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PRINCIPAL ORBITS AND THE YANG-MILLS-HIGGS MODEL

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A B S T R A C T

We generalize, to an arbitrary representation of a compact Lie Group, the definition of topological conserved currents in the Yang-Mills-Higgs model. Related to this we discuss the computation of the second de Rham cohomology of an orbit. A detailed analysis is given in the case of principal orbits and finally we apply this analysis to Lie groups of physical interest.

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1. - INTRODUCTION

Our aim in this paper is to extend, to arbitrary representations of a Lie group G , the analysis which was carried out in Ref. 1) for the Yang-Mills-Higgs (YMH) model with the Higgs field valued in the Lie algebra of G and the action of G given by the adjoint representation. That is, we first show how conserved currents (closed 2-forms) may be defined by pulling back the generators of the second de Rham cohomology of certain G orbits. These conserved currents may be considered as generalizing the 't Hooft field strength ²⁾ and they embody topological information on the YMH field configuration (i.e., this information remains unchanged by small deformations of the fields). In particular, for the adjoint $SU(2)$ YMH model the currents played an essential rôle in establishing properties of the magnetic charge distribution for axially symmetric field configurations ³⁾. As will be seen later on, it is natural to look in detail at principal G orbits (i.e., orbits of maximal dimension). This is mainly due to the fact that the union of such orbits form an open dense set in the carrier space of the representation. Also it turns out that it is a lot less difficult to establish properties for principal orbits than for a general orbit in the case of an arbitrary representation. The second part of the paper will involve these principal orbits.

To outline the contents of the paper precisely, we first explain the notation we will use ⁴⁾. Let $M = \mathbb{R}^3$ and $P = M \times G$ be a trivial principal fibre bundle over M with G being a simple compact connected Lie group. The bundle projection is $\pi : P \ni (x, g) \rightarrow x \in M$ and the group action on P is $g \cdot (x, g') = (x, g'g)$, for all $(x, g') \in P, g \in G$. Let Δ be a linear orthogonal representation of G (not necessarily irreducible) on the real finite dimensional vector space E ^{*}), the scalar product on E being denoted by $(\cdot, \cdot)$, and let $\mathcal{E} = M \times E$ be the vector bundle associated with P with the group action being defined by Δ . We let $\mathfrak{g}$ be the Lie algebra of G and by Ad we mean the adjoint representation of G on $\mathfrak{g}$. The representation of $\mathfrak{g}$ on E induced by Δ we denote by δ and the adjoint representation of $\mathfrak{g}$ we denote by ad . For any $x \in E$, G_x ($\langle G \rangle$) represents the stabilizer of x , $G(x)$ the orbit through x and $S(x)$ the stratum containing x ⁵⁾ $[G(x) \subset S(x) \subset E]$. For the tangent plane to $G(x)$ at x , the tangent plane to $S(x)$ at x and the normal plane to $G(x)$ at x we write : $\mu(x) = x + \mu'(x)$, $\nu(x) = x + \nu'(x)$ and $\kappa(x) = x + \kappa'(x)$, respectively, where $\mu'(x)$, $\nu'(x)$ and $\kappa'(x)$ are vector subspaces of E such that $\mu'(x) \cong T_x(G(x))$, $\nu'(x) \cong T_x(S(x))$ and $\kappa'(x) = \mu'(x)^\perp$.

^{*}) Every finite dimensional real linear representation of a compact Lie group is equivalent to an orthogonal representation ⁵⁾.

The various strata of E may be partially ordered via the partial ordering on the conjugacy classes of subgroups of G . There exists a minimal stratum, S_0 , with respect to this ordering and S_0 is open and dense in E . Any orbit in S_0 is an orbit of maximal dimension and is called a principal orbit and the stabilizer of any point of S_0 is a stabilizer of minimal dimension and called a principal stabilizer. In the YMH model we are given a connection on P (the gauge fields) and a tensorial 0-form ϕ on E (the Higgs field).

In Section 2 we discuss how, for an arbitrary representation Δ of G , the Higgs field can be used to pull back the second de Rham cohomology of a certain orbit of G in E allowing us to define conserved currents associated with the topology of the YMH field configuration. We also discuss how these currents associate integral charges with almost any closed 2-surface. In Section 3, so as to allow it to be computed easily, we relate the second de Rham cohomology space of a G orbit to the centre of the Lie algebra of the stabilizer of a point on the orbit. In Section 4, by specializing to principal orbits, we establish a straightforward necessary condition, on the representation δ , for the Lie algebra of the principal stabilizer to have a non-zero centre. In Section 5, we apply the analysis of Sections 3 and 4 to representations and groups of physical interest. Finally, Section 6 contains a summary of results.

2. - THE DEFORMATION RETRACTION AND PULL BACK OF $H_{DR}(G(x))$

For any $x \in E$ it is known ⁵⁾ that there exists a neighbourhood V_x of x such that $\kappa(x) \cap V_x$ cuts $G(x)$ at x only and that $U_x = G(\kappa(x) \cap V_x)$ ($= \bigcup_{y \in \kappa(x) \cap V_x} G(y)$) is open. The map $r : U_x \rightarrow G(x)$ defined by sending every element of $\kappa(x) \cap V_x$ to x and satisfying $r(\Delta(g)y) = \Delta(g)r$, for all $y \in \kappa(x) \cap V_x$ and $g \in G$ is a deformation retraction and U_x is a tubular neighbourhood of $G(x)$ with fibre map r . Moreover, if A is an open set in E such that it contains the orbit of every one of its elements and if there exists a point x in E and a set U_x of the form specified above with $U_x \supset A$, then A is also a tubular neighbourhood of $G(x)$ with fibre map r . It may be remarked that any A above is a tubular neighbourhood of $G(0) = \{0\}$. However, the choice $x = 0$ is a trivial one and only cases where $G(x) \neq \{0\}$ will be of interest. In the case where $\Delta = \text{Ad}$ and $E = \mathfrak{g}$ the construction of these tubular neighbourhoods has been given in Ref. 1). In general, the construction of these tubular neighbourhoods will be highly dependent on the representation Δ . However, for an arbitrary representation almost all A 's are contained in

the generic structure because it is open and dense in E and hence any such A is a tubular neighbourhood of any of its orbits (since we may choose $V_x = U_x = S_0$).

Letting $\sigma_0 : M \ni y \rightarrow (y, 1) \in P$ define a section of P we can define the map $\phi_0 = \phi \circ \sigma_0 : M \rightarrow E$. Let $\tilde{M}$ be a submanifold of M of the same dimension as M such that there exists an open set A in E containing the orbit of all of its elements which is a tubular neighbourhood of $G(x)$, for some $x \in A$, with fibre map r and such that $\phi_0(\tilde{M}) \subset A^*$. By composing the restriction of ϕ_0 to $\tilde{M}$, $\phi_0|_{\tilde{M}}$, with r we obtain the map $\tilde{\phi} = r \circ \phi_0|_{\tilde{M}} : \tilde{M} \rightarrow G(x)$. The induced mapping on cohomology $\tilde{\phi}_* : H_{DR}^2(G(x)) \rightarrow H_{DR}^2(\tilde{M})$ allows us to pull back the generators of $H_{DR}^2(G(x))$ to define closed 2-forms (conserved currents) on $\tilde{M}$.

It is desirable to know for which orbits, $G(x)$, $H_{DR}^2(G(x))$ is non-zero and in the next section we will show how it is possible to compute $H_{DR}^2(G(x))$ in an easy way. Before ending this section though, let us briefly discuss how the above conserved currents may be used to associate integral charged with a closed 2-cycle in $\tilde{M}$ (i.e., a closed simply connected oriented two-dimensional manifold in $\tilde{M}$) which gives the topological charge inside it. Let N be a 2-cycle in $\tilde{M}$ [$\tilde{M}$, A , $G(x)$ and $\tilde{\phi}$ as above] then $[N]$ ^{**)} generates $H_2(N, \mathbb{Z}) \cong \mathbb{Z}$. The restriction of the map $\tilde{\phi}$ to N , $\tilde{\phi}|_N : N \rightarrow G(x)$ induces the following mapping on homology $\tilde{\phi}|_N^* : H_2(N, \mathbb{Z}) \rightarrow H_2(G(x), \mathbb{Z})$. By letting $\{[\gamma_i] | i=1, \dots, \alpha\}$ be an orthonormal basis for $H_{DR}^2(G(x))$ ^{***)} we may associate a set of integers to N using ϕ by writing

$$n_i(N) = \int_{\tilde{\phi}|_N^* N} \gamma_i, \quad i = 1, \dots, \alpha, \quad (2.1)$$

where $\alpha = \dim H_{DR}^2(G(x))$. Moreover, the integers $n_i(N)$, $i = 1, \dots, \alpha$ are unchanged by deformations of N and ϕ , N' , ϕ' , as long as $\tilde{\phi}'|_{N'}, [N'] = \tilde{\phi}|_N [N]$. In the particular case where we deal with finite energy configurations of the YMH model with the usual sort of Lagrangian being employed, i.e., with a typical spontaneous symmetry breaking Higgs potential, and where $N = S_R^2 = \{x \in M | |x| = R\}$, for sufficiently large R , the integers $\{n_i(S_R^2)\}$ become part of the boundary condition ⁶⁾.

^{*)} $\tilde{M}$ exists if ϕ is sufficiently well behaved, e.g., $\tilde{M} \subset \phi_0^{-1}(S_0)$.

^{**)} $[\cdot]$ denotes homology or cohomology class as appropriate.

^{***)} The inner product being defined by integration, i.e., $H_2(\cdot) \times H^2(\cdot)$
 $[[C], [\gamma]] \rightarrow \int_C \gamma \in \mathbb{R}$.

3. - COMPUTATION OF THE SECOND DE RHAM COHOMOLOGY OF AN ORBIT

Since $G(x)$ may be identified with the coset space G/G_x it is of interest to be able to compute $H_{DR}^2(G/H)$ in a simple way for any subgroup H of G (G is as specified in Section 1). In this section we show that $H_{DR}^2(G/H)$ can be determined from algebraic properties of G and H . In fact $H_{DR}^2(G/H)$ is related to the centre of the Lie algebra of H .

We are able to proceed by elementary methods ^{*)} because of the following result ⁸⁾

$$H_{DR}^k(G/H) \cong \frac{\ker(d: \Omega^k(\mathfrak{H}^\perp)^H \rightarrow \Omega^{k+1}(\mathfrak{H}^\perp)^H)}{d(\Omega^{k-1}(\mathfrak{H}^\perp)^H)} \quad (3.1)$$

Here $\mathfrak{H}$ denotes the Lie algebra of H and $\mathfrak{H}^\perp$ denotes the orthogonal complement of $\mathfrak{H}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, $\Omega^k(V)$ denotes the space of multilinear completely antisymmetric real valued maps on a vector space V , $\Omega^k(\mathfrak{H}^\perp)^H = \{\alpha \in \Omega^k(\mathfrak{H}^\perp) \mid \text{Ad}(h)_* \alpha = \alpha, \text{ for all } h \in H\}$ and $d: \Omega^k(\mathfrak{H}^\perp)^H \rightarrow \Omega^{k+1}(\mathfrak{H}^\perp)^H$ is defined for any $\alpha \in \Omega^k(\mathfrak{H}^\perp)^H$ by

$$\begin{aligned} d\alpha(X_1, \dots, X_{k+1}) \\ = \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j]_{\mathfrak{H}^\perp}, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}), \end{aligned} \quad (3.2)$$

for any $X_1, \dots, X_{k+1} \in \mathfrak{H}^\perp$. Under the decomposition $\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{H}^\perp$, $(\cdot)_{\mathfrak{H}^\perp}$ denotes the $\mathfrak{H}^\perp$ component.

Let $C(\mathfrak{H})$ denote the centre of the Lie algebra $\mathfrak{H}$ ($C(\mathfrak{H}) = \{X \in \mathfrak{H} \mid [X, Y] = 0, \text{ for all } Y \in \mathfrak{H}\}$). Then to apply Eq. (3.1) we need the following proposition.

Proposition 3.1

- (i) $\alpha \in d(\Omega^1(\mathfrak{H}^\perp)^H) \Leftrightarrow$ there exists $\gamma \in \mathfrak{H}^\perp$, which satisfies $\text{Ad}(h)\gamma = \gamma$ for all $h \in H$, such that $\alpha(X_1, X_2) = \langle \gamma, [X_1, X_2] \rangle$ for all $X_1, X_2 \in \mathfrak{H}^\perp$.
- (ii) $\alpha \in \ker(d: \Omega^2(\mathfrak{H}^\perp)^H \rightarrow \Omega^3(\mathfrak{H}^\perp)^H) \Leftrightarrow$ there exists $\gamma \in \mathfrak{H}^\perp \oplus C(\mathfrak{H}) \subset \mathfrak{g}$, which satisfies $\text{Ad}(h)\gamma = \gamma$ for all $h \in H$, such that $\alpha(X_1, X_2) = \langle \gamma, [X_1, X_2] \rangle$, for all $X_1, X_2 \in \mathfrak{H}^\perp$.

^{*)} The result may also be derived by applying the spectral sequence of a fibration and of a covering ⁷⁾.

Proof

For (i) and (ii) ($\Leftarrow$) is easy.

($\Rightarrow$) (i) Since $\alpha \in d(\Omega^1(\mathcal{H}^\perp)^H)$ it follows that $\alpha \in d(\Omega^1(\mathcal{G}))$. Thus $\alpha = d\omega$, for some $\omega \in \Omega^1(\mathcal{G})$. Since there exists $\gamma \in \mathcal{G}$ such that $\omega(X) = \langle \gamma, X \rangle$ for all $X \in \mathcal{G}$, it follows that $\alpha(X_1, X_2) = d\omega(X_1, X_2) = \omega([X_1, X_2]) = \langle \gamma, [X_1, X_2] \rangle$. By requiring $\omega \in \Omega^1(\mathcal{H}^\perp)^H$ we get the stated conditions on γ .

($\Rightarrow$) (ii) Since $\alpha \in \ker(d: \Omega^2(\mathcal{H}^\perp)^H \rightarrow \Omega^3(\mathcal{H}^\perp)^H)$ it follows that $\alpha \in \Omega^2(\mathcal{G})$ and $d\alpha = 0$. Hence there exists $\gamma \in \mathcal{G}$ such that $\alpha(X_1, X_2) = \langle \gamma, [X_1, X_2] \rangle$ for all $X_1, X_2 \in \mathcal{G}$. Writing $\gamma = \gamma_{\mathcal{H}} \oplus \gamma_{\mathcal{H}^\perp}$ and requiring $\alpha \in \Omega^2(\mathcal{H}^\perp)^H$ means $\text{Ad}(h)\gamma_{\mathcal{H}} = \gamma_{\mathcal{H}}$ and $\text{Ad}(h)\gamma_{\mathcal{H}^\perp} = \gamma_{\mathcal{H}^\perp}$ for all $h \in H$ which implies $\gamma_{\mathcal{H}} \in C(\mathcal{H})$.

Letting H_0 be the subgroup of H which is path connected to the identity and is of the same dimension as H , then $H_0 \triangleleft H$ (i.e., H_0 is a normal subgroup of H) and the factor group H/H_0 is a finite group which acts naturally on $C(\mathcal{H})$ by Ad . Since $C(\mathcal{H})^{H/H_0} = C(\mathcal{H})^H = \{X \in \mathcal{H} \mid \text{Ad}(h)X = X, \text{ for all } h \in H\}$ it follows from Eq. (3.1) and Proposition 3.1 that

Theorem 3.2

$$H_{\text{DR}}^2(\mathcal{G}/H) \cong C(\mathcal{H})^{H/H_0}.$$

It follows from Theorem 3.2 that a necessary condition for $H_{\text{DR}}^2(\mathcal{G}/H) \neq 0$ and hence that there exist corresponding conserved currents (if H is the appropriate stabilizer) is $C(\mathcal{H}) \neq 0$. It would be desirable to know for all representations Δ what stabilizers H are such that $C(\mathcal{H}) \neq 0$. This though would be too large a project to undertake from the general standpoint and we shall not do so. However, when H is a principal stabilizer, i.e., a stabilizer of a point in the generic stratum it is possible, quite generally, to decide for which representations $C(\mathcal{H}) \neq 0$, and in the next section we will address this question. Another reason for considering principal orbits and principal stabilizers in some detail is that, for reasonably behaved Higgs fields ϕ , $\phi_0^{-1}(S_0)$ is dense in M . Thus there always exists a manifold $\tilde{M} \subset \phi_0^{-1}(S_0)$, as defined above, so that we can pull back the second de Rham cohomology of a principal orbit onto $\tilde{M}$.

4. - ON PRINCIPAL ORBITS

The main result of this section, stated in Theorem 4.2, is a strong necessary condition on a representation δ of $\mathfrak{g}$ such that the Lie algebra of a principal stabilizer, $\mathfrak{h}$, has a non-zero centre $C(\mathfrak{h})$.

Since Δ is a real orthogonal representation of G it follows that $\delta(Y)$ is skewsymmetric for all $Y \in \mathfrak{g}$, i.e., $(x, \delta(Y)y) = -(\delta(Y)x, y)$ for all $x, y \in E$, and that $\delta(Y)^2$ is a linear self-adjoint transformation for all $Y \in \mathfrak{g}$. Noting that $\mathfrak{g}$ is a simple Lie algebra and letting tr denote the operation of taking the sum of all the eigenvalues of a self-adjoint linear transformation, we have that

$$d_\delta = \text{tr } \delta(X)^2 / \text{tr } \text{ad}(X)^2 > 0, \quad (4.1)$$

called the index of the representation δ of $\mathfrak{g}$, and defined for any non-zero $X \in \mathfrak{g}$, is independent of X (9), *).

For any subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and for all $X \in \mathfrak{h}$ we can block diagonalize $\text{ad}(X)$, i.e.,

$$\text{ad}(X) = \text{ad}_{\mathfrak{h}}(X) \oplus \text{ad}_{\mathfrak{g}}^{\perp}(X), \quad (4.2)$$

corresponding to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Here $\text{ad}_{\mathfrak{h}}$ denotes the adjoint representation of $\mathfrak{h}$. If $\mathfrak{h}$ is the Lie algebra of the stabilizer of a point $x \in E$ then E may be decomposed as follows

$$E = \mu'(x) \oplus (\nu'(x) \cap \kappa'(x)) \oplus \nu'(x)^{\perp}, \quad (4.3)$$

where $\mu'(x)$, $\nu'(x)$ and $\kappa'(x)$ are the vector spaces defined in Section 1. With respect to this decomposition, $\delta(x)$ for each $X \in \mathfrak{h}$ may be written in block diagonal form

$$\delta(X) = \delta_{\mu'(x)}(X) \oplus 0 \oplus \delta_{\nu'(x)^{\perp}}(X), \quad (4.4)$$

where $\delta_{\mu'(x)}(X)y = \delta(X)y$ for all $y \in \mu'(x)$, $\delta_{\nu'(x)^{\perp}}(X)y = \delta(X)y$ for all $y \in \nu'(x)^{\perp}$ and $\delta(X)y = 0$ for all $y \in \nu'(x) \cap \kappa'(x)$.

*) In fact d_δ is a rational number (9).

The linear mapping $\mathcal{H}^\perp \ni X \rightarrow \delta(X)x \in \mu'(x)$ is a vector space isomorphism ($\mathcal{H}$ being the Lie algebra of G_x). Also, since for any $X \in \mathcal{H}$, $\delta_{\mu'(x)}(X)^2$ and $b_{\mathfrak{g}}(X)^2$ are self-adjoint, it follows that $\delta_{\mu'(x)}(X)^2 \delta(Y)x = \lambda \delta(Y)x$ iff $\text{ad}(X)^2 Y = \lambda Y$, where $Y \in \mathcal{H}^\perp$. Hence we have the following result.

Proposition 4.1

For any $X \in \mathcal{H}$, the Lie algebra of the stabilizer of $x \in E$,
 $\text{tr} \delta_{\mu'(x)}(X)^2 = \text{tr} b_{\mathfrak{g}}(X)^2$.

Noting that $\nu'(x)^\perp = 0$ if x is in the generic stratum of E , we may write

$$d_\delta = \text{tr} b_{\mathfrak{g}}(Y)^2 / \{ \text{tr} b_{\mathfrak{g}}(Y)^2 + \text{tr} \text{ad}_{\mathcal{H}}(Y)^2 \}, \quad (4.5)$$

for any non-zero $Y \in \mathcal{H}$ if $\mathcal{H}$ is the Lie algebra of the principal stabilizer G_x (assuming $\mathcal{H} \neq 0$).

Theorem 4.2

If the Lie algebra, $\mathcal{H}$, of a principal stabilizer of the representation δ of $\mathfrak{g}$ on E is non-zero, then $d_\delta \leq 1$ and $d_\delta < 1, = 1$ respectively implies that $\mathcal{H}$ is non-Abelian, or Abelian.

Proof

If $\mathcal{H}$ is non-zero then from Eq. (4.5) $d_\delta \leq 1$. If now $d_\delta < 1$, then from Eq. (4.5) $\text{tr} \text{ad}_{\mathcal{H}}(Y)^2$ is non-zero for any $Y \in \mathcal{H}$, $Y \neq 0$. Thus $[X, Y] \neq 0$ for all $X \in \mathcal{H}$. Finally, if $d_\delta = 1$ then $\text{tr} \text{ad}_{\mathcal{H}}(Y)^2 = 0$ for all $Y \in \mathcal{H}$, in which case $[X, Y] = 0$ for all $X, Y \in \mathcal{H}$.

It follows from this theorem that

Corollary 4.3

Let $\mathcal{H}$ be the Lie algebra of a principal stabilizer of the representation δ of $\mathfrak{g}$ on E . If $d_\delta = 1$ then $C(\mathcal{H}) = \mathcal{H}$ and if $d_\delta \neq 1$ then $C(\mathcal{H}) = 0$.

Finally in this section let us make some remarks on the above analysis. If δ is written as a sum of irreducible representations δ_i , i.e., $\delta = \bigoplus_i \delta_i$ and if d_{δ_i} denotes the index of δ_i then $d_\delta = \sum_i d_{\delta_i}$. Thus to know the index

of any finite dimensional representation of a simple Lie algebra we need only know the index for all its finite dimensional irreducible representations. Moreover, if the Lie algebra is semisimple then we may apply the above analysis to each of its simple components. In the case of the adjoint representation $d_{ad} = 1$ and the Lie algebra of a principal stabilizer (which is equal to its centre) gives a Cartan subalgebra of $\mathfrak{g}$. As the adjoint representation was treated in Ref. 1) we will only consider representations other than the adjoint representation here. Lastly, we point out that the index d_δ , given by Eq.(4.1), is defined for a real representation δ . It is possible that δ may be equivalent to a complex representation δ' , say, in which case a similar expression to Eq. (4.1) can 9) define a different index d_δ , which is related to the former one by $d_\delta = 2d_{\delta'}$.

5. - APPLICATIONS

Because of the strength of the necessary condition in Corollary 4.3, and because of the comments following the Corollary, it is reasonable to try, in connection with Theorem 3.2, to determine all representations ^{*)} for which the Lie algebra of a principal stabilizer, $\mathfrak{H}$, has non-zero centre $C(\mathfrak{H})$. We do this now for all groups G of interest in GUT models. In addition, we give $H, \mathfrak{H}$, and $C(\mathfrak{H})^{H/H_0}$.

First we must determine for what real orthogonal irreducible representations ^{*)} Δ , $d_\delta \leq 1$. With the help of Ref. 9) ^{**)} we give this list of representations in Table I. We use Lie algebra terminology. The basic representations ρ_i correspond to a system of primitive roots ¹⁰⁾. (Note that the representations are real irreducible.) Since the index of any real representation is equal to the sum of the indices of all its irreducible factors we can, from Table I, determine all real orthogonal representations ^{*)}, of the groups involved, which have index 1. We list these representations in Table II. From Corollary 4.3 we know that only these real orthogonal representations can have $C(\mathfrak{H}) \neq 0$, where $\mathfrak{H}$ is the Lie algebra of a principal stabilizer and by Theorem 3.2 this is necessary for the second de Rham cohomology of a principal orbit to be non-zero. It is straightforward but very tedious (we omit details) to directly check what are the corresponding principal stabilizers for these representations [Ref. 10) is only of partial use for this

^{*)} Omitting the adjoint representation (see end of Section 4).

^{**)} Note the remark at the end of Section 4 with reference to the index of real and of complex representations.

task] *). In Table II we also list these principal stabilizers, H , their Lie algebras $\mathfrak{H}$ and $C(\mathfrak{H})^{H/H_0}$. (T^k denotes the k dimensional torus and $\mathfrak{R}^k$ its Lie algebra.)

6. - SUMMARY

We have discussed how the Higgs field, valued in a vector space carrying an arbitrary representation of a compact Lie group G , allows the pull-back of the second de Rham cohomology of certain G orbits to define topological conserved currents with which integral charges may be associated to almost any closed 2-surface. We then show in Theorem 3.2 that this cohomology space is isomorphic to the subalgebra, of the centre of the Lie algebra of the stabilizer (of a point of the orbit) $C(\mathfrak{H})$, whose elements are invariant by the adjoint action of a finite group. By specializing to principal orbits we show that a necessary condition for $C(\mathfrak{H}) \neq 0$, $\mathfrak{H}$ being the Lie algebra of a principal stabilizer, is that the index of the representation of the Lie algebra of G is 1 (Theorem 4.2). This necessary condition is strong enough to allow us to determine all representations, of simple groups of interest to GUT models, for which the second de Rham cohomology of a principal orbit is non-zero. That is, we need only consider all representations with index 1 and directly check their principal stabilizers, etc. Table II gives this information.

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*) With the spin groups we have found Ref. 11) most useful.

Type	Representation Δ	Real dim. of E	Group G	Real dim. of G	Index d_δ
A_r $r \geq 1$	$\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\overset{1}{\circ} \oplus \overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\overset{1}{\circ}$	$2(r+1)$	$SU(r+1)$	$r^2 + r$	$1/(r+1)$
$r = 3$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}$	6	$SO(6)$	15	$1/4$
$r \geq 4$	$\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\overset{1}{\circ} \oplus \overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\overset{1}{\circ}$	$r(r+1)$	$SU(r+1)$	$r^2 + 2r$	$(r-1)/(r+1)$
$r = 5$	$2(\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ})$	40	$SU(6)$	35	1
B_r $r \geq 2$	$\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ} \Leftarrow \overset{1}{\circ}$	$2r+1$	$SO(2r+1)$	$2r^2 + r$	$1/(2r-1)$
$r = 2$	$2(\overset{1}{\circ} \Leftarrow \overset{1}{\circ})$	8	$Spin(5)$	10	$1/3$
$r = 3$	$\overset{1}{\circ}-\overset{1}{\circ} \Leftarrow \overset{1}{\circ}$	8	$Spin(7)$	21	$1/5$
$r = 4$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ} \Leftarrow \overset{1}{\circ}$	16	$Spin(9)$	36	$2/7$
$r = 5$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ} \Leftarrow \overset{1}{\circ}$	32	$Spin(11)$	55	$4/9$
$r = 6$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ} \Leftarrow \overset{1}{\circ}$	64	$Spin(13)$	78	$8/11$
D_r $r \geq 4$	$\overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\overset{1}{\circ} \oplus \overset{1}{\circ}-\overset{1}{\circ}-\dots-\overset{1}{\circ}-\overset{1}{\circ}$	$2r$	$SO(2r)$	$2r^2 - r$	$1/(2r-2)$
$r = 4$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ} \oplus \overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}$	16	$Spin(8)$	28	$1/3$
$r = 5$	$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ} \oplus \overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}$	32	$Spin(10)$	45	$1/2$
$r = 6$	$2(\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ})$	64	$Spin(12)$	66	$4/5$

TABLE I - We give the indices d_δ , $d_\delta \leq 1$, for all irreducible real orthogonal representations Δ (*) of simple Lie groups of physical interest (for notation, see text).
 *) See Footnote *) on p. 8.

Type	Representation Δ	Group G	$\mathcal{H}$	H	$\dim C(\mathcal{H})^{H/H_0}$
A_r $r \geq 1$ $r = 3$	$(r+1)(\rho_1 \oplus \rho_r)$ $4\rho_2$ $3\rho_2 \oplus (\rho_1 \oplus \rho_3)$ $2\rho_2 \oplus 2(\rho_1 \oplus \rho_3)$ $\rho_2 \oplus 3(\rho_1 \oplus \rho_3)$	$SU(r+1)$ $SO(6)$	0 $\mathbb{R}$ 0 0 0	T^1	1
$r \geq 4$ $r = 5$	$(\rho_2 \oplus \rho_{r-1}) \oplus 2(\rho_1 \oplus \rho_r)$ $2\rho_3$	$SU(r+1)$ $SU(6)$	0 $\mathbb{R}^2$	T^2	2
B_r $r \geq 2$ $r = 2$	$(2r-1)\rho_1$ $3(2\rho_2)$ $2(2\rho_2) \oplus \rho_1$ $(2\rho_2) \oplus 2\rho_1$	$SO(2r+1)$ $Spin(5)$	$\mathbb{R}$ 0 0 0	T^1	1
$r = 3$	$5\rho_3$ $4\rho_3 \oplus \rho_1$ $3\rho_3 \oplus 2\rho_1$ $2\rho_3 \oplus 3\rho_1$	$Spin(7)$	0 0 0 0		
$r = 4$	$\rho_3 \oplus 4\rho_1$ $3\rho_4 \oplus \rho_1$ $2\rho_4 \oplus 3\rho_1$	$Spin(9)$	0 0 0		
$r = 5$	$\rho_4 \oplus 5\rho_1$ $2\rho_5 \oplus \rho_1$	$Spin(11)$	0 0		
$r = 6$	$\rho_5 \oplus 5\rho_1$ $\rho_6 \oplus 3\rho_1$	$Spin(13)$	0		
D_r $r \geq 4$ $r = 4$	$(2r-2)\rho_1$ $3(\rho_3 \oplus \rho_4)$ $2(\rho_3 \oplus \rho_4) \oplus 2\rho_1$ $(\rho_3 \oplus \rho_4) \oplus 4\rho_1$	$SO(2r)$ $Spin(8)$	$\mathbb{R}$ 0 0 0	T^1	1
$r = 5$	$2(\rho_4 \oplus \rho_5)$ $(\rho_4 \oplus \rho_5) \oplus 4\rho_1$	$Spin(10)$	0 0		
$r = 6$	$2\rho_5 \oplus 2\rho_1$	$Spin(12)$	0		

TABLE II - We list all real orthogonal representations*) (reducible or irreducible), which have index 1 and that can be constructed from Table I, together with a corresponding stabilizer H (they are all conjugate), its Lie algebra $\mathcal{H}$ and $\dim C(\mathcal{H})^{H/H_0}$.

*) See footnote *) on p. 8.

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