ON THE POSITIVITY OF THE EFFECTIVE ACTION
IN A THEORY OF RANDOM SURFACES

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ABSTRACT

It is shown that the functional $S[\eta] = \frac{1}{2n} \int \frac{1}{2} |\nabla \eta|^2 + 2\eta d\nu_0$, defined on $C^\infty$ functions on the two-dimensional sphere, satisfies the inequality $S[\eta] \geq 0$ if $\eta$ is subject to the constraint $\int (\Phi^n - 1) d\nu_0 = 0$. The minimum $S[\eta] = 0$ is attained at the solutions of the Euler-Lagrange equations. The proof is based on a sharper version of Moser-Trudinger's inequality (due to Aubin) which holds under the additional constraint $\int e^{\eta^2} x d\nu_0 = 0$; this condition can always be satisfied by exploiting the invariance of $S[\eta]$ under the conformal transformations of $S^2$. The result is relevant for a recently proposed formulation of a theory of random surfaces.

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1. - INTRODUCTION

Let $ds^2 = e^n ds_0^2$ denote a Riemannian metric on the two-dimensional sphere $S^2$, conformal to the standard metric $ds_0^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The points of $S^2$ will be parametrized, as usual, by a unit vector $\bar{x}$, by polar co-ordinates $(\theta, \phi)$ or by a complex variable $\xi$ related to $\bar{x}$ by stereographic projection, i.e., $\xi = \cot \frac{\theta}{2} e^{i\phi} = (x_1 + i x_2)/(1 - x_3)$. The conformal factor $e^n$ is assumed to be $C^\infty$. Let $\Delta = e^{-n} \Delta_0$ be the Laplace-Beltrami operator associated to $ds_0^2$ and let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$ be the spectrum of $-\Delta_0$. $\{\lambda_n\}$ will denote the corresponding objects belonging to $ds_0^2$.

It was shown in Ref. [1] that the limit

$$\det \Delta \over \det \Delta_0 = \lim_{n \to \infty} \prod_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k^0} = e^{-S[\eta]}$$

exists provided that $e^n$ is normalized, i.e.,

$$\int (e^n - 1) d\mu_0 = 0$$

where $d\mu_0 = \sin \theta \, d\theta \wedge d\phi$. A closed expression for $S[\eta]$ was obtained, namely

$$S[\eta] = \frac{1}{24 \pi} \int_{S^2} \left\{ \frac{1}{2} |\nabla_0 \eta|^2 + 2 \eta \right\} d\mu_0$$

where $\nabla_0$ is the covariant gradient with respect to $ds_0^2$, i.e.

$$|\nabla_0 \eta|^2 = \left( \frac{\partial \eta}{\partial \theta} \right)^2 + (\sin \theta)^2 \left( \frac{\partial \eta}{\partial \phi} \right)^2$$

The Euler-Lagrange equations for $S[\eta]$ under the constraint Eq. (2) have the simple geometrical meaning that the metric $e^n ds_0^2$ has constant curvature. It follows that the general solution, giving all the stationary points of $S[\eta]$ is the following:

$$\eta = \eta(\xi) = 2 \ln \frac{1 + |\xi|^2}{|\alpha \xi + \beta|^2 + |\gamma \xi + \delta|^2} = -2 \ln (\cosh r + \sinh r \, \hat{n} \cdot \hat{z}) (5)$$
where $g = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{C})$, $\hat{\eta}$ is a unit vector and $\tau \in (0, +\infty)$. $S[\eta]$ vanishes at $\eta^{(o)}_g$ and its expansion around any of these stationary points has a positive semi-definite quadratic part, hence Eq. (5) gives indeed the local minima of $S[\eta]$. Since $S[\eta]$ is interpreted as the classical action of the field $\eta(\xi)$, it is important to know whether $\eta^{(o)}_g$ are merely local minima (metastable states) or whether they are indeed the absolute minima of $S[\eta]$. The problem is less trivial than it might appear at first sight, actually its solution requires some tools from non-linear analysis which are far from trivial.

The answer turns out to be very simple, however, as given by the following theorem:

**Theorem:** $S[\eta]$ is positive semi-definite under the constraint $\int (e^{\eta} - 1) d\mu_0 = 0$ and $S[\eta] = 0$ implies $\eta = \eta^{(o)}_g$ for some $g \in \text{SL}(2, \mathbb{C})$.

2. **Proof of the main theorem**

The proof of the theorem makes essential use of an "exponential" Sobolev inequality due to Aubin, combined with the invariance of $S[\eta]$ under conformal transformations.

Let us dispose of the constraint [Eq. (2)] by introducing

$$\eta = \psi - \frac{1}{3} \ln \int e^\psi \frac{d\mu_0}{4\pi}$$  \hspace{1cm} (6)

($\psi$ is defined up to an additive constant, which we may fix by requiring $\int \psi d\mu_0 = 0$, but this will not be necessary). The unconstrained functional is now

$$S[\eta] = \frac{1}{3} \int \left\{ \frac{1}{4} |\nabla \psi|^2 + \psi \right\} \frac{d\mu_0}{4\pi} - \frac{1}{3} \ln \int e^\psi \frac{d\mu_0}{4\pi}$$

which was introduced long ago in a purely geometrical context [2]. It was shown by Moser [3] that $S[\eta]$ is bounded from below by some absolute constant. A sharper version of the inequality may hold, however, under additional constraints on $\psi$ such as a parity condition [4] $\psi(x) = \psi(-x)$. More generally, Aubin [5] proved that if $\psi$ satisfies

$$\int e^\psi \frac{d\mu_0}{4\pi} = 0$$

(8)
then

$$\int e^{\psi} \frac{d\mu_0}{4\pi} \leq C(\varepsilon) \exp \left\{ \left( \frac{1}{8} + \varepsilon \right) \int |\nabla \psi|^2 \frac{d\mu_0}{4\pi} + \int \psi \frac{d\mu_0}{4\pi} \right\}$$ (9)

for any $\varepsilon > 0$ and some constant $C(\varepsilon)$. Since the coefficient in the exponential is now $\frac{1}{8} + \varepsilon < \frac{1}{4}$, it follows that

$$3 S[\eta] \geq \left( \frac{1}{8} - \varepsilon \right) \int |\nabla \eta|^2 \frac{d\mu_0}{4\pi} - \ln C(\varepsilon)$$ (10)

Under these circumstances it is known that the infimum of $S$ is actually attained at the solutions of Euler-Lagrange equation (see Aubin [5] for details on this point and Berger [6] for the general theory).

At this point, provided $n$ satisfies the additional constraint (8), one has the sharp inequality

$$\begin{cases} S[\eta] \geq 0 \\ S[\eta] = 0 \Rightarrow \eta = 0 \end{cases}$$ (11)

In fact the Euler-Lagrange equation under the constraints (2) and (8) is

$$-\Delta_0 \eta + 2e^\eta = \lambda e^\eta + \vec{\mu} \cdot \vec{z} e^\eta$$ (12)

By integrating over $S^2$ one finds $\lambda = 2$. It is also known (Kazdan and Warner [7]) that the equation

$$\Delta_0 \eta = 2 - (2 + \vec{\mu} \cdot \vec{z}) e^\eta$$ (13)

does not admit any solution except for $\vec{\mu} = 0$, in which case we are led back to the general solution Eq. (5). Only $\eta = 0$ satisfies the constraint (8).

Now we come to the crucial observation that allows us to apply Aubin's result in general:

Lemma: The functional $S[\eta]$ is invariant under the transformations

$$\eta \rightarrow (T_g \eta)(\xi) = \eta(g^{-1}(\xi) + \chi_g(\xi), \xi)$$ (14)
where

$$g \xi = \frac{\alpha \xi + \beta}{\gamma \xi + \delta}, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{C})$$  \hspace{1cm} (15)$$

$$\chi(g, \xi) = 2 \ln \frac{1 + |\xi|^2}{|\alpha \xi + \beta|^2 + |\gamma \xi + \delta|^2}$$  \hspace{1cm} (16)$$

A direct proof is not difficult, but it is rather cumbersome and not particularly enlightening. It is preferable to rely on the link between $S[\eta]$ and the Laplacian [Eq. (1)] and realize that $SL(2, \mathbb{C})$ is the largest connected group of conformal transformations of $S^2$ onto itself, Eq. (14) giving the transformation rule for $\eta$. The spectrum of the Laplacian is clearly the same for $\eta$ and $T_g \eta$.

Now, without changing the value of $S[\eta]$, we can look for a $g \in SL(2, \mathbb{C})$ such that Eq. (8) is satisfied by $T_g \eta$. If such a $g$ exists then, by Eq. (11),

$$S[\eta] = S[T_g \eta] > 0$$  \hspace{1cm} (17)$$

and $S[\eta] = 0 \Rightarrow T_g \eta = 0$ for some $g$ which is the assertion of the theorem. So everything is reduced to the problem of finding a root of the equation

$$\int_{S^2} e^{(T_g \eta)(\xi)} \vec{\chi}(\xi) \, d\mu_0 = 0$$  \hspace{1cm} (18)$$

A simple topological argument will show that such a root actually exists, and the proof of the theorem will be complete. By inserting the definition of $T_g \eta$ and changing the integration variable to $g^{-1} \xi$, we get the equation

$$\int_{S^2} e^{\eta(\xi)} \vec{\chi}(g \xi) \, d\mu_0 = 0$$  \hspace{1cm} (19)$$

where $g$ is the unknown. The function

$$\vec{\chi}(g \xi) = \int_{S^2} e^{\eta(\xi)} \vec{\chi}(g \xi) \, d\mu_0$$  \hspace{1cm} (20)$$
defines a continuous map $\tilde{X} : SL(2, \mathbb{C}) \to \mathbb{R}^3$ the image being contained in the unit ball $\|\tilde{X}\| < 1$. For any $\lambda > 1$ let $B_\lambda$ denote a sphere in $SL(2, \mathbb{C})$ defined by

$$B_\lambda = \{ g \in SL(2, \mathbb{C}) \mid g = u(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix})u^t, u \in SU(2) \}$$

(21)

If $\lambda$ is taken sufficiently large the image of $B_\lambda$ under the map $\tilde{X}$ is close to the sphere $\|\tilde{X}\| = 1$; in fact,

$$\tilde{X}(u(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix})u^t \xi) = D(u) \tilde{X}(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}u^t \xi)$$

(22)

$D : SU(2) \to O(3)$ being the three-dimensional representation of $SU(2)$; but

$$\lim_{\lambda \to +\infty} \tilde{X}(\begin{pmatrix} \lambda^2 & 0 \\ 0 & 1 \end{pmatrix}(u^t \xi)) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(23)

except for a set of measure zero $(u^t \xi = 0)$ which does not contribute to the integral. Hence

$$\lim_{\lambda \to +\infty} \tilde{X}(u(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix})u^t) = D(u) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(24)

This shows that for sufficiently large $\lambda$ the map $\tilde{X} : B_\lambda \to \mathbb{R}^3 - \{0\}$ is homotopically non-trivial. Since $B_\lambda$ is contractible (it shrinks to the identity as $\lambda \to 1$) this implies the existence of a root. A similar argument holds in a much more general setting (Gluck [8]).

3. CONCLUDING REMARKS

We have shown that the action functional introduced in [1] in the context of Polyakov's theory of random surfaces [9] is indeed bounded from below and attains its absolute minimum at the "classical solutions" Eq. (5). Let us recall that the symmetry of $S[\eta]$ under conformal transformations is a reflection of the fact that Polyakov's "gauge choice" $a_{ab} = \rho_{ab}$ does not completely fix the gauge in the case of simply connected surfaces. Our result shows that the residual gauge freedom can be consistently eliminated by imposing the additional
constraint \( \int_0^{\eta} x \, du_0 = 0 \), which near \( \eta = 0 \) reduces to the condition that \( \eta \) be orthogonal to the zero modes. All these problems are peculiar of the simply connected surfaces. For surfaces with Euler characteristic \( \chi \leq 0 \) there is no residual gauge freedom, no zero modes and the effective action is manifestly positive definite.

From a mathematical point of view, we have obtained the best constant in the Moser-Trudinger inequality, which now reads

\[
\int_{S^2} e^{\frac{\psi}{4\pi}} \frac{d\mu_0}{4\pi} \leq \exp \left\{ \frac{1}{4\pi} \int \left[ \psi + \frac{1}{4} |\nabla_0 \psi|^2 \right] \, d\mu_0 \right\}
\]

(25)

If \( \psi \) is independent of \( \phi \), this reduces to the elementary inequality

\[
\int_0^1 e^{\int_0^t \frac{\psi(t)}{4\pi}} \, dt \leq \exp \left\{ \int_0^1 \frac{\psi(t)}{4\pi} \, dt + \frac{1}{4} \int_0^1 t (1-t) \psi(t)^2 \, dt \right\}
\]

(26)

the equality sign implying

\[
\psi(t) = \frac{\ln \left[ \frac{c_1}{(1 + c_2 t)^2} \right]}{c_1 > 0, c_2 > -1}
\]

(27)

The inequality (26) is "complementary" to the arithmetic-geometric-mean inequality \([10]\).

Finally, the result of the theorem implies the following bound on the spectrum of \( A \), which does not seem to have been noticed previously

\[
\lim_{n \to \infty} \prod_{k=1}^{n} \frac{\lambda_k}{\lambda_k^0} = \frac{e^{-S[\eta]}}{1}
\]

(28)

the bound being saturated only by the standard metric (up to isometries).

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