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DIVERGENCES AND ANOMALIES IN KALUZA-KLEIN THEORIES ⁺

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A B S T R A C T

The quantization of Kaluza-Klein theories is analyzed from both the higher and lower dimensional standpoints. Equivalence is established using zeta-function techniques to handle the ultra-violet divergences. Certain special features emerge by compactifying an odd number of dimensions. For example, the resulting four-dimensional theory has vanishing anomalies.

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1. - INTRODUCTION

The original motivation for Kaluza-Klein /1/ field theories was to provide a unification of gravitation and electromagnetism. The basic idea is that instead of having a theory which is based upon a four-dimensional space-time manifold, there is an extra fifth dimension. In order to get around the question of why this fifth dimension is not observed, it was proposed that the fifth direction corresponded to a cyclic co-ordinate whose periodicity length was sufficiently small to place it beyond the range of available energies. [Thus the five-dimensional manifold was taken to be of the form $M(4) \times S^1$ where $M(4)$ is any four-dimensional manifold.] A further requirement is that the five-dimensional metric have no dependence on the fifth co-ordinate. The extra off-diagonal components $g_{\mu 5}$ ($\mu = 1, \dots, 4$) of the five-dimensional metric are taken to be proportional to the four gauge potentials A_μ . If the five-dimensional action is taken to be the Einstein-Hilbert gravitational action, then the equivalent four-dimensional theory consists of the usual Einstein-Maxwell system. Extra matter fields may also be included in this scheme /2/. Souriau /2/ contains many references to the older literature on unified field theories of the Kaluza-Klein type.

These days, any theory which unifies only gravity and electromagnetism is to be viewed as deficient. However, by generalizing the approach of Kaluza and Klein it is possible to obtain theories which unify gravity and Yang-Mills fields. This may be done by taking a higher dimensional space-time of the form $M(4) \times N$ where N is a compact manifold of sufficiently small dimensions to place it below the limits of present day

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energies. N is usually taken to be a group manifold G /3/ or a coset space G/H /4/. By a suitable generalization of the Kaluza-Klein ansatz for the higher dimensional metric (see Refs. /3,4/ or Section 3 below) the equivalent four-dimensional theory becomes Einstein-Yang-Mills with gauge group G .

Another place where higher dimensional field theories play a rôle is in dimensional reduction /5/. (See Cremmer /5/ for a review.) The idea here is that fields are defined on a product manifold $M(4) \times N$, where N is compact, which inherits the natural product metric. N is usually taken to be a n -torus $S^1 \times S^1 \times \dots \times S^1$. Fields are then expanded in a suitable set of harmonics on N with expansion coefficients which depend only on the co-ordinates of $M(4)$. The extra dimensions are then integrated over resulting in an infinite set of fields on $M(4)$. The masses of these fields will be inversely proportional to the length scales which characterize N . Since the length scales must be small in order to explain why N is unobservable, the resulting masses must be very large, and hence are not seen at present energies. In the limit where the length scales of N are taken to zero, only the lowest mode which in general is massless will retain a finite mass and the rest are usually ignored. This is dimensional reduction. (In the case of the n -torus, an equivalent way of obtaining only a single massless mode, for all spins, is to assume that the fields do not depend at all on the extra dimensions.)

In order to illustrate this, consider the simplest example where we take $N = S^1$ and have a massless real scalar field on $M(4) \times S^1$. The five-dimensional action may be taken to be

$$S = -\frac{1}{2} \int d^5x \sqrt{g} \Phi(x) \hat{\square} \Phi(x), \quad (1.1)$$

where a caret denotes quantities formed using the five-dimensional metric. Allow the metric on $M(4) \times S^1$ to be

$$\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \quad (1.2)$$

where $g_{\mu\nu}$ is the metric on $M(4)$ which does not depend on x^5 . Fourier expand the field as

$$\Phi(x, x^5) = L^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{inMx^5}, \quad (1.3)$$

where $0 \leq x^5 \leq L = 2\pi/M$. By substituting (1.2), (1.3) into (1.1) and performing the integration over x^5 it is found that

$$S = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d^4x \sqrt{g} \overline{\phi_n(x)} (-\square + n^2 M^2) \phi_n(x), \quad (1.4)$$

where $\square$ is the d'Alembertian on $M(4)$. The equivalent four-dimensional theory therefore consists of a single real massless scalar field and an infinite number of complex massive modes with quantized masses. In the limit in which $L \rightarrow 0$ (i.e., $M \rightarrow \infty$) only the $n = 0$ mode retains a finite mass. Since this would be the only mode observable at low energies it is commonly supposed that the other modes may be dropped. However, if gravitational effects are considered, the more massive particles would be expected to be as important as the massless modes, and it is not at all clear that the $M \rightarrow \infty$ limit may be used to recover the massless quantum theory.

Dimensional reduction plays an important rôle in supergravity. In particular it has been used to obtain $N = 8$ supergravity in four-dimensions from $N = 1$ supergravity in eleven-dimensions /6/. It has also been used to discuss models with broken supersymmetry /7/. One view which may be adopted here is that dimensional reduction is merely a useful trick for obtaining four-dimensional theories.

The other possibility, as in Kaluza-Klein theories, is to regard the extra dimensions as real. Furthermore, rather than imposing the requirement that the manifold be of the form $M(4) \times N$ with compact N in an ad hoc manner, the manifold is required to be a solution of the higher dimensional field equations. This is called spontaneous compactification /8/. The idea of spontaneous compactification is an attractive one within the context of $N = 1$ supergravity in eleven-dimensions. The reason is that this theory contains an antisymmetric tensor $A_{\mu\nu\rho}$ (in addition to the elfbein e_μ^α and gravitino ψ_μ^α) which introduces a cosmological constant /9/ and leads to the spontaneous compactification of the seven extra dimensions /10/. Witten /4/ has also shown that eleven dimensions is the minimum which allows for a Kaluza-Klein theory with a symmetry $SU(3) \times SU(2) \times U(1)$ for the extra dimensions. This presents the possibility of viewing $N = 1$ eleven-dimensional supergravity in terms of a Kaluza-Klein theory.

With regard to ultra-violet divergences, if one takes the view that the extra dimensions are real, then the relevant question is not: "Is $N = 8$ supergravity finite in four-dimensions?", but rather: "Is $N = 1$ supergravity finite in eleven-dimensions?". Since any field theory is automatically odd-loop finite in an odd-dimensional space-time (barring surface terms if boundaries are present) (see Section 2 below), we know that $N = 1$ supergravity is odd-loop finite in eleven-dimensions. The reduced four-dimensional theory will therefore also be odd-loop finite if all of the modes are kept in the dimensional reduction. If only a finite number of modes are kept, then there are no guarantees that this will be true.

[Consider pure gravity in d dimensions. By power counting the degree of divergence of the effective action is $D = (d-2)L+2$ at order L in the loop expansion. Thus odd loops have an odd degree of divergence if d is odd. Since there are no curvature

invariants with an odd dimension which can form a possible counterterm, pure gravity in an odd dimensional space-time would be expected to be odd-loop finite. Any reduced theory of the Kaluza-Klein type will therefore be odd-loop finite.]

These comments lead to the following somewhat puzzling result. Consider the real scalar field in $M(4) \times S^1$ discussed earlier. Quantize this theory on $M(4) \times S^1$ which is regarded as a fixed background. From the five-dimensional viewpoint the theory must be finite. From the four-dimensional viewpoint the theory consists of an infinite number of scalar fields on a curved four-dimensional background. In order to form the counterterm for this four-dimensional theory we would first write down the counterterm for the n^{th} mode and then sum them to obtain the total counterterm. But the counterterm for a single mode is not zero and involves invariants such as $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$. The puzzle is how this four-dimensional theory with an infinite number of fields can be one-loop finite when a single scalar field gives divergences. Equally perplexing is what happens when the four-dimensional theory is obtained by reduction from a higher dimensional space of even dimension. In the case of six-dimensions for example, the theory is not finite and the counterterm involves curvature invariants of dimension six such as $R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\lambda\tau} R^{\lambda\tau}_{\mu\nu}$. But invariants of dimension six do not occur in four-dimensional counterterms. How can a four-dimensional theory with an infinite number of fields give rise to counterterms which have a dimension greater than four? Similar puzzles occur for the axial and the trace anomalies. The main purpose of this paper is to investigate the renormalizability properties of such higher dimensional field theories.

The outline of the paper is the following. In Section 2, we review zeta-functions and the heat kernel expansion, presenting a number of results which are used in subsequent sections. In Section 3 the reduction of a scalar field is discussed. Resolutions of the puzzles above are given. In Section 4 the reduction of a Dirac spinor for a five-dimensional Kaluza-Klein theory is considered, and the axial anomaly is discussed. A discussion of our results is presented in Section 5.

2. - ZETA-FUNCTIONS AND THE HEAT KERNEL EXPANSION

For quantum field theory in general space-times, the most efficient way of discussing renormalizability and computing counterterm is to evaluate the effective action using the background field method (DeWitt /3/) and regularize using heat kernel techniques. This approach has been used for free /11/ as well as interacting /12/ fields in curved space-time, and is also useful for computing anomalies /13/. The one-loop contribution to the effective action arises from performing a Gaussian functional integral over that part of the classical action which is quadratic in the fields. If the theory is a free one then no higher loop contributions arise. If the theory is an interacting one then it is necessary to expand about an arbitrary background field.

Let Δ be the operator representing the quadratic part of the action in either case. [For example, in the case of a free massless scalar field as in Eq. (1.1), $\Delta = -\square$.] The one-loop contribution to the effective action is:

$$\Gamma^{(1)} = \frac{1}{2} \ln \text{Det } \Delta \quad (2.1)$$

$$= \frac{1}{2} \text{Tr } \ln \Delta. \quad (2.2)$$

The determinant and trace in (2.1), (2.2) are understood to be functional. We are assuming here that the manifold is equipped with a positive definite Riemannian metric. In order that the heat kernel expansion used below is valid we assume that the manifold is compact without boundary /14/ or else non-compact with negligible boundary /15/. This method may also be used when boundaries are present /16/ but we do not consider this further here.

By varying (2.2) with respect to the background field,

$$\begin{aligned} \delta \Gamma^{(1)} &= \frac{1}{2} \text{Tr} [\Delta^{-1} \delta \Delta] \\ &= -\frac{1}{2} \delta \left\{ \text{Tr} \left[\int_0^\infty \frac{dt}{t} e^{-t\Delta} \right] \right\}, \end{aligned}$$

where the second line follows by introducing a parameter integration /17/. This leads to the definition

$$\Gamma^{(1)} = -\frac{1}{2} \text{Tr} \left[\int_0^\infty \frac{dt}{t} e^{-t\Delta} \right]. \quad (2.3)$$

Assume that Δ is a self-adjoint operator and let $\{\lambda_N, \phi_N\}$ be a spectral decomposition of Δ ; that is,

$$\Delta \phi_N = \lambda_N \phi_N, \quad (2.4)$$

where $\{\phi_N\}$ is a complete orthonormal set. Any field ϕ may then be expanded in terms of the ϕ_N :

$$\phi(x) = \sum_N c_N \phi_N(x). \quad (2.5)$$

From (2.4) it may be seen that

$$(e^{-t\Delta} \phi_N)(x) = e^{-t\lambda_N} \phi_N(x), \quad (2.6)$$

and therefore from (2.5)

$$(e^{-t\Delta} \phi)(x) = \sum_N c_N e^{-t\lambda_N} \phi_N(x). \quad (2.7)$$

Let $K(t, x, y, \Delta)$ be the kernel for $e^{-t\Delta}$

$$(e^{-t\Delta}\phi)(x) = \int dV_y K(t, x, y, \Delta) \phi(y) \quad (2.8)$$

($dV_y = \sqrt{g_y} d^n y$ is the invariant volume element.) It may be expressed as

$$K(t, x, y, \Delta) = \sum_N e^{-t\lambda_N} \phi_N(x) \otimes \bar{\phi}_N(y). \quad (2.9)$$

[Substitution of (2.9) into (2.8) is seen to give (2.7).] In terms of the kernel function, (2.3) becomes

$$\Gamma^{(1)} = -\frac{1}{2} \int dV_x \int_0^\infty \frac{dt}{t} \text{tr} [K(t, x, x, \Delta)], \quad (2.10)$$

where the functional trace has been performed leaving only the trace over any indices which might be present.

It is also useful to define

$$Y(t, \Delta) = \int dV_x \text{tr} K(t, x, x, \Delta) \quad (2.11)$$

to be the integrated heat kernel. From (2.9) it may be seen that

$$Y(t, \Delta) = \sum_N e^{-t\lambda_N}. \quad (2.12)$$

As $t \downarrow 0^+$, $K(t, x, x, \Delta)$ has the following asymptotic expansion (we follow the second paper of Ref. /14/)

$$K(t, x, x, \Delta) \sim (4\pi t)^{-n/2} \sum_{m=0}^{\infty} t^m E_m(x, \Delta). \quad (2.13)$$

(n represents the dimension of the manifold.) The $E_m(x, \Delta)$ are local invariants constructed from quantities occurring in Δ . If we define

$$a_m(\Delta) = \int dV_x \text{tr} E_m(x, \Delta), \quad (2.14)$$

then the integrated heat kernel (2.11) has the asymptotic expansion

$$Y(t, \Delta) \sim (4\pi t)^{-n/2} \sum_{m=0}^{\infty} t^m a_m(\Delta). \quad (2.15)$$

For large t , $Y(t, \Delta) \sim e^{-t\lambda_0}$ where $\lambda_0 = \min \{\lambda_N\}$. It is assumed here that any zero modes are projected out. Then the divergences of $\Gamma^{(1)}$ come only from the $t = 0$ limit in the parameter integration in (2.10). Split up the integration over t into

$$\int_0^\infty \frac{dt}{t} K(t, x, x, \Delta) = \int_0^{t_0} \frac{dt}{t} K(t, x, x, \Delta) + \int_{t_0}^\infty \frac{dt}{t} K(t, x, x, \Delta), \quad (2.16)$$

where $0 < t_0 < 1$ is arbitrary. We may take t_0 to be as small as we choose and use the asymptotic expansion (2.13) in the first integral of (2.16). Adopt dimensional regularization where the space-time dimension n is treated as a regularizing parameter /18/. Equation (2.16) is analytic for $\text{Re}(n) < 0$ and we have

$$\int_0^{t_0} \frac{dt}{t} K(t, x, x, \Delta) = (4\pi)^{-n/2} \sum_{m=0}^{\infty} \left(\frac{-2}{n-2m} \right) t_0^{m-\frac{1}{2}n} E_m(x, \Delta). \quad (2.17)$$

Simple poles are seen to occur at $n = 0, 2, 4, \dots$ corresponding to space-times of even dimension. Let n_0 be the physical space-time dimension and put $n = n_0 + \epsilon$. If $\delta\Gamma$ represents the pole part of the one-loop effective action, then

$$\delta\Gamma^{(0)} = (4\pi)^{-n_0/2} \epsilon^{-1} \int dV_x \text{tr} E_{\frac{1}{2}n_0}(x, \Delta) \quad (2.18)$$

if n_0 is even, and $\delta\Gamma^{(1)} = 0$ if n_0 is odd. That is, in an odd-dimensional space-time the effective action is one-loop finite.

Define a generalized zeta-function by

$$\zeta(s, \Delta) = \sum'_N \lambda_N^{-s} \quad (2.19)$$

for $\text{Re}(s) > \frac{1}{2}n$, where the prime denotes that only non-zero eigenvalues are included in the sum (2.19). From (2.12) it is seen that $\zeta(s, \Delta)$ and $Y(t, \Delta)$ are related by a Mellin transform

$$\zeta(s, \Delta) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Y(t, \Delta) dt. \quad (2.20)$$

By use of the asymptotic expansion (2.15) for $Y(t, \Delta)$ it is seen [splitting up the integration as in (2.16)] that

$$\zeta(s, \Delta) = (4\pi)^{-n/2} \sum_{m=0}^{\infty} \frac{a_m(\Delta)}{\Gamma(s)} (s+m-\frac{n}{2})^{-1} + \frac{F(s)}{\Gamma(s)}, \quad (2.21)$$

where $F(s)$ is an analytic function of s . This proves that $\zeta(s, \Delta)$ is analytic for $\text{Re}(s) > \frac{1}{2}n$ and that only simple poles can be present in the complex s plane. Recall that

$$\Gamma(s) = \frac{(-1)^k}{k!} (s+k)^{-1} + O(1), \quad (2.22)$$

for s in a neighbourhood of $s = -k$, $k = 0, 1, 2, \dots$. Then for n even, $\zeta(s, \Delta)$ has $n/2$ simple poles at $s = (n/2) - k$ for $k = 0, 1, 2, \dots, (n/2) - 1$ with residue

$$(4\pi)^{-n/2} \frac{1}{\Gamma(\frac{n}{2} - k)} a_k(\Delta). \quad (2.23)$$

For n odd, $\zeta(s, \Delta)$ has an infinite number of simple poles at $s = (n/2) - k$, $k = 0, 1, 2, \dots$ with residue (2.23). From (2.21) we see also that if n is odd,

$$\zeta(s = -k, \Delta) = 0 \quad (2.24)$$

for $k = 0, 1, 2, \dots$. If n is even,

$$\zeta(s = -k, \Delta) = (-1)^k k! (4\pi)^{-n/2} a_{k+\frac{1}{2}n}(\Delta). \quad (2.25)$$

Suppose that $M = M_1 \times M_2$ is a product manifold of dimension n . Let $\Delta = \Delta_1 + \Delta_2$ where in local co-ordinates (x_1, x_2) on M Δ_1 acts trivially on the x_2 dependence of a field, and Δ_2 acts trivially on the x_1 dependence. Then from

$$e^{-t\Delta} = e^{-t\Delta_1} \otimes e^{-t\Delta_2} \quad (2.26)$$

it follows from (2.8) that

$$K(t, (x_1, x_2), (x_1, x_2), \Delta) = K(t, x_1, x_1, \Delta_1) \otimes K(t, x_2, x_2, \Delta_2), \quad (2.27)$$

and from (2.11) that

$$Y(t, \Delta) = Y(t, \Delta_1) Y(t, \Delta_2). \quad (2.28)$$

By equating coefficients of like powers of t in the asymptotic expansions on both sides of (2.27) it follows that /14/

$$E_m((x_1, x_2), \Delta) = \sum_{k=0}^m E_k(x_1, \Delta_1) \otimes E_{m-k}(x_2, \Delta_2). \quad (2.29)$$

Similarly, from (2.28),

$$a_m(\Delta) = \sum_{k=0}^m a_k(\Delta_1) a_{m-k}(\Delta_2). \quad (2.30)$$

For operators acting on scalars of the form

$$\Delta = -\square + Q(x), \quad (2.31)$$

where $Q(x)$ is in general not a constant, and is matrix-valued, the first three coefficients in the asymptotic expansion are (we use the curvature conventions of /19/ which are opposite to those of /14/).

$$E_0(x, \Delta) = 1, \quad (2.32)$$

$$E_1(x, \Delta) = \frac{1}{6}R - Q, \quad (2.33)$$

$$E_2(x, \Delta) = \frac{1}{360} [12 \square R + 5R^2 - 2R^{\mu\nu}R_{\mu\nu} + 2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 60RQ + 180Q^2 - 60 \square Q]. \quad (2.34)$$

For fields of higher spin a useful reference is /13/.

3. - THE REDUCTION OF SCALARS

Let $M(d_1)$, $M(d_2)$ be Riemannian manifolds without boundary of dimensions d_1 , d_2 respectively. Consider the direct product manifold $M(d) = M(d_1) \times M(d_2)$ which has dimension $d = d_1 + d_2$. Let $x^{\hat{\mu}} = (x^\mu, y^i)$ be co-ordinates in a local chart on $M(d)$ where x^μ , y^i are local co-ordinates on $M(d_1)$, $M(d_2)$ respectively. We use the index conventions that $\mu = 1, 2, \dots, d_1$; $i = 1, 2, \dots, d_2$; $\hat{\mu} = 1, 2, \dots, d = d_1 + d_2$.

(A) The Product Case

First of all consider the case where the metric on the product manifold is given by

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x, y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{ij}(y) \end{pmatrix}. \quad (3.1)$$

Let $\Phi(x, y)$ denote a real scalar field on $M(d)$ with action

$$S = \int dv \left[-\frac{1}{2} \Phi \hat{\square} \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{1}{2} \xi \hat{R} \Phi^2 \right]. \quad (3.2)$$

Here $dv = d^d x \sqrt{\hat{g}}$ represents the invariant volume element on $M(d)$. $\hat{R}$ and $\hat{\square}$ are computed using $\hat{g}_{\hat{\mu}\hat{\nu}}$. In (3.2) we have allowed for the possibility of a non-minimal scalar coupling. By substitution of the metric (3.1) we find that

$$\hat{\square} = \square_x + \square_y, \quad (3.3)$$

where $\square_x$ is the d'Alembertian on $M(d_1)$, and $\square_y$ is the d'Alembertian on $M(d_2)$. Also

$$\hat{R} = R_x + R_y, \quad (3.4)$$

$$dv = dv_x + dv_y, \quad (3.5)$$

where R_x (R_y) is the scalar curvature on $M(d_1)$ ($M(d_2)$), and dv_x (dv_y) is the invariant volume element on $M(d_1)$ ($M(d_2)$).

Let $\{\psi_N(y)\}$ be a complete orthonormal set of eigenfunctions of $-\square_y + \xi R_y + m^2$ with eigenvalues $\{\lambda_N\}$; that is,

$$[-\square_y + m^2 + \xi R_y] \psi_N(y) = \lambda_N \psi_N(y), \quad (3.6)$$

$$\int dv_y \bar{\psi}_N(y) \psi_M(y) = \delta_{NM}. \quad (3.7)$$

Expand the field $\Phi(x, y)$ as

$$\Phi(x, y) = \sum_N \phi_N(x) \psi_N(y) \quad (3.8)$$

for expansion coefficients $\phi_N(x)$. Since $\Phi(x, y)$ is assumed to be real ($\bar{\Phi} = \Phi$) we have

$$\sum_N \bar{\phi}_N(x) \bar{\psi}_N(y) = \sum_N \phi_N(x) \psi_N(y).$$

Multiply both sides by $\psi_M(y)$ and integrate over $M(d_2)$ using the orthonormality condition (3.7) :

$$\bar{\phi}_M(x) = \sum_N \phi_N(x) \int dv_y \psi_N(y) \psi_M(y).$$

This leads to the following identity

$$\sum_{N, M} \int dv_x dv_y \phi_M(x) O_x \phi_N(x) \psi_N(y) \psi_M(y) = \sum_N \int dv_x \bar{\phi}_N(x) O_x \phi_N(x), \quad (3.9)$$

where O_x is any operator involving only the x co-ordinates.

Substitution of the expansion (3.8) into the expression for the action (3.2), and making use of (3.3)-(3.5) and the identity (3.9) leads to

$$S = \frac{1}{2} \sum_N \int dv_x \bar{\phi}_N(x) [-\square_x + \lambda_N + \xi R_x] \phi_N(x). \quad (3.10)$$

This represents the reduced theory on $M(d_1)$. (Typically we are interested in the case $d_1 = 4$.) It is seen to consist of an infinite number of fields whose masses are given by $\lambda_N^{\frac{1}{2}}$. In general, the fields ϕ_N would be expected to be complex.

The mass spectrum of the reduced theory is seen to reflect the choice for $M(d_2)$. For example, if $d_2 = 2$, $M(d_2) = S^2$ and we take $m^2 = \xi = 0$ in (3.2), then $\lambda_{\ell m} = \ell(\ell+1)\mu^2$ where $\ell = 0, 1, \dots$ give the eigenvalues in (3.6) where μ is a unit of mass which is the reciprocal of the radius of S^2 . The four-dimensional fields $\phi_{\ell m}(x)$ are in a $(2\ell+1)$ dimensional representation of $SO(3)$. If we had taken $M(d_2=2) = S^1 \times S^1$, then we would have $\lambda_{n_1 n_2} = n_1^2 \mu_1^2 + n_2^2 \mu_2^2$ for mass scales μ_1, μ_2 inversely proportional to the two periods on the double torus. In either of these cases the lowest mode in the reduced theory consists of a single massless uncharged scalar.

The possibilities are even richer if it is noted that in general there will be inequivalent real scalar fields /20/. If we take $M(d_2=2) = RP^2$ which may be regarded as the two-sphere with antipodal points identified, then the eigenfunctions ψ_N in (3.6) may either take the same value at antipodal points on S^2 in which case it is a standard or untwisted field, or take the opposite value at antipodal points in which case it is a twisted field. The eigenvalues are again given by $\lambda_{\ell m} = \ell(\ell+1)\mu^2$, except that for the untwisted field $\ell = 0, 2, 4, \dots$, while for the twisted field $\ell = 1, 3, 5, \dots$. If the higher-dimensional field is a twisted one, all of the fields in the reduced theory are seen to be charged and massive. None of the fields would retain a finite mass in the limit $\mu \rightarrow \infty$. This is a general result for twisted fields. Twisted fields have been discussed in Kaluza-Klein theories in /21/. They are also implicit in the earlier work of Cremmer and Scherk /5/ and Scherk and Schwarz /7/.

This makes the puzzle concerning renormalizability mentioned in the introduction even more perplexing from the viewpoint of the reduced theory. Since the mass spectrum is seen to depend heavily upon the form of $M(d_2)$ we want to know if the counterterms also reflect this choice. In addition, the mass spectrum depends upon whether the field is twisted or not, and from the d_1 dimensional viewpoint, the counterterm would be expected to reflect these twists. But the twists are non-local and related to the global topology of the fibre bundle of which the field is a cross-section. The heat kernel expansion discussed in Section 2 proves that the divergent part of the one-loop effective action depends only on local quantities, so that from the higher dimensional viewpoint the divergences cannot reflect information concerning the bundle topology. Before resolving these questions we consider the reduction of scalars in generalized Kaluza-Klein theories.

(B) Kaluza-Klein Case

Suppose that we now take the case $M(d_2) = G$ where G is a compact semi-simple Lie group /3/. [The case where $M(d_2) = G/H$ is a coset space has recently been examined by Salam and Strathdee /4/.] Take the Einstein-Hilbert gravitational action on $M(d)$ to be

$$S_G = - \frac{1}{\kappa^2 V_G} \int dv (\hat{R} - 2\Lambda), \quad (3.11)$$

where $V_G = \int_G dv_y$ is the volume of G and $\kappa^2 = 16\pi G_N$ with G_N the Newtonian gravitational constant. We have also allowed for the possibility of a cosmological constant.

In order to reduce this theory adopt the Kaluza-Klein ansatz for the metric on $M(d)$; this is,

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} g_{\mu\nu} + g_{ij} B_\mu^i B_\nu^j & B_\mu^i g_{ij} \\ \hline g_{ij} B_\nu^j & g_{ij} \end{array} \right) \quad (3.12)$$

$$\hat{g}^{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} g^{\mu\nu} & -g^{\mu\nu} B_\nu^j \\ \hline -B_\mu^i g^{\mu\nu} & g^{ij} + g^{\mu\nu} B_\mu^i B_\nu^j \end{array} \right) \quad (3.13)$$

Here $g^{\mu\nu}$, g^{ij} are the inverses of $g_{\mu\nu}$, g_{ij} respectively. This form holds only for a particular choice of basis called the local direct product basis. (See Cho /3/ for details.) In this basis, the line element, in local co-ordinates, may be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ij} (dx^i + B_\mu^i dx^\mu) (dx^j + B_\nu^j dx^\nu), \quad (3.14)$$

which is a direct generalization of the Kaluza-Klein five-dimensional line element. (For calculational purposes it must be noted that the local direct product basis is a non-co-ordinate basis.)

The aim is to now identify the fields B_μ^i with the gauge fields. Since B_μ^i is dimensionless, and d_1 dimensional gauge fields have dimensions of $(\text{mass})^{\frac{1}{2}d_1-1}$, this entails the introduction of a mass parameter μ . Let $\{L_i\}$ be a set of tangent vector fields which generate the Lie algebra of G . Take the structure constants of the Lie algebra to be dimensionless numbers f_{ij}^k . Then we may write

$$[L_i, L_j] = \mu f_{ij}^k L_k. \quad (3.15)$$

(The dimensions on both sides agree since tangent vector fields have dimensions of mass, or inverse length.) We then define

$$B_\mu^i = e \mu^{-1} A_\mu^i, \quad (3.16)$$

where e is the non-Abelian charge dimension $(\text{mass})^{2-\frac{1}{2}d_1}$ and A_μ^i are the d_1 dimensional gauge fields. If field strengths $F_{\mu\nu}^i$ are defined in terms of A_μ^i in the usual manner, viz.

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + e f_{jk}^i A_\mu^j A_\nu^k, \quad (3.17)$$

then it may be shown that /3/

$$\hat{R} = R + R_G - \frac{1}{4} e^2 \mu^{-2} F_{\mu\nu}^i F_i^{\mu\nu} \quad (3.18)$$

where R is the scalar curvature of $M(d_1)$, R_G is the scalar curvature of $M(d_2) = G$ and $F_{\mu\nu}^i F_i^{\mu\nu} = g_{ij} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu}^i F_{\rho\sigma}^j$. Since R_G will be constant, the gravitational action (3.11) reduces to

$$S_G = -\frac{1}{\kappa^2} \int dV_x (R + R_G - 2\Lambda) + \frac{e^2}{4\kappa^2 \mu^2} \int dV_x F_{\mu\nu}^i F_i^{\mu\nu}. \quad (3.19)$$

In order that the last term be identified with the Yang-Mills action it is necessary that

$$\frac{e^2}{\kappa^2 \mu^2} = 1. \quad (3.20)$$

This fixes μ in terms of the Planck mass (take $d_1 = 4$) by

$$\mu = \frac{|e| m_P}{4\sqrt{\pi}} \quad (3.21)$$

where $m_P = G_N^{-\frac{1}{2}}$ is the Planck mass. From (3.18), (3.20), the scalar curvature is seen to be

$$\hat{R} = R + R_G - \frac{1}{4} \kappa^2 F_{\mu\nu}^i F_i^{\mu\nu}. \quad (3.22)$$

Suppose that we now look at the scalar field whose action is given in (3.2). The d'Alembertian may be shown to reduce as

$$\hat{\square} = g^{\mu\nu} (\nabla_\mu - \kappa A_\mu^i \partial_i) (\nabla_\nu - \kappa A_\nu^j \partial_j) + \square_G, \quad (3.23)$$

where $\square_G$ is the d'Alembertian on G , and ∇_μ is the covariant derivative on $M(d_1)$. Mode expansions similar to (3.7) may be made using the knowledge of the group representations of G . (See for example Salam and Strathdee /4/ or Chockalingham /21/.) Because the present paper is intended to be merely illustrative we shall work out only the simplest example of $M(4) \times S^1$ [i.e., $G = U(1)$]. Expand

$$\Phi(x, y) = L^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iM(n+\alpha)y}, \quad (3.24)$$

where $M = 2\pi/L$ as before. The constant α is zero if the field is untwisted (i.e., periodic) in the fifth dimension, and $\alpha = \frac{1}{2}$ if the field is twisted (i.e., antiperiodic) in the fifth direction. By making use of the identity (3.9) the reduced four-dimensional theory is found to be

$$S = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int dV_x \bar{\phi}_n(x) \left[-D_n^2 + m_n^2 + \xi R - \frac{1}{4} \xi \kappa^2 F_{\mu\nu} F^{\mu\nu} \right] \phi_n(x), \quad (3.25)$$

where

$$D_n^2 = g^{\mu\nu} (\nabla_\mu - i q_n A_\mu) (\nabla_\nu - i q_n A_\nu), \quad (3.26)$$

$$q_n = \kappa M(n + \alpha), \quad (3.27)$$

$$m_n^2 = m^2 + (n + \alpha)^2 M^2. \quad (3.28)$$

This describes an infinite number of fields of mass m_n and charge q_n interacting in a gauge invariant way with the electromagnetic field. Once again, if $\alpha \neq 0$ all of the four-dimensional fields are massive, complex, charged scalars none of which retain a finite mass in the $M \rightarrow \infty$ limit. Even in the usual case where $\alpha = 0$ so that there is a single massless neutral field, there is an unusual interaction with the electromagnetic field coming from the $-\frac{1}{4} \xi \kappa^2 F_{\mu\nu} F^{\mu\nu} \phi_0^2$ term in (3.25).

(C) Renormalization

Consider first the case of a product manifold as in part (A) of this section. The one-loop counterterm is given from (2.14), (2.18) as

$$\delta \Gamma = (4\pi)^{-d/2} \varepsilon^{-1} a_{\frac{1}{2}d}(\Delta). \quad (3.29)$$

This vanishes for d odd. Here,

$$\Delta = -\hat{\square} + m^2 + \xi \hat{R}. \quad (3.30)$$

From (3.3), (3.4), this splits into $\Delta = \Delta_x + \Delta_y$ where

$$\Delta_x = -\square_x + \xi R_x, \quad (3.31)$$

$$\Delta_y = -\square_y + m^2 + \xi R_y. \quad (3.32)$$

We may therefore use the identity (2.30) to write

$$\delta \Gamma = (4\pi)^{-d/2} \varepsilon^{-1} \sum_{k=0}^{d/2} a_k(\Delta_x) a_{\frac{1}{2}d-k}(\Delta_y). \quad (3.33)$$

This gives the counterterm from the higher dimensional viewpoint.

From the d_1 dimensional viewpoint of the reduced theory (3.10) it might be expected that the total counterterm would just be the sum of counterterms for each N ; that is,

$$\delta\Gamma = \sum_N \delta\Gamma_N, \quad (3.34)$$

where

$$\delta\Gamma_N = (4\pi)^{-d_1/2} \varepsilon^{-1} a_{\frac{1}{2}d_1}(\Delta_x + \lambda_N). \quad (3.35)$$

This brings us to the puzzles mentioned earlier. From (3.35), it is seen that if d_1 is odd, each $\delta\Gamma_N$ vanishes, and so the counterterm in (3.34) will vanish. But (3.29) only vanishes if d is also odd. Conversely, if d_1 is even, each $\delta\Gamma_N$ is non-zero, so that $\delta\Gamma$ formed as in (3.34) would be expected to be non-zero in general. This would not be consistent with (3.29) in the case of odd d . Furthermore, $\delta\Gamma_N$ in (3.35) depends upon the eigenvalues λ_N of Δ_y , which depend in turn on non-local properties of fibre bundles (i.e., the boundary conditions satisfied by the fields). But $\delta\Gamma$ in (3.29) is formed using local objects and therefore cannot reflect any such global properties. What is wrong here?

It is clear that nothing is wrong with (3.29), so that the fault must be with (3.34), (3.35). We may use the identity (2.30) to write

$$a_{\frac{1}{2}d_1}(\Delta_x + \lambda_N) = \sum_{k=0}^{d_1/2} a_k(\Delta_x) a_{\frac{1}{2}d_1-k}(\lambda_N). \quad (3.36)$$

For any positive constant operator, such as λ_N , the heat kernel expansion is trivial since

$$Y(t, \lambda_N) = e^{-t\lambda_N} = \sum_{m=0}^{\infty} t^m \frac{(-1)^m}{m!} \lambda_N^m. \quad (3.37)$$

We may therefore read off

$$a_m(\lambda_N) = \frac{(-1)^m}{m!} \lambda_N^m. \quad (3.38)$$

From (3.36), (3.38), (3.35), we have that

$$\delta\Gamma_N = (4\pi)^{-d_1/2} \varepsilon^{-1} \sum_{k=0}^{d_1/2} \frac{(-1)^{\frac{1}{2}d_1-k}}{\Gamma(\frac{1}{2}d_1-k+1)} \lambda_N^{\frac{1}{2}d_1-k} a_k(\Delta_x). \quad (3.39)$$

Since all powers of λ_N occurring in this result are seen to be positive there will be problems when the summation over N is taken in (3.34) since they will diverge. Instead they must be regularized, and a natural choice is to replace them by their ζ function regularized values [see (2.19)]. This leads to

$$\delta \Gamma = (4\pi)^{-d_1/2} \varepsilon^{-1} \sum_{k=0}^{d_1/2} \frac{(-1)^{\frac{1}{2}d_1-k}}{\Gamma(\frac{1}{2}d_1-k+1)} a_k(\Delta_x) \zeta(k-\frac{1}{2}d_1, \Delta_y). \quad (3.40)$$

If (2.25) is used, we find that

$$\delta \Gamma = (4\pi)^{-d/2} \varepsilon^{-1} \sum_{k=0}^{d_1/2} a_k(\Delta_x) a_{\frac{1}{2}d-k}(\Delta_y). \quad (3.41)$$

This looks similar to the previous result (3.33) except that there are terms missing here. (The summation extends only as far as d_1 .)

The discrepancy has to do with the ζ function regularization of the divergent summations over N . In the approach above we have expanded

$$\sum_N e^{-\lambda_N t} = \sum_N \sum_{k=0}^{\infty} \frac{(-\lambda_N)^k}{k!} t^k,$$

interchanged the order of summation and defined

$$\sum_N \lambda_N^k = \zeta(-k, \Delta_y).$$

Use of (2.25) leads to

$$\sum_N e^{-\lambda_N t} = (4\pi)^{-d_2/2} \sum_{k=0}^{\infty} t^k a_{k+\frac{1}{2}d_2}(\Delta_y). \quad (3.42)$$

However, we know that $\sum_N e^{-\lambda_N t}$ is just the integrated heat kernel $Y(t, \Delta_y)$ containing negative powers of t :

$$\sum_N e^{-\lambda_N t} = (4\pi)^{-d_2/2} \sum_{k=0}^{d_2/2-1} t^{k-\frac{1}{2}d_2} a_k(\Delta_y) + (4\pi)^{-d_2/2} \sum_{k=0}^{\infty} t^k a_{k+\frac{1}{2}d_2}(\Delta_y). \quad (3.43)$$

Comparison of (3.43) with (3.42) shows that we have missed out all of the terms involving t to negative powers. If these missing terms are reinstated, in place of (3.39), we have

$$\delta \Gamma_N = (4\pi)^{-d_1/2} \varepsilon^{-1} \sum_{k=0}^{d_1/2} \frac{(-1)^{\frac{1}{2}d_1-k}}{\Gamma(\frac{1}{2}d_1-k+1)} \lambda_N^{\frac{1}{2}d_1-k} a_k(\Delta_x). \quad (3.44)$$

The first terms $k = 0, \dots, \frac{1}{2}d_1$ are just those in (3.39). The terms for $k = \frac{1}{2}d_1+1, \dots, \frac{1}{2}d$ are seen to involve negative powers of λ_N and coefficients $a_k(\Delta_x)$ for $k > \frac{1}{2}d_1$. From the standpoint of the d_1 dimensional theory these are certainly unconventional counter-terms. An additional complication is that $1/\Gamma(\frac{1}{2}d_1-k+1)$ vanishes for $k = \frac{1}{2}d_1+1, \dots, \frac{1}{2}d$. To avoid this replace d_1 by $d_1+\delta$ and take $\delta \rightarrow 0$ at the end. Substitution of (3.44) into (3.34), and defining the summations over N by ζ function regularization as before leads to

$$\delta\Gamma = (4\pi)^{-d_1/2} \varepsilon^{-1} \sum_{k=0}^{d_1/2} \frac{(-1)^{\frac{1}{2}d_1-k}}{\Gamma(\frac{1}{2}d_1-k+1+\frac{1}{2}\delta)} a_k(\Delta_x) \zeta(k-\frac{1}{2}d_1-\frac{1}{2}\delta, \Delta_y). \quad (3.45)$$

For terms $k = 0, \dots, \frac{1}{2}d_1$ we may use (2.25). For the remaining terms, the ζ -functions have a pole at $\delta = 0$ which is cancelled by the zero of the reciprocal of the gamma function. For these terms, from (2.22), (2.23) it is seen that as $\delta \rightarrow 0$,

$$\frac{\zeta(k-\frac{1}{2}d_1-\frac{1}{2}\delta, \Delta_y)}{\Gamma(\frac{1}{2}d_1-k+1+\frac{1}{2}\delta)} \rightarrow (4\pi)^{-d_1/2} (-1)^{k-\frac{1}{2}d_1} a_{\frac{1}{2}d_1-k}(\Delta_y). \quad (3.46)$$

Thus, we find that

$$\delta\Gamma = (4\pi)^{-d_1/2} \varepsilon^{-1} \sum_{k=0}^{d_1/2} a_k(\Delta_x) a_{\frac{1}{2}d_1-k}(\Delta_y), \quad (3.47)$$

in complete agreement with (3.29), (3.33).

The conclusion is that the correct counterterms may be found either from the higher-dimensional viewpoint or else the viewpoint of the reduced theory. If this latter approach is followed then the divergent sums which arise by adding up the counterterms for each mode must be regularized using ζ -function techniques. Furthermore, the counterterm for a given mode must contain unconventional invariants from the standpoint of the d_1 -dimensional theory. If this is done the puzzles mentioned earlier are seen to be solved.

From the expression (3.33) the form of the counterterm is seen to depend upon the choice of $M(d_2)$. For example, consider a massless, minimally coupled ($\xi = 0$) scalar field in $M(4) \times M(d_2)$. Choosing $M(d_2) = S^{d_2}$ or RP^{d_2} will make a difference as far as four-dimensional counterterms are concerned, but on RP^2 it does not matter whether the field is twisted or not. Choosing $M(d_2) = S^1 \times S^1 \times \dots \times S^1$ (d_2 factors) moreover will give yet another counterterm, since all of the $a_{\frac{1}{2}d_2-k}(\Delta_y)$ coefficients vanish except for $a_0(\Delta_y)$ which gives the volume of $M(d_2)$.

Finally we turn to the Kaluza-Klein theory discussed in part (B) above. The only additional puzzle to be solved is what happens to the term in the reduced theory depending on the gauge field. The counterterm for each mode must be of the form [see (2.18)]

$$\delta\Gamma_n = (4\pi)^{-2} \varepsilon^{-1} \int dV_x \operatorname{tr} E_2(x, \Delta_x + m_n^2), \quad (3.48)$$

where m_n^2 is defined in (3.28). E_2 contains in addition to the terms in (2.34), a term $\frac{1}{24} F_{\mu\nu} F^{\mu\nu}$ which arises because the theory is gauged. The expression for Q is

$$Q_n = m_n^2 + \xi R - \frac{1}{4} \xi \kappa^2 F_{\mu\nu} F^{\mu\nu}. \quad (3.49)$$

If we take $\Delta_y = -(\partial^2/\partial y^2)$, then the eigenvalues are $\lambda_n = (n+\alpha)^2 M^2$, and the generalized ζ function is

$$\zeta(s, \Delta_y) = \sum_{n=-\infty}^{\infty} \lambda_n^{-s} = M^{-2s} [\zeta_R(2s, \alpha) + \zeta_R(2s, 1-\alpha)], \quad (3.50)$$

where $\zeta_R(s, z)$ is the generalized Riemann ζ function /22/. It satisfies $\zeta_R(s=0, z) = \frac{1}{2} - z$; therefore from (3.50) we find that $\zeta(s=0, \Delta_y) = 0$ which is a special case of (2.24). When $E_2(x, \Delta_x + m_n^2)$ is formed it is clear that

$$\delta \Gamma_n = (4\pi)^{-2} \varepsilon^{-1} \int dV_x [A_0(x) + A_1(x) \lambda_n + A_2(x) \lambda_n^2], \quad (3.51)$$

where $A_0(x)$, $A_1(x)$, $A_2(x)$ involve the curvature and the field strengths. When we take $\sum_n \delta \Gamma_n$ and replace the divergent sums by their ζ -function regularized values it is found that

$$\delta \Gamma = (4\pi)^{-2} \varepsilon^{-1} \int dV_x [A_0(x) \zeta(0, \Delta_y) + A_1(x) \zeta(-1, \Delta_y) + A_2(x) \zeta(-2, \Delta_y)]. \quad (3.52)$$

But either by making use of (2.24), or else (3.50) and the properties of the Riemann ζ functions, the right-hand side of (3.52) gets regularized to zero. Thus we find $\delta \Gamma = 0$ in agreement with the five-dimensional viewpoint. The troublesome term in $\xi \kappa^2 F_{\mu\nu} F^{\mu\nu}$ disappears. This would not be the case if only a finite number of modes had been kept in the reduced theory.

4. - REDUCTION OF THE DIRAC EQUATION

It is natural, in Kaluza-Klein theories, to have additional matter fields. In this section we discuss a Dirac spinor. The reduction of such a field is quite complicated. First of all, since a Dirac spinor in d dimensions has $2^{[d/2]}$ components, where $[d/2]$ is the integer part of $\frac{1}{2}d$, it is necessary to write the higher dimensional spinor as a set of 4-component spinors when reducing to $d_1 = 4$. Secondly the Dirac matrices must be expressed in a suitable form for reduction. Finally, mode expansions on the higher dimensional space must be used. The reduction of the Dirac equation in various cases is discussed in Refs. /2,4,5,23/, usually in the case of $M(4)$ flat. Because the present paper is intended to illustrate the effects arising at the quantum level, we shall consider only Dirac spinors on $M(4) \times S^1$, but where $M(4)$ is a general four-dimensional space-time. The quickest way of obtaining the reduction is to follow

Thirring /2/ and use the Cartan formalism. Since the curved space-time generalization is relatively straightforward we shall not discuss it here.

Define γ -matrices by

$$\{\gamma^{\hat{a}}, \gamma^{\hat{b}}\} = 2\eta^{\hat{a}\hat{b}}, \quad (4.1)$$

where $\eta^{\hat{a}\hat{b}} = \text{diag}(\text{-----})$, and Latin indices denote components taken with respect to a local orthonormal frame. As before, a caret over an index denotes that it runs over 1,...,5. Define a Hermitian γ^5 by

$$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 \quad (4.2)$$

whis is related to $\gamma^{\hat{5}}$ by $\gamma^{\hat{5}} = i\gamma^5$. The Dirac action in five dimensions is

$$S_D = \int dV \bar{\Psi}(x,y) (i\hat{\not{D}} - m) \Psi(x,y), \quad (4.3)$$

where $\hat{\not{D}} = \gamma^{\hat{a}} \hat{\nabla}_{\hat{a}}$. ($\hat{\nabla}_{\hat{a}}$ is the five-dimensional covariant derivative which includes the spin correction.) By use of the Cartan calculus one may show that

$$\hat{\not{D}} = \not{D} - \kappa \not{A} \frac{\partial}{\partial y} + i\gamma^5 \frac{\partial}{\partial y} + \frac{i}{8} \kappa F_{ab} \gamma^a \gamma^b \gamma^5, \quad (4.4)$$

where $A = \gamma^a A_a$, $\not{D} = \gamma^a \nabla_a$ are the four-dimensional expressions.

In order to obtain the reduced theory we perform a mode expansion as in the scalar field case. Again the five-dimensional spinor may be periodic or antiperiodic in the fifth co-ordinate, this time because inequivalent spin connections exist /24/. Write

$$\Psi(x,y) = L^{-1/2} \sum_{n=-\infty}^{\infty} \psi_n(x) e^{i(n+\alpha)My}, \quad (4.5)$$

where $M = 2\pi/L$, $\alpha = 0$ or $\frac{1}{2}$. The equivalent four-dimensional action is therefore

$$S_D = \sum_{n=-\infty}^{\infty} \int dV_x \bar{\psi}_n(x) \left[i\not{D} + \kappa(n+\alpha)M\not{A} - m - i(n+\alpha)M\gamma_5 - \frac{1}{8} \kappa F_{ab} \gamma^a \gamma^b \gamma^5 \right] \psi_n(x). \quad (4.6)$$

Define

$$\not{D}_n = \not{D} - i q_n \not{A} \quad (4.7)$$

so that

$$q_n = \kappa(n+\alpha)M \quad (4.8)$$

represents the charge of ψ_n . For the untwisted spinor, ($\alpha = 0$) there is one neutral and an infinite number of charged four-dimensional spinors. For the twisted five-dimensional spinor ($\alpha = \frac{1}{2}$) all of the four-dimensional spinors are charged. This is similar to the scalar field case.

There is an interesting effect due to the axial anomaly. The clearest way of seeing this is to follow the approach of Ref. /25/. There it was noted that the functional measure for fermions was not invariant under a chiral transformation. If $d\mu[\bar{\psi}_n, \psi_n]$ represents the functional measure for fermions, then under $\psi_n \rightarrow \exp i\theta\gamma^5 \cdot \psi_n$

$$d\mu[\bar{\psi}_n, \psi_n] \rightarrow e^{-2i\theta\nu_n} d\mu[\bar{\psi}_n, \psi_n], \quad (4.9)$$

where

$$\nu_n = \frac{q_n^2}{16\pi^2} \int dV_x {}^*F_{\mu\nu} F^{\mu\nu} - \frac{1}{384\pi^2} \int dV_x {}^*R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \quad (4.10)$$

is a topological invariant. The stars denote the dual tensors. The complete functional measure in our case is $\prod_{n=-\infty}^{+\infty} d\mu[\bar{\psi}_n, \psi_n]$. From (4.9) we find

$$\prod_{n=-\infty}^{\infty} d\mu[\bar{\psi}_n, \psi_n] \rightarrow e^{-2i\theta \sum_{n=-\infty}^{\infty} \nu_n} \prod_{n=-\infty}^{\infty} d\mu[\bar{\psi}_n, \psi_n]. \quad (4.11)$$

But $\sum_{n=-\infty}^{+\infty} \nu_n$ is clearly divergent. We may regularize it using the same ζ -function regularization technique as in the last section. Define

$$\sum_{n=-\infty}^{\infty} \nu_n = \frac{-1}{384\pi^2} \zeta(0) \int dV_x {}^*R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{16\pi^2} \zeta(-1) \int dV_x {}^*F_{\mu\nu} F^{\mu\nu}, \quad (4.12)$$

where

$$\zeta(s) = \sum_{n=-\infty}^{\infty} (q_n^2)^{-s}$$

defines a "charge zeta-function". From (4.8),

$$\zeta(s) = (\kappa M)^{-2s} [\zeta_R(2s, \alpha) + \zeta_R(2s, 1-\alpha)], \quad (4.13)$$

where $\zeta_R(s, z)$ is the generalized Riemann zeta-function. This gives $\zeta(s=0) = \zeta(s=-1) = 0$. Thus, the regularization scheme defines $\sum_{n=-\infty}^{+\infty} v_n = 0$. From (4.11) it follows that the functional measure for this infinite number of fermions is invariant under a chiral transformation even though that for a single fermion is not. The axial anomaly therefore disappears which is consistent with the five-dimensional viewpoint. If the original space-time has a dimension higher than five, then it may be necessary to include unconventional contributions to v from the standpoint of the lower dimensional theory. (Theories with an infinite number of fermions have also been considered in a different context by Nicolai /26/.)

The fact that the anomalies may disappear if there are an infinite number of fermions in a theory which is reduced from one in an odd dimension means that the θ parameter is zero. This has implications for the CP problem.

5. - CONCLUSIONS

The main conclusion reached here is that in dimensional reduction it makes a great difference whether or not all of the modes are kept in the reduced theory when it is quantized. If the extra dimensions are taken to be real, but unobservable at low energies, then the correct procedure is to keep all of the modes. It is false to argue that only the modes of lowest mass are important. The procedure followed makes a drastic effect on the conclusions concerning renormalizability and the presence of anomalies. We have found, expectedly, that if all of the modes are kept the conclusions are the same regardless of whether the analysis is carried out in the higher dimensional space-time or using the reduced theory. However, if the reduced theory is used it may be necessary in general to include unconventional counterterms.

Since the unobserved extra dimensions are necessarily characterized by small length scales, it has been remarked by Unwin /21/ that the resulting Casimir vacuum energy density, due to the compactification, will be enormous. (For a massless d dimensional theory characterized by a length scale L , the vacuum energy density will be proportional to L^{-d} .) This provides another reason for taking the d dimensional theory to be supersymmetric: vanishing vacuum energy density can be achieved through mutual cancellation of boson and fermion contributions /27/. (However, there is always a potential problem with the cosmological constant even at the classical level, when the extra dimensions are not flat.)

Finally, when the gravitational field is itself quantized in these Kaluza-Klein theories, one must expand the metric itself about the ground state background using a suitable mode expansion. This will result in an infinite number of massive spin-2 fields in the reduced

theory (which as Salam and Strathdee have recently pointed out belong to infinite dimensional representations of some non-compact group). If the original theory is odd-dimensional, moreover, including these states in the quantum theory will yield a four-dimensional theory which is finite at odd-loop order. The crucial question remaining, of course, is whether supergravity would also take care of the even loops!

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