Ref.TH.3257-CERN
ERRATUM

ON EXCLUSIVE PROCESSES INVOLVING HEAVY-FLAVOURED MESONS

H.F. Jones

CERN -- Geneva

The conclusions of this paper are essentially correct, but for the wrong reasons. The scale at which the simple delta function ansatz for the heavy meson wave function is applicable is of the order of Λ rather than M . With this factorization scale the radiative corrections [Eq. (4)] constitute a double logarithmic suppression. If a higher factorization scale is chosen, in order to minimize the radiative corrections, there is again a suppression, this time due to the development of the wave function [the sign in Eq. (19), and consequently in the exponent of Eq. (22), is incorrect and should be reversed]. It should also be noted that the dominant contribution to the result of Ref. 5) comes from non-leading mass terms.

ERRATUM
24 May 1982

Ref.TH.3257-CERN

ON EXCLUSIVE PROCESSES INVOLVING HEAVY
FLAVOURED MESONS

H.F. Jones *)

CERN -- Geneva

ABSTRACT

Lowest order calculations of certain exclusive processes involving mesons made up of a heavy quark and a light antiquark can lead to large enhancements due to the peaking of the wave function near the limit of phase space. We discuss how higher-order corrections indicate that the wave function should be evaluated at a scale lower than the heavy meson mass, and show that the bigger the initial enhancement the more significant is the damping thereby induced.

*) Permanent address: Physics Department, Imperial College, London SW7.

A. A fundamental difficulty in the calculation of hard exclusive processes in the framework of perturbative QCD¹⁾ is our lack of knowledge of the hadron wave functions (distribution amplitudes). The asymptotic form is indeed determined, as $\varphi(x, Q^2) \rightarrow (\sqrt{3}/4)f_M(1-x^2)$ in the case of a pseudoscalar meson^{*}, but the approach to the limit is very slow, by a fractional power of $\ln(Q^2/\Lambda^2)$. In principle the wave function of a meson can be measured by its isolated production, e.g. in e^+e^- annihilation, as was first emphasized by Baier and Grozin²⁾. However, such an experimental determination still remains to be carried out.

Faced with this situation various authors have turned to the consideration of exclusive processes involving mesons containing heavy quarks, whose wave functions are simpler and better understood than those of the light mesons. In this vein Shifman and Vysotsky³⁾ have calculated the processes $H \rightarrow \gamma\psi$ and $\Upsilon^* \rightarrow \gamma\eta_c$ in a framework whereby the wave function of the charmonium state is taken as $\phi(x, M^2) \propto \delta(x)$ at the scale of its mass M . Its behaviour for larger scales is then completely determined by the Brodsky-Lepage integral equation, or by the standard anomalous dimensions in a Gegenbauer decomposition¹⁾. The physical picture giving rise to the above ansatz is of a bound state with negligible binding energy in which the heavy quark and antiquark each carry half the momentum of the meson as a whole.

In the case of mesons bearing a heavy flavour quantum number the corresponding picture is that the heavy quark and the light antiquark carry momentum fractions in proportion to their mass: $x_1 = M/(M+m)$, $x_2 = m/(M+m)$. The delta function form $\varphi(x, M^2) \propto \delta(x - (M-m)/(M+m))$ may be regarded as a convenient idealization of the harmonic oscillator wave function used, for example, by Chao⁴⁾ in the calculation of the charmonium decays $\psi \rightarrow K\bar{K}$, $D\bar{D}$. It is the flavour asymmetry of the wave function that allows these decays to take place and enhances the $D\bar{D}$ mode, in which the wave functions are peaked nearer to $x = 1$. The same feature occurs, to an even more marked degree, in the calculation of Horgan et al.⁵⁾, who evaluate the decay width for $Z^0 \rightarrow T\bar{T}$ using a wave function of this type. The extreme asymmetry of the wave function, that is the closeness of x to the kinematic limit, gives rise to a large enhancement, due to the fact that the hard diagram, which on dimensional grounds must be proportional to $1/Q^2$, in fact contains a factor $((1-x)(1-y))^{-1}$, where x and y are the relative momentum fractions for the two mesons, both assigned the value $(M-m)/(M+m)$ in the above approximation. Although for the range of masses for

^{*}) Here x is the scaling variable associated with the relative momentum of the quark and antiquark. Their individual momentum fractions are given by $x_1 = (1+x)/2$, $x_2 = (1-x)/2$.

the top quark used in Ref. 5) it does not occur, it is possible to imagine a situation in which the enhancement would be so large as to violate unitarity by giving a width for the exclusive decay $Z^0 \rightarrow T\bar{T}$ in excess of the inclusive decay $Z^0 \rightarrow Q\bar{Q}$. It is to the resolution of this apparent paradox that the present note is addressed.

B. Let us in fact consider the electromagnetic form factor of the pseudoscalar T^+ meson for large timelike Q^2 . This is to all intents and purposes the same problem, simplified slightly by having a pure vector coupling to the gauge boson. In the general framework established by Brodsky and Lepage¹⁾ the form factor is given by

$$F_T(Q^2) = \int_{-1}^1 dx dy \varphi(x, Q^2) T_H(x, y, Q^2) \varphi(y, Q^2) \quad (1)$$

where the hard diagrams, illustrated in the figure, sum to

$$T_H(x, y, Q^2) = \frac{16\pi\alpha_s C_F}{Q^2} \left[\frac{e_b}{(1-x)(1-y)} - \frac{e_d}{(1+x)(1+y)} \right] \quad (2)$$

If one ignores the (logarithmic) development of the wave function from M^2 to Q^2 , taking

$$\varphi(x, Q^2) = \frac{1}{\sqrt{3}} f_T \delta(x - (m - m)/(m + m)), \quad (3)$$

one obtains the large enhancement referred to above. In fact $Q^2(1-x)(1-y)/4$ is precisely the virtuality of the exchanged gluon in Figs. 1a, 1b, where the photon couples to the heavy quark, so that the enhancement arises because the gluon is actually much softer than the nominal Q^2 , though for perturbative QCD to be applicable at all we must still have $Q^2(1-x)^2 \gg \Lambda^2$.

This naturally leads one to wonder about the importance of multiple "soft" gluon exchanges which might possibly sum to provide a Sudakov-type damping of the effect. Let us therefore look at the next-to-leading order corrections to the hard amplitude, which have been calculated^{*)} by Dittes and Radyushkin⁶⁾ and by Field et al.⁷⁾. Significantly the second order amplitude contains double

*) The calculations were actually performed for the pion form factor, but may be adapted to the present case by supplying the appropriate quark charges to the separate contributions related by $x \leftrightarrow -x$, $y \leftrightarrow -y$.

logarithms of $\bar{x} \equiv 1-x$, $\bar{y} \equiv 1-y$ which have been isolated by Dittes and Radyushkin in the form^{*)}

$$T_H^1(x, y, Q^2) = T_H^0(x, y, Q^2) \left[1 + \frac{\alpha_s(\mu_R)}{4\pi} \left(\ln^2 \left(\frac{\bar{x}\bar{y}Q^2}{\mu^2} \right) - \ln^2 \left(\frac{Q^2}{\mu^2} \right) \right) + \dots \right] \quad (4)$$

Here μ_R is the ultra-violet subtraction scale, while μ is the infra-red subtraction scale, or the scale at which the factorization between the hard subprocess and the wave function is performed. The $(\log)^2$ terms provide a significant constraint on the choice of μ : if chosen inappropriately there will be large corrections which will very likely exponentiate⁶⁾.

The choice $\mu^2 = M^2$, which one would like to make in order to exploit the simple form of the lowest order wave function, fixes $\bar{x}$ and $\bar{y}$ at $\bar{x} = \bar{y} \approx 2m/M$ and gives a correction $(\alpha_s/\pi) \ln(m/M) \ln(mQ^2/M^2)$, which constitutes a double logarithmic enhancement when m/M is sufficiently small, as is the case for the masses considered in Ref. 5). The choice $\mu^2 = Q^2$ would equally give a large enhancement, provided that the logarithmic development of the wave function did not significantly change $\langle \bar{x} \rangle$, $\langle \bar{y} \rangle$ from the value $\delta \equiv 2m/M$. What these corrections are clearly telling us is that the appropriate scale is one much less than Q^2 , in agreement with the observation made above that the propagators of the hard diagram are in fact much softer than the nominal Q^2 . The correction can actually be made to vanish by the choice⁶⁾ $\mu^2 = (\langle \bar{x} \rangle \langle \bar{y} \rangle)^{1/2} Q^2$. Note, however, that this is in general a complicated iterative procedure, since the averages $\langle \bar{x} \rangle$, $\langle \bar{y} \rangle$, defined in Ref. 6), depend on the wave function, and hence on μ . Nonetheless the fundamental point is clear: when δ becomes small, giving rise to a $1/\delta$ enhancement in the naïve lowest order calculation, the "soft" gluon corrections will be very significant unless the factorization scale is also taken small, of the order of δQ^2 .

It might be thought that the influence of the choice of factorization scale on the form of the wave function was only weak, since its development is logarithmic. However, as we now show, the development is anomalously rapid precisely in the case when the initial wave function is peaked very close to the phase space limit $x = 1$.

*) There appears to be a sign error in Ref. 7), namely that of $H(u,v)$ in Eq. (3.13f). If this is corrected the $(\log)^2$ terms agree.

C. Since it is rather difficult to deal with a distribution let us in fact consider the integral

$$f(x, \mu^2) \equiv \int_{-1}^1 \frac{dx'}{1-x'} \varphi(x', \mu^2), \quad (5)$$

which is the integral that actually occurs in Eq. (1). The first argument of f refers to the initial wave function, which for simplicity we normalize to one:

$$\varphi(x', \mu^2) = \delta(x' - x) \quad (6)$$

with $x = 1 - \delta$. The development of the wave function away from this form is governed by the Brodsky-Lepage kernel, which can be diagonalized by an expansion in terms of Gegenbauer polynomials $C(x') \equiv C^{3/2}(x')$:

$$\varphi(x', \mu^2) = (1-x'^2) \sum_{n=0}^{\infty} a_n C_n(x') L^{-\gamma_n} \quad (7)$$

where $L = \alpha_s(M^2)/\alpha_s(\mu^2) = \ln(\mu^2/\Lambda^2)/\ln(M^2/\Lambda^2)$ and the γ_n are the standard non-singlet anomalous dimensions

$$\gamma_n = \frac{C_F}{\beta} \left[-3 - \frac{2}{(n+1)(n+2)} + 4 \sum_{j=1}^{n+1} \frac{1}{j} \right] \quad (8)$$

The coefficients a_n are just the Gegenbauer coefficients of the initial delta function and are given by

$$a_n = C_n(x)/\eta_n \quad (9)$$

where $\eta_n = 2(n+1)(n+2)/(2n+3)$ is the normalization constant occurring in the orthogonality relation

$$\int_{-1}^1 dx' (1-x'^2) C_m(x') C_n(x') = \eta_n \delta_{mn} \quad (10)$$

Inserting the expansion (7) into Eq. (5) gives

$$f(x, \mu^2) = \sum_{n=0}^{\infty} f_n C_n(x) L^{-\gamma_n} \quad (11)$$

with $f_n = 2/\eta_n = (2n+3)/((n+1)(n+2))$.

We are interested in the development of $f(x, \mu^2)$ away from its initial value $f(x, M^2) = 1/\bar{x}$ in the limit of small $\bar{x}$. The summation of the full series (11) proves somewhat intractable, even in that limit. However, a strong indication of the nature of the development can be obtained from the first two terms of the perturbation expansion in $\ell \equiv \ln(\mu^2/M^2)/\ln(M^2/\Lambda^2) = \beta(\alpha_s(M^2)/4\pi) \ln(\mu^2/M^2)$

$$f(x, \mu^2) = f^{(0)}(x) - \ell f^{(1)}(x) + \frac{1}{2} \ell^2 f^{(2)}(x) + \dots \quad (12)$$

with

$$f^{(0)}(x) = \sum_{n=0}^{\infty} f_n C_n(x) = 1/\bar{x} \quad (13)$$

$$f^{(1)}(x) = \sum_{n=0}^{\infty} f_n C_n(x) \gamma_n \quad (14)$$

$$f^{(2)}(x) = \sum_{n=0}^{\infty} f_n C_n(x) \gamma_n (\gamma_n + 1) \quad (15)$$

The first and second derivatives (14) and (15) can be evaluated by a method akin to that used by Shifman and Vysotsky³⁾ for the case $x = 0$. The method applies provided one knows a more general series than (13), namely

$$g(x, z) = \sum_{n=0}^{\infty} f_n C_n(x) z^n \quad (16)$$

After some manipulations on one of the relations⁸⁾ defining the $C_n(x)$ by means of a generating function, this can be evaluated as

$$g(x, z) = \frac{1}{(1-x^2)z^2} \left[(1+xz) - \frac{1-z^2}{R} \right] \quad (17)$$

where^{*}) $R(x, z) = (z^2 - 2zx + 1)^{1/2}$. By representing

$$\hat{\gamma}_n \equiv \frac{4C_F}{\beta} \sum_{j=1}^{n+1} \frac{1}{j}$$

as

$$(4C_F/\beta) \int_0^1 dt (1-t^{n+1})/(1-t)$$

^{*}) The geometrical interpretation of R as the third side of a triangle in which the other two have lengths $z, 1$ and contain an angle $\theta = \cos^{-1}x$ proves very useful in the evaluation of subsequent integrals.

one can then write

$$\hat{f}^{(1)}(x) \equiv \sum_{n=0}^{\infty} f_n C_n(x) \hat{\gamma}_n$$

in the form

$$\hat{f}^{(1)}(x) = \frac{4C_F}{\beta} \int_0^1 \frac{g(x, z) - t g(x, zt)}{1-t} dt \Big|_{z=1} \quad (18)$$

which can be integrated exactly to give

$$\hat{f}^{(1)}(x) = \frac{4C_F}{\beta} \frac{\ln(\frac{1}{2}\bar{x})}{1-x^2} \quad (19)$$

This is the first hint of the accelerated development referred to above: for small $\bar{x}$, $\hat{f}^{(1)}(x)$ is logarithmically enhanced, viz. $\hat{f}^{(1)}(x) \sim (2C_F/\beta)(\ln \bar{x})/\bar{x}$. The first two terms in the expression for γ_n do not give such an enhancement, so the limiting expression for the full $f^{(1)}(x)$ is the same.

The result can perhaps be understood as follows. The series (13) only just converges by virtue of the alternating signs of the $C_n(x)$ for large n . The introduction of a factor $\hat{\gamma}_n$, behaving like $\ln n$, makes the cancellation more difficult, resulting in an extra factor of $\ln \bar{x}$.

The calculation of $f^{(2)}(x)$ involves the double integral $\int_0^1 ds dt [g(x, 1) - tg(x, t) - sg(x, s) + stg(x, st)]/(\bar{s}\bar{t})$, that is $\int_0^1 dz (G(x, 1) - G(x, z))/\bar{z}$, where $G(x, z) \equiv \int_0^1 dt (g(x, z) - tg(x, zt))/\bar{t}$ is the integral occurring in Eq. (18), evaluated for general $z \leq 1$. Keeping only the dominant terms as $\bar{x} \rightarrow 0$ the result is

$$f^{(2)}(x) \sim \left(\frac{2C_F}{\beta}\right)^2 \frac{\ln^2 \bar{x}}{\bar{x}} \quad (20)$$

The method becomes rapidly more cumbersome in higher orders and a calculation of $f^{(3)}(x)$ is not practicable. However, the results for $f^{(1)}$ and $f^{(2)}$ are extremely suggestive and are consistent with the conjecture that

$$f^{(n)}(x) \sim \left(\frac{2C_F}{\beta}\right)^n \frac{(\ln \bar{x})^n}{\bar{x}} \quad (21)$$

If this is indeed the case then the leading $\ln \bar{x}$ behaviour exponentiates to give

$$f(x, \mu^2) \sim \frac{1}{\bar{x}} \exp \left(-\frac{\alpha_s(M^2)}{2\pi} C_F \ln \bar{x} \ln \frac{\mu^2}{M^2} \right) \quad (22)$$

Recall that the higher order corrections to the coefficient function indicated that μ^2 should be taken as $\mu^2 = \bar{x}Q^2$. Thus the singular behaviour of the tree diagram as $\bar{x} \rightarrow 0$ is rendered harmless by a Sudakov type of suppression when the wave function is evaluated at the lower mass scale.

D. Given the validity of the conjectured exponentiation we have provided a resolution of the paradox posed at the end of Section A. The answer is that the wave function should be evaluated at a mass scale different from M , where the delta function approximation applies, and that the change this makes is very significant precisely when the naïve calculation becomes dangerously large. There seems to be a kind of uncertainty principle operating similar to that for the spread of a quantum mechanical wave packet, in that the closer the initial wave function is peaked to $x = 1$ the more rapidly it departs from that configuration when the scale is changed.

For the range of masses used in Ref. 7), where indeed the unitarity bound was not exceeded, the effect is not exceptionally large. Thus with $M = 30 \text{ GeV}/c^2$ and taking $\Lambda = 300 \text{ MeV}/c^2$, the exponent in Eq. (22) is approximately -0.5 , which in fact should be doubled since there are two wave functions involved, giving a modest renormalization. However, the effect will be there and will be important in any calculation in which a highly asymmetric wave function apparently gives anomalously large results.

ACKNOWLEDGEMENT

I should like to thank Dr. P.V. Landshoff for initially bringing this problem to my attention, and for subsequent correspondence.

REFERENCES

- 1) G.P. Lepage and S.J. Brodsky, Phys. Rev. D22 (1980) 2157.
- 2) V.N. Baier and A.G. Grozin, Phys. Lett. 96B (1980) 181.
- 3) M.A. Shifman and M.I. Vysotsky, Moscow preprint ITEP-147 (1980).
- 4) S.-C. Chao, Columbia preprint, CU-TP-199 (1981).
- 5) R.R. Horgan, P.V. Landshoff and D.M. Scott, Cambridge preprint DAMTP 81/24 (1981).
- 6) F.-M. Dittes and A.V. Radyushkin, Dubna preprint E2-80-688 (1980); Yad. Fiz. 34 (1981) 529.
- 7) R.D. Field et al., Nucl. Phys. B186 (1981) 429.
- 8) W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer (Berlin 1966)

FIGURE CAPTION

The hard scattering diagrams for the electromagnetic form factor of the T^+ meson at large Q^2 . In the diagrams explicitly shown the virtual photon couples to the t quark, which has momentum fractions x_1, y_1 .

(a)

(b)

+ (1 ↔ 2)

Fig. 1