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RELATIVE MOTION OF TWO HEAVY IONS

NUMERICAL STUDIES ON THE PHASE-SPACE EVOLUTION OF

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the world-first approach (LM) which can overcome the difficulties
in describing radial motion in deep-neutron reactions. As a replacement
for the description of phase-space density, the LM is not useful for
with the PPE. No approach to phase-space density has yet been performed.

It is the aim of this publication to show that the LM is not appropriate for
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approximate solutions for the two-dimensional problem are available for
approximate calculations in the driving potential due to shell effects are found to
 collisions, described by a one-dimensional [5,6] formula. It is applicable to
its applicability [5,6]. Also for the nuclear mass transport in deep-neutron
the density function is slightly different from zero, the LM should lose
soon as the energy of the potential changes significantly within the domain where
approximate equations in the case of a slightly perturbed potential, and
approximate solutions for harmonic driving potentials and
approached (CM) yields exact solutions for harmonic driving potentials and
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1. Introduction

NUMERICAL STUDIES ON THE PHASE-SPACE EVOLUTION OF RELATIVE MOTION
OF TWO HEAVY IONS
2. Validity of the Global Moment Approach

One of the most interesting applications of classical transport equations is the description of the relative motion of two colliding heavy ions. Restricting ourselves to a single collective degree of freedom, i.e. the radial motion, the time evolution of the classical phase-space density \( f(r,v;t) \) is given by

\[
\frac{\partial}{\partial t} f(r,v;t) = -v \frac{\partial}{\partial r} f(r,v;t) + \frac{1}{\mu} \frac{\partial}{\partial v} \left[ \gamma(r) v + D(r) \frac{\partial^2}{\partial v^2} \right] f(r,v;t),
\]

where the coordinates \( r, v \) correspond to the radial distance and radial velocity. The density function takes the initial values

\[
f(r,v;0) = \delta(r-r_0) \delta(v-v_0) \]

and has to fulfill the boundary conditions

\[
f(r,v;t) = 0 \text{ for } r = 0, \infty, |v| = \infty.
\]

In eq. (1) the conservative potential is described by \( U(r); \gamma(r) \) and \( D(r) \) denote the formfactors for friction and diffusion while the quantity \( \mu \) corresponds to the reduced mass of the system. The conservative potential is a superposition of the attractive nuclear, the repulsive Coulomb, and the centrifugal potential, which yields a pocket for lower angular momenta \( \ell \) to allow the description of fusion. Such a potential is depicted in fig. 1 for different angular momenta \( \ell \), generated by a Gaussian for the nuclear part, the Coulomb contribution corresponding to \( Z_1 Z_2 = 3800 \) and the reduced mass given by \( \mu = 100 \text{ MeV } \text{ bs}^{-2} \text{ fm}^{-2} \) (bs = 10^{-23}s). The formfactor for friction is supposed to be a Gaussian

\[
\gamma(r) = \gamma_0 \cdot \exp \left\{ -r^2 / 2a^2 \right\},
\]

with \( \gamma_0 = 7 \text{ bs}^{-1} \) and \( a = 6 \text{ fm} \). This choice leads to the mean trajectories shown in fig. 1 (dashed lines) for an incident collective energy of \( \varepsilon_{\text{CM}} = 400 \text{ MeV} \). In this model the critical angular momentum is given by \( \ell_{\text{crit}} = 23 \text{ H} \); trajectories with lower angular momentum are trapped and lead to fusion in the classical sense. The formfactor for diffusion is chosen according to an Einstein relation with constant temperature

\[
D(r) = \gamma(r) \cdot D_0
\]

with \( D_0 = 5 \cdot 10^{-2} \text{ fm}^2 \text{bs}^{-2} \). In this way all coefficients in eq. (1) are specified and supposed to approximately simulate the dynamics of a heavy-ion collision. The standard procedure for the solution of eq. (1) is to expand the function \( f(r,v;t) \) in moments up to 2nd order in \( r \) and \( v \) [2-4] which yields

\[
\begin{align*}
\frac{d}{dt} \langle r \rangle &= \langle v \rangle, \\
\frac{d}{dt} \langle v \rangle &= -\frac{1}{\mu} \frac{dU}{dr} \langle r \rangle - \gamma \langle r \rangle \langle v \rangle, \\
\frac{d}{dt} \langle \frac{v^2}{2} \rangle &= 0, \\
\frac{d}{dt} \langle \frac{r^2}{2} \rangle &= 2D \langle r \rangle \langle v \langle r \rangle \rangle + 2D \langle r \rangle \langle v \rangle - \frac{2}{\mu} \frac{dU}{dr} \langle v \rangle \langle r \rangle \langle v \rangle, \\
\frac{d}{dt} \langle \frac{r^2}{2} \rangle &= 2 \frac{dU}{dr} \langle v \rangle \langle r \rangle \langle v \rangle - \frac{1}{\mu} \frac{dU}{dr} \langle v \rangle \langle v \rangle \langle r \rangle \langle v \rangle.
\end{align*}
\]

with

\[
\begin{align*}
\langle r \rangle &= \langle r \rangle, \\
\langle \frac{v^2}{2} \rangle &= \langle v \rangle \langle v \rangle \\
\langle \frac{r^2}{2} \rangle &= \langle r \rangle \langle r \rangle.
\end{align*}
\]

The approach is expected to break down if the second derivative of \( U(r) \) changes sign. In this case the variance \( \langle \frac{r^2}{2} \rangle \) increases according to eqs. (4) much too strong because the mean value \( \langle r \rangle \) remains in the domain of
By $d^{0}$ we mean the elastic interaction rule. This is obtained by the formula

$$\frac{d^{0}}{d\Omega} \propto \frac{1}{s^{2}} \frac{d\Omega}{d^{0}}$$

where $s$ is the center-of-mass energy and $d\Omega/d^{0}$ is the differential cross section in the laboratory frame.

It is worthwhile to note that the reaction under consideration is a three-body reaction, involving two pions and a nucleon. The pion-nucleon interaction is described by the optical potential model, which takes into account the spin and isospin degrees of freedom.

The reaction cross-sections for the process $p + p \rightarrow n + n + 
\nu \bar{\nu}$ are calculated using an optical potential model. The model parameters are chosen to fit the experimental data on nucleon-nucleon scattering.

The results of the calculations are shown in Figs. 1 and 2. In addition, the results of elastic scattering are shown in Fig. 3. The agreement between theory and experiment is satisfactory, indicating the validity of the optical potential model.
3. The Local Moment Approach

The main impediment for the solution of partial differential equations is the dimensionality of the problem. Equations with more than four independent variables (time included) seem to be unaccessible by computations on a grid. Physics, however, forces us to take more degrees of freedom into account than just \( v \) and \( r \). Therefore we propose a method to reduce the dimensionality of the original FPE, the local moment approach (LMA), and examine its usefulness for the problem of section 2, where we can compare it with the guaranteed approximations from ADIP.

The idea of introducing a local moment approach to eq. (1) is guided by the results for \( f(r,v;\lambda) \) for fixed \( r \) depicted in fig. 4 (\( \lambda = 23 \lambda, t = 23 \) bs). The distribution very much resembles Gaussians with different centers and widths, and a harmonic expansion of eq. (1) with respect to \( v \) appears promising. This is, of course, connected with the fact that all coefficients in eq. (1) are linear with respect to \( v \). Defining the density \( \rho \), the local velocity \( V \), and the local spreading \( \sigma_v^2 \) by

\[
\begin{align*}
\rho(r,t) &= \int dv f(r,v;t), \\
V(r,t) &= \int dv \frac{v f(r,v;t)}{\rho(r,t)}, \quad (6a) \\
\sigma_v^2(r,t) &= \int dv \frac{(v-V(r,t))^2}{\rho(r,t)} f(r,v;t), \quad (6b)
\end{align*}
\]

and we obtain

\[
\frac{3}{\lambda^2} \rho(r,t) = - V(r,t) \frac{3}{\partial r} \rho(r,t) - \rho(r,t) \frac{3}{\partial r} V(r,t), \quad (7)
\]

\[
\frac{3}{\lambda^2} \sigma_v^2(r,t) = - V(r,t) \frac{3}{\partial r} \sigma_v^2(r,t) - \sigma_v^2(r,t) \frac{3}{\partial r} \ln \rho(r,t),
\]

(8)

and

\[
\begin{align*}
\frac{3}{\lambda^2} \sigma_v^2(r,t) &= -2 \sigma_v^2(r,t) \frac{3}{\partial r} V(r,t) - V(r,t) \frac{3}{\partial r} \sigma_v^2(r,t) \\
&= -2 V(r,t) \sigma_v^2(r,t) + 2 D(r)
\end{align*}
\]

(9)

by neglecting 3rd and higher moments of \( f(r,v;\lambda) \) with respect to \( v \).

Furthermore we have to determine the initial values and the boundary conditions for the functions \( \rho(r,t), V(r,t) \) and \( \sigma_v^2(r,t) \). The initial conditions are simply derived from (1a) and the definitions (6)

\[
\begin{align*}
\rho(r,0) &= \delta(r-r_0), \quad (11a) \\
V(r,0) &= v_0, \quad (11b) \\
\sigma_v^2(r,0) &= 0. \quad (11c)
\end{align*}
\]

The boundary conditions can be obtained by the requirement of compatibility at \( t = 0 \) and by the physical conditions at \( r = 0 \), where the density distribution must vanish due to the repulsive core of the potential, and at \( r \to \infty \), which the system cannot reach within a finite time;

\[
\begin{align*}
\rho(0,t) &= \rho(\lambda,t) = 0, \quad (12a) \\
V(0,t) &= V(\lambda,t) = v_0, \quad (12b) \\
\sigma_v^2(0,t) &= \sigma_v^2(\lambda,t) = 0. \quad (12c)
\end{align*}
\]

However, (11a,b) and (12b) are not suitable for practical purposes where we wish to limit all calculations to a grid as small as possible. We apply physical arguments to ameliorate the situation.
Formally, we can not observe experiments.

the distribution is much
sampled is too steep. The velocity fields \( v(r,z) \), however, agree
within an error of \( 5\% \). The distribution function for
and the experimental results for constant \( r(z) \). It may be
and apparently yields similar shapes for constant \( r(z) \).

isolates the effects of the distribution.

The constant \( r(z) \) corresponds to the same values of \( r(z) \) as in Fig. 2 - C. The
constant \( r(z) \) together with the anti-solution of the PP (1.1.4 \( \alpha \)). The
and initial values \( (g\alpha) \) for \( r = 1 \) to be displayed in Fig. 5 (r.4 \( \alpha \)).

The maximal solution of (a) - (g) with the boundary conditions (1)

\[ f(r,z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

(1)

where a direct insertion of \( f(z) \) into \( \Omega \) yields the desired function.

The boundary conditions for \( f(z) \) and \( \Omega \) are

\[ \frac{\partial f(r,z)}{\partial r} \bigg|_{r=0} = 0, \quad \frac{\partial f(r,z)}{\partial z} \bigg|_{z=\Omega} = 0 \]

In order to compactly display our results within the LWa, we choose the

(2)

\[ f(r,z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

from the boundary conditions (1) - (g) as the boundary conditions

\[ f(r,z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

In eq. (18), \( f(z) \) is given by

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

for \( \Omega = 0 \)

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

For \( \Omega > 0 \)

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

for which the solution is given by

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

well approximated by the solution given by eq. (4).

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

Furthermore, since \( f(z) \) is so sharply localized, we can

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

are clearly insufficient. Choosing a small domain within which all coefficients

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]

For short times, \( f(z) \) is a good approximation, since the density function

\[ f(z) = \frac{1}{2} z \left( \frac{r}{z} \right)^2 \]
4. Summary

We would like to point out that the conventional global moment approach (GMA) (4) appears to be no approximation to the correct solution of the Fokker-Planck equation (1) in case of deeply inelastic heavy-ion collisions because the moment expansion tends to diverge if the physical solution splits into parts describing partial fusion and inelastic scattering. Unfortunately (w.r.t. the GMA) the transition from inelastic scattering to complete fusion is very soft in angular momentum in accordance with experimental findings. Thus the Fokker-Planck equation seems to be a reliable description of deeply inelastic heavy-ion collisions and to be applicable to the dynamics of fusion, but does not permit the conventional approximation schemes. Therefore we proposed a local moment approach (LMA) formally close to hydrodynamics which yielded approximative solutions for the Fokker-Planck equation in this case.

Appendix

Peaceman and Rachford [14] developed the alternating directions implicit procedure (ADIP) for the numerical solution of the two-dimensional heat equation. We applied their method to the two-dimensional Fokker-Planck equation

\[ \partial_t P(x_1, x_2; t) = \frac{1}{1 + \lambda} \partial_{x_1} \left[ V^I P(x_1, x_2; t) \right] + \frac{1}{\lambda} \partial_{x_1} \left[ d^I P(x_1, x_2; t) \right], \]  

(A.1)

where the drift coefficients \( V^I(x_1, x_2; t) \) and the diffusion coefficients \( d^I(x_1, x_2) \) are given functions of the coordinates \( x_1 \) and \( x_2 \).

We want to compute the time evolution of the probability density \( P(x_1, x_2; t) \) provided the initial distribution \( P(x_1, x_2; 0) \) is known. In our problem (1) it is given by

\[ P(x_1, x_2; 0) = \delta(x_1 - x_{10}) \cdot \delta(x_2 - x_{20}) \]  

(A.2)

and the boundary values are specified by

\[ P(x_1, x_2; t) = 0 \quad \text{for} \quad x_1^2 + x_2^2 = \infty. \]  

(A.3)

In a sufficiently small neighborhood of \( (x_{10}, x_{20}) \), the drift and diffusion coefficients can be approximated by linear expressions

\[ V^i = a_i + \sum_{j=1}^{2} b_{ij} (x_j - x_{j0}), \quad d^i = c_i \quad \text{if} \quad x_i = x_{i0}, \]  

(A.4)

\( a_i, b_{ij}, \) and \( c_i \) being constants. In this case the Fokker-Planck equation (A.1) is solved in closed form

\[ P(x_1, x_2; t) = \exp(-\text{arg}/\det)/(2\pi/\det) \]  

(A.5)

where

\[ \text{arg} = \frac{\alpha_{11}^2}{\alpha_{22}^2} (x_1 - x_{10})^2 + \frac{\alpha_{12}^2}{\alpha_{11}^2} (x_2 - x_{20})^2 - 2\alpha_{12}(x_1 - x_{10})(x_2 - x_{20}), \]  

(A.6)
This time the direction is treated explicitly.

\[ (2/n)(d^2/dy^2) = (2/n)(d^2/dy^2 + 2) \]

The result is that the explicit treatment of the differential operator yields a simpler expression for \( f \).

\[ (3/n)(d^3/dy^3) = (3/n)(d^3/dy^3 + 3) \]

In this case, the differential operator is treated explicitly.

\[ (4/n)(d^4/dy^4) = (4/n)(d^4/dy^4 + 4) \]

For the next term in the series, the differential operator is treated explicitly.

\[ (5/n)(d^5/dy^5) = (5/n)(d^5/dy^5 + 5) \]

In this case, the differential operator is treated explicitly.

\[ (6/n)(d^6/dy^6) = (6/n)(d^6/dy^6 + 6) \]

For the next term in the series, the differential operator is treated explicitly.

\[ (7/n)(d^7/dy^7) = (7/n)(d^7/dy^7 + 7) \]

In this case, the differential operator is treated explicitly.

\[ (8/n)(d^8/dy^8) = (8/n)(d^8/dy^8 + 8) \]

For the next term in the series, the differential operator is treated explicitly.
The equations (A.12) and (A.13) represent ADIP. The method is stable and accurate. The error of the solution of the difference equations (A.12,13) relative to that of the differential equation (A.1) is $O(\Delta x_1^2 + \Delta x_2^2 + \Delta t^2)$. However, this is only true for the solution at the full time steps $t, t+\Delta t, t+2\Delta t, \ldots$. The values at the intermediate steps $t + \Delta t/2, t + 3\Delta t/2, \ldots$ are auxiliary and should not be used for the calculation of moments and similar purposes.

We are aware that ADIP loses its unconditional stability if the drift coefficients become too large:

$$|V^i| > 2d/\Delta x_i.$$  \hspace{1cm} (A.14)

For this case, the linear systems in (A.12) or (A.13) forfeit diagonal dominance, and the standard methods to solve them, being basically all Gaussian eliminations without pivoting, cease to be stable. But ADIP is still conditionally stable if the inequalities

$$2\Delta x / \Delta t > |V_i|$$  \hspace{1cm} (A.15)

are fulfilled. This is nothing else than the Courant-Friedrichs-Lewy criterion [16] for time step size $\Delta t/2$. We are aware of means which reintroduce unconditional stability [17], but in the present application we could do without them.

References


P. Grange and H.A. Weidenmüller, Fission Probability and the Nuclear Friction Constant, Preprint CRN/PN 80-24


In the sequence (0.2, 0.5, 1.0, 2.0, 1.0, 2.0, 1.0, 1.0, 1.0), the contour lines correspond to constant (r/2)\sqrt{t}.

4. Distribution function (r/2)\sqrt{t} for r = 2, 2.5, 3 at various fixed distances R (m).

3. Comparison with sharp cut off time (dashed line) and probability for fusion as a function of angular momentum in 2c-10 (r = 1.0, 2.0, 3.0, 4.0, 5.0, 6.0). The contour lines correspond to constant (r/2)\sqrt{t} in the sequence 10 (r = 1.0, 2.0, 3.0, 4.0, 5.0, 6.0). The dashed line in the upper part indicates the mean trajectory of the GM. The dashed line at r = 2, 3, 4, 5, 6, 7 for the global moment approach.

2c-10 (r = 1.0, 2.0, 3.0, 4.0, 5.0, 6.0) for various times of different angular momentum. The dashed line indicates the mean trajectory obtained with the friction model potential U for some initial angular momenta. The dashed line.

Figure captions