad (\mu, \nu, \rho, \sigma, \tau, \eta = 0, 1, 2, 3, 5, 6) . \quad (2.13)$$

Then, it is

$$Z_0 = 0 \quad (2.14)$$

and

$$Z_n = \frac{1}{4} \int d^5 \vec{x} F^{lma} \tilde{F}_{lmn}^a , \quad (2.15)$$

where

$$\tilde{F}_{lmn}^a = \frac{1}{2} \epsilon_{lmnpq} F^{pqa} \quad (l, m, n, p, q = 1, 2, 3, 5, 6) . \quad (2.16)$$

As the antisymmetric tensor (2.13) is a divergence of a totally antisymmetric tensor $I_{\rho\nu\mu}$,⁵ Z_μ is the integral of a spatial divergence

$$Z_\mu = \int d^5 \vec{x} \partial^\rho I_{\rho 0 \mu} , \quad (2.17)$$

and it is conserved

$$\partial_0 Z_\mu = 0 . \quad (2.18)$$

By (2.17), Z_μ can be written as a surface integral

$$Z_\mu = \int d\sigma^\lambda I_{\lambda\mu} . \quad (2.19)$$

For the surface $x_0 = \text{const}$, it is

$$d\sigma^\lambda = \delta_{\lambda 0} dx_1 \cdots dx_6 \quad (2.20)$$

and (2.14) becomes the constraint

$$Z_\lambda n^\lambda = 0 , \quad (2.21)$$

where n^λ is a vector orthogonal to the surface $x_0 = \text{const}$.

The $(5+1)$ -dimensional USp(2) supersymmetry algebra (2.11) (notice that it is an $N=1$ supersymmetry²) can be rewritten as the usual one,

$$\{Q_\alpha, \bar{Q}_\beta\} = [(1 + \Gamma_7) \Gamma^\mu]_{\alpha\beta} \tilde{P}_\mu , \quad (2.22)$$

$$\{Q_\alpha, Q_\beta\} = 0$$

(where Q_α is a single Weyl spinor) if, following Olive,⁵ we take

$$\tilde{P}_\mu = P_\mu + \int d^5 \vec{x} I_{0\mu} . \quad (2.23)$$

Anyway, I will not consider $I_{\nu\mu}$, given in (2.13) as an antisymmetric part of the energy-momentum tensor $\theta_{\nu\mu}$. I will look for nontrivial topological configurations that contribute to the surface integral (2.19) at infinity. Then, Z_μ will not be included in the momentum, but will play the role of a central

charge of topological origin in the supersymmetry algebra.

Let us consider the surface integral (2.19) at infinity,

$$d\sigma^\lambda \xrightarrow[\infty]{} dS^4 , \quad (2.24)$$

where dS^4 is the surface element of the boundary of the five-dimensional Euclidean space R^5 . As we will see in the following, there are no “pseudoparticles” in R^5 that can contribute to the surface integral (2.19), and it then vanishes at infinity.

The surface integral (2.19) plays a relevant role in the dimensional reduction to $4+1$ dimensions (and to $3+1$ dimensions). The dimensional reduction from $5+1$ to $4+1$ dimensions breaks the Lorentz group,

$$O(5,1) \rightarrow O(4,1) \times U(1) \quad (2.25)$$

(i.e., take the X_6 axis around a circle), and in the surface integral at infinity

$$dS^4 = dS^3 \times dS^1 ,$$

where S^3 is the boundary of the four-dimensional Euclidean space R^4 . As we will see in Sec. IV, there are “strings” along S^1 which contribute to the surface integral Z_6 at infinity. By dimensional reduction to $4+1$ dimensions, Z_6 will give the topological charge of the solitons (instantons in R^4).

III. DIMENSIONAL REDUCTION OF THE SUPERSYMMETRY ALGEBRA FROM $5+1$ TO $4+1$ DIMENSIONS

The dimensional reduction of the $(5+1)$ -dimensional theory (2.1) to $4+1$ dimensions is given in Ref. 2. Now, let us perform the dimensional reduction of the $(5+1)$ -dimensional supersymmetry algebra (2.11) to $4+1$ dimensions.

Let us decompose the vector field A_μ^a ($\mu=0,1,2,3,5,6$) as

$$A_\mu^a \equiv A_\mu^a \quad (\mu=0,1,2,3,5) , \quad (3.1)$$

$$A_6^a \equiv \phi^a ,$$

and compactify the sixth dimension,

$$\partial_6 = 0 . \quad (3.2)$$

Z_μ , for $\mu=6$ ($Z_6 = \int d^5 x I_{06}$) in (2.11) is reduced to

$$Z_T = -\frac{1}{4} \int d^4 \vec{x} F_{lm}^a \tilde{F}^{lma} \quad (l, m = 1, 2, 3, 5) \quad (3.3)$$

in the phase $\langle \phi \rangle = 0$.

For $\mu \neq 6$, Z_μ gives identically zero in the phase $\langle \phi \rangle = 0$. The sixth component of P_μ in (2.11) is reduced to

$$Z_E = \frac{1}{\langle \phi \rangle} \int d^4 \vec{x} \partial m(\phi^a F_{0m}^a) \quad (m=1,2,3,5) \quad (3.4)$$

in the phase $\langle \phi \rangle \neq 0$. Thus we recover the supersymmetry algebra in $4+1$ dimensions² in both phases $\langle \phi \rangle = 0$ and $\langle \phi \rangle \neq 0$.

Phase $\langle \phi \rangle \neq 0$ [the gauge group $SU(2)$ is broken down to an Abelian subgroup $U(1)$]:

$$\{\bar{Q}_\alpha^i, Q_\beta^j\} = \Omega^{ij}(\gamma^\mu)_{\beta\alpha} P_\mu + \Omega^{ij} \delta_{\alpha\beta} Z_E, \quad (3.5)$$

where Z_E , given in (3.4), is the electric charge of the unbroken Abelian subgroup $U(1)$.

Phase $\langle \phi \rangle = 0$ [the gauge group $SU(2)$ is unbroken]:

$$\{\bar{Q}_\alpha^i, Q_\beta^j\} = \Omega^{ij}(\gamma^\mu)_{\beta\alpha} P_\mu + \Omega^{ij} \delta_{\alpha\beta} Z_T \quad (\alpha, \beta = 1, 2, 3, 4, \quad \mu = 0, 1, 2, 3, 5, \quad i, j = 1, 2) \quad (3.6)$$

where Z_T , given in (3.3), is the topological charge of the solitons that modifies the supersymmetry algebra as a central charge in this phase.

IV. TOPOLOGICAL CONTENT OF THE DIMENSIONAL REDUCTION FROM $5+1$ TO $4+1$ DIMENSIONS

Let us consider the $(4+1)$ -dimensional Minkowski space $R^5 = R^1 \times R^4$, where R^1 is the time that will be taken constant and R^4 is the four-dimensional Euclidean space.

Then let us consider the boundary of R^4 , i.e., the surface of the four-dimensional sphere at infinity:

$$S_3 = \left\{ (x_1, x_2, x_3, x_5) \in R^4 \mid \sum_i x_i^2 = 1 \right\}. \quad (4.1)$$

In the phase $\langle \phi \rangle = 0$, the gauge group $SU(2)$ is unbroken and there is the instanton map

$$S_3 \xrightarrow{\pi_3(S_3)=Z} S_3. \quad (4.2)$$

Now let us go back to the $(5+1)$ -dimensional theory and consider the six-dimensional Minkowski space $R^6 = R^1 \times \tilde{R}^5$. The boundary of $\tilde{R}^5$ is the surface of the five-dimensional sphere at infinity,

$$S_4 = \left\{ (x_1, x_2, x_3, x_5, x_6) \in \tilde{R}^5 \mid \sum_i x_i^2 = 1 \right\}. \quad (4.3)$$

Now let us look at localized topological objects in $\tilde{R}^5$, i.e., the gauge fields $A_\mu^{a'}(x_1, x_2, x_3, x_5, x_6)$ that fall in the homotopy classes of the maps,

$$S_4 \xrightarrow{A_\mu^{a'}} SU(2) \quad (\mu=1,2,3,5,6),$$

i.e., the maps

$$S_4 \xrightarrow{\pi_4(S_3)=Z_2} S_3. \quad (4.4)$$

As was shown by Nahm,⁷ this topological object $A_\mu^{a'}$ is an instanton—anti—instanton pair in S_3 embedded in S_4 .

The distance d between the instanton and the anti-instanton is taken along x_5 , for example, and ϕ is the angle between their $SU(2)$ orientations. See Fig. 1.

But the distance d can be continuously deformed to zero (see Fig. 2), and the instanton—anti—instanton pair annihilates. Thus, the topological object $A_\mu^{a'}$ localized in $\tilde{R}^5$ has a topological number that is identically zero.

Now let us take $A_\mu^{a'}$ to be periodic on x_6 ,

$$A_\mu^{a'} = A_\mu^a(x_1, x_2, x_3, x_5, \bar{x}_6) \quad (\mu=1,2,3,5), \quad A_6^a = 0. \quad (4.5)$$

The solution (4.5) is defined on $R^4 \times R (=x_6)$. It is a “string” in $\tilde{R}^5$. The boundary of $\tilde{R}^5$, S_4 is deformed into the cylinder $S_3 \times S_1$ and the map (4.4) becomes the projection

$$\pi: S_3 \times S_1 \rightarrow S_3. \quad (4.6)$$

If we make a slice of the periodic solution (4.5) through x_6 we get the instanton in R^4 (see Fig. 3).

Now let us show how we get the instanton map (4.2) by dimensional reduction. The two steps of the dimensional reduction from $5+1$ to $4+1$ dimen-

FIG. 1. Instanton—anti—instanton pair in S_3 embedded in S_4 .

FIG. 2. Annihilation of the instanton-anti-instanton pair.

sions are as follows.

(i) $A_6^a = \phi^a$. In the phase $\langle \phi \rangle = 0$ the gauge group $SU(2)$ is left unbroken when passing from $5 + 1$ to $4 + 1$ dimensions. There is the map

$$\varphi_n: S_3 \rightarrow S_3. \quad (4.7)$$

(ii) $\partial_6 = 0$. Compactification of the sixth dimension. There is the injection

$$i: S_3 \rightarrow S_3 \times S_1. \quad (4.8)$$

The maps (4.2), (4.6), (4.7), and (4.8) close in the commutative diagram

$$\begin{array}{ccc} S_3 \times S_1 & \xrightarrow{\pi} & S_3 \\ i \uparrow & & \downarrow \varphi_n \\ S_3 & \xrightarrow{\varphi_n} & S_3 \end{array} \quad (4.9)$$

where

$$\begin{aligned} i: P &\rightarrow (P, x_6), \\ \pi: (P, x_6) &\rightarrow P, \quad P \in R^4 \end{aligned} \quad (4.10)$$

FIG. 3. Dimensional reduction of the "string" in R^5 which gives rise to the instanton in R^4 .

and

$$\varphi_n = \varphi_n \circ \pi \circ i \quad (4.11)$$

V. DIMENSIONAL REDUCTION OF THE SUPERSYMMETRY ALGEBRA FROM $4 + 1$ TO $3 + 1$ DIMENSIONS AND ELECTRIC-MAGNETIC DUALITY

Let us perform the dimensional reduction of the $(4 + 1)$ -dimensional supersymmetry algebra (3.6) to $3 + 1$ dimensions. Let us decompose the vector field A_μ^a ($\mu = 0, 1, 2, 3, 5$) as

$$\begin{aligned} A_\mu^a &\equiv A_\mu^a \quad (\mu = 0, 1, 2, 3), \\ A_5^a &\equiv \phi^a \end{aligned} \quad (5.1)$$

and compactify the fifth dimension

$$\partial_5 = 0. \quad (5.2)$$

P_5 gives

$$Z'_E = \frac{1}{\langle \phi' \rangle} \int d^3 \vec{x} \partial_i (\phi'^a F_{0i}^a) \quad (5.3)$$

for $\langle \phi' \rangle \neq 0$, and identically gives zero for $\langle \phi' \rangle = 0$.

Z_T gives

$$G' = \frac{1}{\langle \phi' \rangle} \int d^3 \vec{x} \epsilon_{ijk} \partial_i (\phi'^a F_{jk}^a) \quad (i, j, k = 1, 2, 3) \quad (5.4)$$

for $\langle \phi' \rangle \neq 0$, and identically gives zero for $\langle \phi' \rangle = 0$. We get the following supersymmetry algebra in $3 + 1$ dimensions.

If $\langle \phi' \rangle = 0$ ($\langle \phi \rangle = 0$),

$$\{\bar{Q}_{i\alpha}, Q_{j\beta}\} = \delta_{ij} (\gamma^\mu)_{\alpha\beta} P_\mu. \quad (5.5)$$

If $\langle \phi' \rangle \neq 0$ ($\langle \phi \rangle = 0$),

$$\begin{aligned} \{\bar{Q}_{i\alpha}, Q_{j\beta}\} &= \delta_{ij} (\gamma^\mu)_{\alpha\beta} P_\mu + \epsilon_{ij} [(\gamma^5)_{\alpha\beta} Z'_E + \delta_{\alpha\beta} G'] \\ (\mu = 0, 1, 2, 3, \alpha, \beta = 1, 2, 3, 4, i, j = 1, 2), \end{aligned} \quad (5.6)$$

where Z'_E , given in (5.3), and G' , given in (5.4), are the electric charge of the unbroken Abelian subgroup $U(1)$ and the magnetic charge of the monopoles, respectively.

In conclusion, we can get the results of Witten and Olive in the $(3 + 1)$ -dimensional supersymmetry algebra (in both phases) by dimensional reduction from the $(4 + 1)$ -dimensional supersymmetry algebra in the phase of unbroken gauge symmetry [i.e., the phase in which the $(4 + 1)$ -dimensional theory has solitons]. Thus, we recover the electric-magnetic duality in the $(3 + 1)$ -dimensional supersymmetry algebra by dimensional reduction from $4 + 1$ dimensions where there is no duality.

Electric-magnetic duality is a peculiar feature of the $(3+1)$ -dimensional theory and it can be recovered by dimensional reduction from a higher dimensional theory which has nontrivial topological features.

VI. TOPOLOGICAL CONTENT OF THE DIMENSIONAL REDUCTION FROM $4+1$ TO $3+1$ DIMENSIONS

Now let us look at the topological content of the dimensional reduction from the $(4+1)$ -dimensional theory to the $(3+1)$ -dimensional theory, each one considered in a phase where there is a nontrivial topology.

Consider the $N=2$ supersymmetric $SU(2)$ Yang-Mills theory in the $(3+1)$ -dimensional Minkowski space $R^4 = R^1 \times R^3$. The boundary of R^3 is the surface of the three-dimensional sphere at infinity S_2 :

$$S_2 = \left\{ (x_1, x_2, x_3) \in R^3 \mid \sum_i x_i^2 = 1 \right\}. \quad (6.1)$$

For a nonvanishing vacuum expectation value of the scalar field ϕ' , the gauge group $SU(2)$ is broken down to an Abelian subgroup $U(1)$, and there is the nontrivial map

$$S_2 \rightarrow \frac{SU(2)}{U(1)},$$

i.e., the map

$$\psi_n^{(2)}: S_2 \xrightarrow{\pi_2(S_2)=Z} S_2. \quad (6.2)$$

The map (6.2) of the 't Hooft-Polyakov monopoles (solitons in $3+1$ dimensions) can be obtained by dimensional reduction from $4+1$ dimensions, as we will see in the following.

Let us go back to the $N=2$ supersymmetric $SU(2)$ Yang-Mills theory in the $(4+1)$ -dimensional Minkowski space $R^5 = R^1 \times \tilde{R}^4$. In the phase of unbroken gauge symmetry there are solitons (instantons in the four-dimensional Euclidean space $\tilde{R}^4$).

Let us now consider periodic instantons. A single instanton solution periodic in time was constructed by Harrington and Shepard⁸ from a multi-instanton solution⁹ describing an infinite string of instantons with identical size aligned along the time axis and regularly spaced. Then, Rossi¹⁰ showed that the static monopole solution in the Prasad-Sommerfield-Bogomolny limit¹¹ is equivalent to a periodic instanton. In this context, however, we will consider an instanton solution periodic on x_5 . This periodic instanton is defined on $R^3 \times R (=x_5)$. The

boundary of $\tilde{R}^4$, S_3 , given in (4.1), is deformed into the cylinder $S_2 \times S_1$ and the map (4.2) becomes

$$\varphi_n^{(3)}: S_2 \times S_1 \rightarrow S_3. \quad (6.3)$$

It was shown by Hopf¹² that the group of homotopy classes of maps (6.3) is again $\pi_3(S_3) = Z$. If we take a slice of this periodic instanton (a chain of instantons aligned along x_5), through x_5 , we get the monopole in R^3 (see Fig. 4).

Now let us consider the two steps of the dimensional reduction from $4+1$ to $3+1$ dimensions.

(i) $A_5^a = \phi^a$: For $\langle \phi' \rangle \neq 0$ the gauge group $SU(2)$ is broken down to $SU(2)/U(1)$. There is the Hopf map¹²

$$h: S_3 \rightarrow S_2. \quad (6.4)$$

The breaking of the gauge group $SU(2)$ involves the exact short sequence of sets

$$U(1) \xrightarrow{i} SU(2) \xrightarrow{\pi} \frac{SU(2)}{U(1)}, \quad (6.5)$$

i.e., the Puppe sequence¹³ of the Hopf bundle,

$$S_1 \xrightarrow{i} S_3 \xrightarrow{\pi} S_2, \quad (6.6)$$

that induces the exact sequence of homotopy groups

$$\begin{array}{ccccc} \pi_1(S_1) & \xrightarrow{i} & \pi_3(S_3) & \xrightarrow{\pi} & \pi_2(S_2) \cup \pi_3(S_2) \\ \parallel & & \parallel & & \parallel \\ Z & & Z & & Z \cup Z \end{array} \quad (6.7)$$

(ii) $\partial_5 = 0$. Compactification of the fifth dimension. There is the injection

$$i: S_2 \rightarrow S_2 \times S_1. \quad (6.8)$$

And the maps (6.2), (6.3), (6.4), and (6.8) close in the commutative diagram

FIG. 4. Dimensional reduction of the chain of instantons which gives rise to the monopole in R^3 .

$$\begin{array}{ccc}
 S_2 \times S_1 & \xrightarrow{\varphi_n^{(3)}} & S_3 \\
 \uparrow i & & \downarrow h \\
 S_2 & \xrightarrow{\psi_n^{(2)}} & S_2
 \end{array} , \quad (6.9)$$

where

$$i: P \rightarrow (p, 0), \quad P \in R^3, \quad (6.10)$$

h is the Hopf map, and

$$\psi_n^{(2)} = h \circ \varphi_n^{(3)} \circ i. \quad (6.11)$$

Similar arguments were used by Pak and Percacci¹⁴ to show the topological equivalence of dyons and periodic instantons.

VII. DIMENSIONAL REDUCTION OF THE SUPERSYMMETRY ALGEBRA FROM 5 + 1 TO 3 + 1 DIMENSIONS

Now let us perform the dimensional reduction of the (5 + 1)-dimensional supersymmetry algebra (2.11) directly to 3 + 1 dimensions. Thus let us decompose the vector field A_μ^a ($\mu = 0, 1, 2, 3, 5, 6$) as

$$\begin{aligned}
 A_\mu^a &\equiv A_\mu^a \quad (\mu = 0, 1, 2, 3), \\
 A_5^a &\equiv \phi_1^a, \\
 A_6^a &\equiv \phi_2^a,
 \end{aligned} \quad (7.1)$$

and compactify the fifth and sixth dimensions

$$\begin{aligned}
 \partial_5 &= 0, \\
 \partial_6 &= 0.
 \end{aligned} \quad (7.2)$$

We consider the following cases.

(i) $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$. P_5 , P_6 , and Z_μ (for any $\mu = 0, 1, 2, 3, 5, 6$) in (2.11) give, by dimensional reduction to 3 + 1 dimensions, identically zero. We get the following supersymmetry algebra in 3 + 1 dimensions:

$$\begin{aligned}
 \{\bar{Q}_{i\alpha}, Q_{j\beta}\} &= \delta_{ij} (\gamma^\mu)_{\alpha\beta} P_\mu \\
 (\mu &= 0, 1, 2, 3, \quad \alpha, \beta = 1, 2, 3, 4, \quad i, j = 1, 2). \quad (7.3)
 \end{aligned}$$

(ii) $\langle \phi_1 \rangle \neq 0$, $\langle \phi_2 \rangle = 0$. P_5 and Z_5 in (2.11) give, respectively, by dimensional reduction to 3 + 1 dimensions, the electric charge Z_{E_1} and the magnetic charge of the monopoles G_1 ,

$$Z_{E_1} = \frac{1}{\langle \phi_1 \rangle} \int d^3 \vec{x} \partial_i (\phi_1^a F_{0i}^a), \quad (7.4)$$

$$G_1 = \frac{1}{\langle \phi_1 \rangle} \int d^3 \vec{x} \epsilon_{ijk} \partial_i (\phi_1^a F_{jk}^a) \quad (i, j, k = 1, 2, 3), \quad (7.5)$$

while P_6 and Z_6 give identically zero. Thus, we get the following supersymmetry algebra in 3 + 1 dimensions:

$$\begin{aligned}
 \{\bar{Q}_{i\alpha}, Q_{j\beta}\} &= \delta_{ij} (\gamma^\mu)_{\alpha\beta} P_\mu + \epsilon_{ij} [(\gamma^5)_{\alpha\beta} Z_{E_1} + \delta_{\alpha\beta} G_1] \\
 (\mu &= 0, 1, 2, 3 \quad \alpha, \beta = 1, 2, 3, 4 \quad i, j = 1, 2). \quad (7.6)
 \end{aligned}$$

(iii) $\langle \phi_1 \rangle = 0$, $\langle \phi_2 \rangle \neq 0$. P_6 and Z_6 in (2.11) give by dimensional reduction to 3 + 1 dimensions, the electric charge Z_{E_2} and the magnetic charge of the monopoles G_2 , respectively:

$$Z_{E_2} = \frac{1}{\langle \phi_2 \rangle} \int d^3 \vec{x} \partial_i (\phi_2^a F_{0i}^a), \quad (7.7)$$

$$G_2 = \frac{1}{\langle \phi_2 \rangle} \int d^3 \vec{x} \epsilon_{ijk} \partial_i (\phi_2^a F_{jk}^a) \quad (i, j, k = 1, 2, 3), \quad (7.8)$$

while P_5 and Z_5 give identically zero. Thus we get the following supersymmetry algebra in 3 + 1 dimensions:

$$\begin{aligned}
 \{\bar{Q}_{i\alpha}, Q_{j\beta}\} &= \delta_{ij} (\gamma^\mu)_{\alpha\beta} P_\mu + \epsilon_{ij} [\delta_{\alpha\beta} Z_{E_2} + (\gamma^5)_{\alpha\beta} G_2] \\
 (\mu &= 0, 1, 2, 3 \quad \alpha, \beta = 1, 2, 3, 4, \quad i, j = 1, 2). \quad (7.9)
 \end{aligned}$$

VIII. TOPOLOGICAL CONTENT OF THE DIMENSIONAL REDUCTION FROM 5 + 1 TO 3 + 1 DIMENSIONS

Let us take the field configuration $A_\mu^{a'}$ (x_1, x_2, x_3, x_5, x_6) ($\mu = 1, 2, 3, 5, 6$), that is defined on $\bar{R}^5$, to be periodic on x_5 and x_6 ,

$$A_\mu^{a'} = A_\mu^a(x_1, x_2, x_3, \bar{x}_5, \bar{x}_6) \quad (\mu = 1, 2, 3),$$

$$A_5^a = \phi_1^a(x_1, x_2, x_3, \bar{x}_5, \bar{x}_6), \quad (8.1a)$$

$$A_6^a = 0;$$

$$A_\mu^{a'} = A_\mu^a(x_1, x_2, x_3, \bar{x}_5, \bar{x}_6) \quad (\mu = 1, 2, 3),$$

$$A_5^a = 0, \quad (8.1b)$$

$$A_6^a = \phi_2^a(x_1, x_2, x_3, \bar{x}_5, \bar{x}_6).$$

The solutions (8.1a) and (8.1b) are defined on $R^3 \times R (=x_5) \times R (=x_6)$. They are surfaces in $\tilde{R}^5$ and contribute to the surface integrals Z_5 and Z_6 at infinity, respectively. If we take a slice of the solution (8.1) through the plane (x_5, x_6) , we get the monopole defined on R^3 (see Fig. 5). The boundary of R^5 , S_4 , given in (4.3), is deformed into $S_2 \times S_1 \times S_1$ and the map (4.4) is changed into the two maps,

$$\psi'_n: S_2 \times S_1 \times S_1 \rightarrow S_3 \quad (8.2a)$$

with

$$\psi'_n: (P, x_5, x_6) \rightarrow (P, x_5), \quad P \in R^3$$

$$\psi''_n: S_2 \times S_1 \times S_1 \rightarrow S_3 \quad (8.2b)$$

with

$$\psi''_n: (P, x_5, x_6) \rightarrow (P, x_6), \quad P \in R^3.$$

The map

$$\sigma_n: S_2 \rightarrow S_2 \quad (8.3)$$

of the 't Hooft-Polyakov monopoles can be obtained, by dimensional reduction, from maps (8.2). In fact, let us consider the two steps of the dimensional reduction from $5 + 1$ to $3 + 1$ dimensions.

(i) $A_5^a = \phi_1^a$, $A_6^a = \phi_2^a$. For $\langle \phi_1 \rangle \neq 0$ (or $\langle \phi_2 \rangle \neq 0$), the gauge group $SU(2)$ is broken down to

FIG. 5. Dimensional reduction of the "surface" in R^5 which gives rise to the monopole in R^3 .

$SU(2)/U(1)$. There is the Hopf map

$$h: S_3 \rightarrow S_2. \quad (8.4)$$

(ii) $\partial_5 = \partial_6 = 0$. Compactification of the fifth and sixth dimensions. There is the injection

$$i: S_2 \rightarrow S_2 \times S_1 \times S_1. \quad (8.5)$$

Each map (8.2a) and (8.2b) closes with the other maps (8.3), (8.4), and (8.5) in a commutative diagram

$$\begin{array}{ccc} & \xrightarrow{\psi'_n} & S_3 \\ S_2 \times S_1 \times S_1 & \xrightarrow{i} & \downarrow h \\ S_2 & \xrightarrow{\sigma_n} & S_2 \end{array} \quad \sigma_n = h \circ \psi'_n \circ i, \quad (8.6a)$$

$$\begin{array}{ccc} & \xrightarrow{\psi''_n} & S_3 \\ S_2 \times S_1 \times S_1 & \xrightarrow{i} & \downarrow h \\ S_2 & \xrightarrow{\sigma_n} & S_2 \end{array} \quad \sigma_n = h \circ \psi''_n \circ i, \quad (8.6b)$$

where $i: p \rightarrow (p, 0, 0)$, h is the Hopf map, and ψ'_n , ψ''_n are given in (8.2a) and (8.2b). The commutative diagrams (8.6a) and (8.6b) show that G_1 and G_2 are obtained by dimensional reduction from Z_5 and Z_6 , respectively.

IX. COMMENTS

(i) The geometrical features of dimensional reduction have been investigated so far in the context of Kaluza-Klein theories¹⁵ in general and in supergravity (spontaneous compactification¹⁶) in particular.

In this paper it is shown that even in global supersymmetry, the geometrical interpretation of the dimensional-reduction technique is very relevant when we consider supersymmetric theories which have solitons. In fact, it is shown that a topological charge that appears in a $(D + 1)$ -dimensional supersymmetry algebra is not obtained by dimensional reduction from one extra component of the momentum, but from another topological charge in a $[(D + n) + 1]$ -dimensional supersymmetry algebra. [This fact is most evident in the dimensional reduction of the supersymmetry algebra from $4 + 1$ to $3 + 1$ dimensions (Sec. V).]

This means that the dimensional-reduction technique preserves the topological properties of group manifolds. In fact, it is shown that, if the $(D + 1)$ -

dimensional theory (with a gauge group G) has solitons, then the nontrivial map

$$S_{D-1} \rightarrow G \quad (\text{or } G/H) \quad (9.1)$$

(where S_{D-1} is the boundary of the D -dimensional Euclidean space), is obtained, by dimensional reduction, from the map

$$S_{D-1} \times S_1 \times S_1 \times \cdots \times S_1 \rightarrow G, \quad (9.2)$$

where S_1 appears n times. The topological content of dimensional reduction consists in the fact that it makes maps (1) and (2) close in a commutative diagram of spheres (Secs. IV, VI, VIII).

(ii) Topological central charges in higher dimensions are not, in general, the generators of an unbroken Abelian subgroup H of a gauge group G . (One example is the topological charge Z_T that appears in the $(4+1)$ -dimensional supersymmetry algebra, in the phase of unbroken gauge symmetry²).

Consider a $(D+1)$ -dimensional supersymmetric theory with a gauge group G . The theory can have a nontrivial topology in the phase of unbroken gauge symmetry (a) or in the phase of broken gauge symmetry (b) or in both of them (c), or in none of them (d). [We will not take into account case (d), as we are considering only those supersymmetric theories which involve topological central charges.]

Obviously, it depends on the dimension D of the Euclidean space, and on the particular choice of the gauge group G that (a) the map

$$S_{D-1} \rightarrow G \quad (9.3)$$

is not trivial, while the map

$$S_{D-1} \rightarrow G/H \quad (9.4)$$

is trivial, or vice versa (b), or both of them are not trivial (c).

In case (a) the $(D+1)$ -dimensional supersymmetry algebra has a topological charge (and no electric charge) in the phase of unbroken gauge symmetry, while it has an electric charge (and no topological charge) in the phase of broken gauge symmetry.

In case (b) the supersymmetry algebra has both an electric charge and a topological charge in the phase of broken gauge symmetry, and it has no central charges at all in the phase of unbroken gauge symmetry.

In case (c) the supersymmetry algebra has a topological charge Z_{T_1} (and no electric charge) in the phase of unbroken gauge symmetry, while it has both a topological charge Z_{T_2} ($Z_{T_2} \neq Z_{T_1}$) and an electric charge in the phase of broken gauge symmetry.

Thus we do not expect, in general, electric-

magnetic duality in a $(D+1)$ -dimensional ($D > 3$) supersymmetry algebra. Nevertheless, we recover electric-magnetic duality in the $(3+1)$ -dimensional supersymmetry algebra, by dimensional reduction from a higher dimensional supersymmetry algebra chosen in the phase in which it has a topological central charge (Sec. V, VII).

(iii) One example of nonscalar topological charge is Z_μ in the six-dimensional supersymmetry algebra (Sec. II). Another example is the tensor $Z_{\mu_1 \cdots \mu_5}$ that appears in the ten-dimensional supersymmetry algebra.¹⁷ Nonscalar central charges have also been considered in $N=8$ supergravity, in the group-manifold approach.¹⁸

(iv) Nonscalar topological charges are not central charges in the usual sense, as they do not commute with the Lorentz generators. Moreover, their topological meaning, even if it is not trivial, is not the usual one. Take as an example Z_μ in the $(5+1)$ -dimensional supersymmetry algebra. Z_μ is not related to any "localized topological object" in six dimensions, but, in the dimensional reduction to five dimensions, Z_6 becomes the flux of strings along the extra (X_6) spatial dimension. A cut of a string produces a soliton in five dimensions (Sec. IV, Fig. 3). Nonscalar topological charges can give, by dimensional reduction, scalar topological charges related to localized solitons in the lower dimensions.

Thus, the appearance of nonscalar topological charges in higher-dimensional supersymmetry algebras must not be ignored, in order to recover the right results (mass formula, representations of the supersymmetry algebra, and electric-magnetic duality) in $3+1$ dimensions. As was shown in Sec. VII, the fifth and the sixth components of Z_μ give the two possible magnetic charges in the $N=2$ supersymmetry algebra in $3+1$ dimensions.

Moreover, the tensorial central charge

$$Z_{\mu\nu\rho\sigma\tau} = \int d^9\vec{x} \epsilon_{\mu\nu\rho\sigma\tau\epsilon\eta\alpha\beta} F^{\epsilon\eta\alpha} F^{\alpha\beta\sigma} \quad (9.5)$$

with $\mu, \nu, \rho, \sigma, \tau, \epsilon, \eta, \alpha, \beta = 0, 1, 2, 3, 5, \dots, 10$; $a = 1, 2, 3$ that appears in the 10-dimensional supersymmetry algebra, gives, by dimensional reduction, the three magnetic charges in the $N=4$ supersymmetry algebra in $3+1$ dimensions.

(v) As we have seen, topological central charges are not extra components of the momentum. Then, in order to recover the mass formula by dimensional reduction, it will be necessary to redefine the Casimir constants of the Poincaré group in the higher dimensions, including nonscalar central charges. It would be interesting to look for the representations of supersymmetry algebras with vectorial central charges.

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