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DETERMINATION OF THE CORRELATION LENGTH BY VARYING
THE BOUNDARY CONDITIONS IN LATTICE GAUGE THEORIES

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ABSTRACT

By Monte Carlo techniques we study the response of the expectation values of Wilson loops (sampled from the interior of a lattice) to the change of boundary conditions. In the case of the pure SU(2) gauge theory, we find, within reasonable computer time, a clear numerical signal for the correlation between the boundary and the interior plaquettes over a distance of up to four lattice spacings. As a result of our computations on 8^4 and 10^4 lattices, we verify the asymptotic freedom behaviour of the correlation length for β values up to 2.3 - 2.4 and estimate the glueball mass to be

$$m_g = (2.35 \pm 0.30) \sqrt{\sigma}$$

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Ever since the exciting discovery of asymptotic freedom behaviour of the string tension calculated from the area decrease of large Wilson loops in SU(2) and SU(3) lattice gauge theories¹⁾ it has been a challenge to establish a scaling region of the correlation length $\xi(\beta)$ as well. The practical problem turned out to be that Monte Carlo simulations would need extremely long computer time before yielding any significant signal for the correlation function $\rho(r)$ between plaquettes at distances r more than two lattice spacings apart²⁾. The short distance behaviour of $\rho(r)$ is, however, rapidly dominated by perturbative effects as the correlation length increases with β . Therefore, the β -region over which the asymptotic freedom behaviour

$$\xi(\beta) = \frac{\Lambda_L}{m_g} \left(\frac{6\pi^2}{11} \beta \right)^{-\frac{51}{121}} \exp \left(\frac{3\pi^2}{11} \beta \right) \quad (\text{for the SU(2) case}) \quad (1)$$

can be tested, is very limited ($2. \lesssim \beta \lesssim 2.2$) so far.

There appear to exist two possibilities to obtain a reasonably large scaling domain in Monte Carlo calculations for the correlation functions³⁾:

1. Concentrate on short distances $r = 1$ and $r = 2$, but include the correlations of more complicated Wilson loops than simple plaquettes and disentangle the true long range behaviour due to the lowest lying 0^+ glueball state.

³⁾ A more speculative third method has been recently advanced by Brower et al³⁾. They propose to use finite size scaling à la Fischer⁴⁾ to determine the correlation length. Engels et al⁵⁾ proposed to extract the glueball mass (without verification of the asymptotic freedom behaviour of ξ) from finite temperature lattice calculations by describing energy density and specific heat in terms of an ideal gas of resonances.

The method is to expand the glueball state in terms of certain Wilson loop operators acting on the vacuum and varying the coefficients in this expansion until the state with lowest energy is reached. This variational method was proposed by Wilson a year ago and performed most recently in refs. 6.

2. Search for a situation where the long range ($r = 3$ or 4) correlation is large enough to be detectable in Monte Carlo simulation.

In this paper we pursue the second option by exploiting the sensitivity of a Wilson loop average to variations of the boundary conditions in finite lattices. The maximal signal is achieved by changing from free to fixed boundary conditions. As we showed in a recent paper⁷⁾ the Wilson loop average in the thermodynamical limit ($V \rightarrow \infty$) is bounded from above and below by its expectation value calculated in a finite lattice V with fixed and free boundary conditions, respectively:

$$L(r) = \langle W(C) \rangle_{0,V} \leq \langle W(C) \rangle_{V \rightarrow \infty} \leq \langle W(C) \rangle_{1,V} = U(r) \quad (2)$$

Here the index 0 and 1 denotes free and fixed boundary conditions respectively. The quality of these bounds of course crucially depends on the distance r between the loop C and the boundary ∂V .

It will be shown below that up to third and higher order correlations, the difference $\Delta = U - L$ between the two bounds has the following structure in terms of second order correlation functions:

$$\frac{\Delta}{L} = \left(1 + \sum_{P, P' \in \partial V} \tilde{\rho}(P, P') \right)^{-1} \sum_{P \in \partial V} \rho(P, C) \quad (3)$$

where

$$\rho(P, C) = \frac{\langle O(P) W(C) \rangle}{\langle O(P) \rangle \langle W(C) \rangle} - 1 \quad (4a)$$

and

$$\tilde{\rho}(P, P') = \frac{\langle O(P) O(P') \rangle}{\langle O(P) \rangle \langle O(P') \rangle} - 1 \quad (4b)$$

In eq.(3) the sums extend over the plaquettes P on the boundary ∂V . $O(P)$ is a local operator and the averages in eq.(4) have to be performed with free boundary conditions. In this paper we shall demonstrate that Monte Carlo calculations yield a significant signal for the quantity Δ even if all the distances $r(P, C)$ in the correlation functions $\rho(P, C)$ (cf. eq.(3)) exceed three lattice spacings.

As an example we have chosen the pure $SU(2)$ gauge theory - approximated by its 120 element icosaheder subgroup $T^{(8)}$ - on a hypercubical lattice $V = N^4$ ($N=8, 10$) with the Wilson action⁹⁾:

$$S = \frac{1}{2} \sum_{P \in V} \text{Tr} (u(P) + u^+(P)) \quad (5)$$

where

$$u(P) = \prod_{l \in P} u(l)$$

stands for the ordered product of the four link variables $U(l)$, $l \in P$. Expectation values are calculated with respect to the partition function

$$Z(\beta) = \prod_{l \in V} \int du(l) e^{\beta S} \quad (6)$$

For the subgroup T , the integral in eq.(6) turns into a sum which is calculated by the standard Metropolis process. Free boundary conditions are defined by simply integrating over all links inside the boundary ∂V . Fixed boundary conditions are realized by freezing all plaquette variables on the boundary ∂V to the unit element

$$u(p) = 1 \quad \text{for } p \in \partial V \quad (7)$$

Let us now sketch the derivation of eqs. (3) and (4): the condition (7) for the SU(2) case is equivalent to the vanishing of the plaquette angles $\theta(p)$, defined by

$$\cos \theta(p) = \frac{\text{Tr}(u(p) + u^\dagger(p))}{2 \text{Tr} 1} \quad (8)$$

It is obvious that any expectation value with fixed boundary conditions can be converted into another such quantity with free boundary conditions. In particular one derives

$$\frac{\Delta}{L} = \frac{\langle W(c) \rangle_{\pi}^{\theta(p)} - 1}{\langle W(c) \rangle_0 \langle \pi \theta(p) \rangle_0} \quad (9)$$

where

$$\theta(p) = \delta(\theta(p)) \quad (10)$$

Now we perform a cluster expansion of the expectation values encountered in eq. (9):

$$\langle \pi \theta(p) \rangle_0 = \pi \langle \theta(p) \rangle_0 + \quad (11)$$

$$+ \sum_{p_1 + p_2 \in \partial V} \{ \langle \theta(p_1) \theta(p_2) \rangle_0 - \langle \theta(p_1) \rangle_0 \langle \theta(p_2) \rangle_0 \} \pi \langle \theta(p_3) \rangle_0$$

+ ...

$$\begin{aligned}
 & \langle W(C) \prod_{P \in \partial V} \theta(P) \rangle_0 - \langle W(C) \rangle_0 \langle \prod_{P \in \partial V} \theta(P) \rangle_0 = \\
 & = \sum_{P_1 \in \partial V} \{ \langle \theta(P_1) W(C) \rangle_0 - \langle \theta(P_1) \rangle_0 \langle W(C) \rangle_0 \} \prod_{P_2 \neq P_1 \in \partial V} \langle \theta(P_2) \rangle_0 \quad (12) \\
 & + \dots
 \end{aligned}$$

In view of the fact that in the β region of interest the correlation length ξ is of order $\lesssim 1$ we drop the higher order correlations in Eq. (9). As a result Eq. (3) is easily recovered.

In our computations, we have chosen the loops C to be simple plaquettes Q sampled from concentric cubes

$$K_d = (N - 2d)^4 \quad (13)$$

with distance d to the boundary ∂V . In a given lattice $V = N^4$ the bounds $U(d)$ and $L(d)$ are expected to become tighter with increasing d . The numerical situation is exemplified for $N = 8$ in table 1, where the most precise previous values for the mean plaquette obtained with periodic boundary conditions are also given for reference. Comparing these results one observes that the averages performed with free and fixed boundary conditions are always below and above those with periodic boundary conditions, which is a numerical support of the inequalities eq. (2).

In fig. 1 we have plotted for an 8^4 lattice the difference $\Delta(d) = U(d) - L(d)$ as a function of β and for distances $d = 1, 2, 3$ (connected by full lines to guide the eye) and the corresponding results for the 10^4 lattice (connected by dotted lines). This figure illustrates that our method indeed extends the accessible range for correlation measurements from $d = 2$ to $d = 4$.

In order to extract the correlation length from these data, we assume an exponential behaviour for the correlation function

appearing in eq.(3):

$$\rho(P, Q) \approx \rho_0 \exp \left[-r(P, Q) / \xi(\beta) \right] \quad (14)$$

where $r(P, Q)$ denotes the Euclidean distance between the plaquettes P in the boundary ∂V and the measured plaquettes Q in the concentric cube K_d (cf. eq.(13)). With this ansatz the quantity $\Delta(d)$ is related to the correlation length $\xi(\beta)$ in the following way:

$$\Delta(d) = n(d)^{-1} \rho_0 \frac{\sum_{Q \in K_d} \sum_{P \in \partial V} \exp \left[-r(P, Q) / \xi(\beta) \right]}{1 + \sum_{P_1, P_2 \in \partial V} \tilde{\rho}(P_1, P_2)} \quad (15)$$

where $n(d)$ is the number of plaquettes Q in the cube K_d . By means of eq.(10) we extract the inverse correlation length $m(\beta) = \xi(\beta)^{-1}$ from the measured ratios $\Delta(d)/\Delta(d-1)$ in a 8^4 lattice with $d = 2, 3$ and in a 10^4 lattice with $d = 2, 3, 4$. The results in figs. 2a and 2b display the pattern familiar since Creutz's famous results¹⁾ for the string tension: a high temperature regime fairly well accounted for by the high temperature expansion in lowest order^{10) 11)}, a rapid turnover into the asymptotic freedom domain and a breakoff into a spin wave dominated region, whose position depends strongly on the distance d involved in the measurement. We emphasize that in both lattices the scaling domain - where the correlation length $\xi(\beta)$ follows the prediction of asymptotic freedom (cf. eq.(1)) - is now clearly visible. While the data taken at a short distance ($d = 2$) start deviating from the asymptotic freedom behaviour near $\beta = 2.1$, the long distance measurements ($d = 4$) provide us with a considerably larger scaling domain, extending up to $\beta = 2.4$.

In figs. 2a and 2b the full straight lines - with slopes predicted by asymptotic freedom - mark the band-widths of our data. According to eq.(1) we obtain for the glueball mass:

$$m_g = (2.2 \pm .3) \sqrt{\sigma} \quad \text{from the } 8^4 \text{ lattice} \quad (16)$$

and

$$m_g = (2.5 \pm .3)\sqrt{\sigma} \text{ from the } 10^4 \text{ lattice} \quad (17)$$

in terms of the string tension σ , using $\Lambda_L = .0119\sqrt{\sigma}^{11}$). Our best estimate for the glueball mass is thus $m_g = (2.35 \pm .3)\sqrt{\sigma}$, which is in beautiful agreement with the most advanced calculation using the variational method by Berg et al⁶⁾. They quote a value of $m_g = (2.4 \pm .6)\sqrt{\sigma}$. Let us mention that the computer effort to obtain correlation signals over four lattice spacings by our method is quite reasonable; the 10^4 lattice results presented here were obtained with a total of 21000 sweeps in 20 hours of CDC 7600.

In conclusion, the method of varying the boundary conditions appears to be very useful in the study of lattice gauge theories. It would be very desirable to try it in the SU(3) case as well. Another open question is how the results obtained here depend on the specific choice of the action. Work along this line is in progress.

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REFERENCES

- 1) M. Creutz, Phys. Rev. D21 (1980) 2308; Phys. Rev. Lett. 45 (1980) 313;
E. Pietarinen, Nucl. Phys. B190 [FS3] (1981) 349.
- 2) B. Berg, Phys. Lett. 97B (1980) 401;
B. Berg and I. Stehr, Z. Phys. C9 (1981) 333;
G. Bhanot and C. Rebbi, Nucl. Phys. B180 [FS2] (1981) 469.
- 3) R. Brower, M. Nauenberg and T. Schalk, Phys. Rev. D24 (1981) 548.
- 4) M.E. Fisher, in Proc. Int. School of Physics, Enrico Fermi, Varenna (1970) Course No. 51, ed. M.S. Green, Academic Press (New York).
- 5) J. Engels, F. Karsch, I. Montvay and H. Satz, Phys. Lett. 102B (1981) 332 and Bielefeld preprint BI TP 81/29.
- 6) K.G. Wilson, talk given at the Abingdon meeting on Lattice Gauge Theories (March 1981);
B. Berg, A. Billoire and C. Rebbi, Brookhaven preprint, "Monte Carlo Estimate of the SU(2) Massgap" (1982);
K. Ishikawa, G. Schierholz and M. Teper, DESY preprint 81-089 (1981);
M. Falcioni, E. Marinari, M.L. Paciello, G. Parisi, F. Rapuano, B. Taglienti and Zhang Yi-cheng, On the Masses of the Glueballs in pure SU(2) Lattice Gauge Theory, Ist. de Fisica "G. Marconi" preprint 277;
B. Berg and A. Billoire, CERN preprint TH-3230 (1982).
- 7) K.H. Mütter and K. Schilling, Nucl. Phys. B200 [FS4] (1982) 362.
- 8) G. Bhanot, C.B. Lang and C. Rebbi, CERN preprint TH-3178 (1980).
- 9) K. Wilson, Phys. Rev. D19 (1975) 2445.
- 10) G. Münster, Nucl. Phys. B190 [FS3] (1981) 548.
- 11) A thorough discussion of the numerical situation of high temperature series expansion results for m will be given by B. Berg, A. Billoire and C. Rebbi in a Brookhaven preprint (1982).

SUBTITLE TABLE 1

Upper and lower bounds $U(d)$ and $L(d)$ in the 8^4 lattice.

The row labelled BNS refers to the mean plaquette in the infinite lattice calculated in ref.(3) by extrapolating periodic boundary results in various finite lattices to $N \rightarrow \infty$. BS quotes the periodic boundary condition results for the mean plaquette obtained by Berg and Stehr²⁾ in an 8^4 lattice. Our errors are calculated from the fluctuations between subsamples over a 200 sweeps each over the whole lattice $K_0 = 8^4$. The accuracy for $L(2)$, $U(2)$, $L(3)$, $U(3)$ was reached by updating and measuring over K_2 , K_3 more frequently in between each sweep over $K_0 = V$. The relative frequencies for these sweeps were adjusted to be $f_0:f_2:f_3 = 1:5:60$.

TABLE 1:

β	# free b.c.	$L(1) \cdot 10^5$	$L(2) \cdot 10^5$	$L(3) \cdot 10^5$	BNS $\cdot 10^5$	BS $\cdot 10^5$	$U(3) \cdot 10^5$	$U(2) \cdot 10^5$	$U(1) \cdot 10^5$	# fixed b.c.
1.5	4600	36375 $\pm$ 59	36263 $\pm$ 13	36235 $\pm$ 16			36258 $\pm$ 16	36301 $\pm$ 14	38106 $\pm$ 56	5400
1.8	7200	44050 $\pm$ 40	44094 $\pm$ 12	44097 $\pm$ 19		44075 $\pm$ 10	44145 $\pm$ 24	44573 $\pm$ 14	48350 $\pm$ 40	6400
1.9	8000	46834 $\pm$ 60	47005 $\pm$ 14	47031 $\pm$ 21		46980 $\pm$ 12	47156 $\pm$ 19	47975 $\pm$ 15	51972 $\pm$ 29	7200
1.95	4600		48524 $\pm$ 27	48518 $\pm$ 20			48844 $\pm$ 29	49845 $\pm$ 16		5600
2.0	5200	49862 $\pm$ 32	50057 $\pm$ 18	50083 $\pm$ 27		50095 $\pm$ 25	50764 $\pm$ 35	51809 $\pm$ 32	55447 $\pm$ 65	5600
2.05	4200		51702 $\pm$ 22	51735 $\pm$ 36	517400		52731 $\pm$ 32	53820 $\pm$ 21		5400
2.10	7000	53060 $\pm$ 60	53395 $\pm$ 20	53409 $\pm$ 25	534500	53461 $\pm$ 16	54670 $\pm$ 35	55686 $\pm$ 30	58760 $\pm$ 50	4200
2.15	4600		55021 $\pm$ 15	55113 $\pm$ 27	551800		56635 $\pm$ 40	57507 $\pm$ 25		3600
2.2	2800	56329 $\pm$ 77	56683 $\pm$ 30	56836 $\pm$ 53	569200	56877 $\pm$ 13	58374 $\pm$ 36	59061 $\pm$ 23	61390 $\pm$ 120	3400
2.25	2200		58327 $\pm$ 31	58480 $\pm$ 53	586200		59892 $\pm$ 34	60445 $\pm$ 22		2800
2.30	3000	59476 $\pm$ 60	59903 $\pm$ 32	60033 $\pm$ 44	602000	60183 $\pm$ 14	61227 $\pm$ 23	61691 $\pm$ 24	63690 $\pm$ 50	2600
2.4	2800	62150 $\pm$ 30	62686 $\pm$ 23	62819 $\pm$ 38	630000	63032 $\pm$ 12	63553 $\pm$ 23	63905 $\pm$ 15	65530 $\pm$ 20	2800
2.5	3000	64400 $\pm$ 30	64880 $\pm$ 18	64969 $\pm$ 21	651900	65135 $\pm$ 34	65536 $\pm$ 24	65779 $\pm$ 16	67130 $\pm$ 20	3000

FIGURE CAPTIONS

Fig. 1: The quantities $\Delta = U - L$ vs. β . Full lines connect the results obtained in an 8^4 lattice, for various distances d of the interior cube to the boundary. Dashed lines connect respective results in a 10^4 lattice.

Fig. 2a: Inverse correlation length $m(\beta)$ obtained in an 8^4 lattice. Open points are obtained from $\Delta(2)/\Delta(1)$, full points from $\Delta(3)/\Delta(2)$ according to eq.(10). The dotted line represents $m = -4 \ln I_2(\beta)/I_1(\beta)$, the lowest order strong coupling result. The full lines have the asymptotic freedom slope $-3\pi^2/11$ and normalization $m_g = (2.2 \pm .3)\sqrt{\sigma}$ (with $\Lambda_L = .0119\sqrt{\sigma}$), according to eq.(1).

Fig. 2b: Inverse correlation length $m(\beta)$ obtained in a 10^4 lattice. Open and full points as in Fig. 2a, squares obtained from $\Delta(4)/\Delta(3)$. Full lines present asymptotic freedom predictions normalized to $m_g = (2.5 \pm .3)\sqrt{\sigma}$ (with $\Lambda_L = .0119\sqrt{\sigma}$).

Fig. 1

Fig. 2a

Fig. 2b