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POTENTIAL AND RESTORATION OF ROTATIONAL SYMMETRY
IN SU(2) LATTICE GAUGE THEORY

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ABSTRACT

We study by numerical techniques the restoration of rotational invariance in SU(2) lattice gauge theory. Results on the potential between two static sources in the fundamental representation are also represented.

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During the last few years numerical simulations of lattice gauge theories¹⁾⁻³⁾ have emerged as a very powerful technique for obtaining information about the properties of the corresponding continuum quantum systems. Although the lattice introduces a distortion of space-time, one expects that by a renormalization procedure whereby the lattice spacing is sent to zero while the bare coupling constant is simultaneously readjusted, a continuum limit can be obtained where the theory exhibits full Poincaré invariance. In the all-important non-Abelian models, where the continuum limit corresponds to a vanishing value of the bare coupling constant g , numerical computations of several physical quantities⁴⁾⁻¹⁰⁾ as functions of g reveal a rather sudden transition (crossover) from a strong coupling behaviour to the scaling behaviour predicted by the renormalization group. One expects then that the symmetry properties of the models should also undergo a rapid change in the crossover region, from the symmetries of the lattice to an approximate invariance under the full $O(4)$ group of Euclidean rotations. This recovery of rotational invariance has been related to the occurrence of a roughening transition^{11),12)} in the expectation value of some observables.

In this letter we wish to present numerical results, obtained by Monte Carlo simulations, which show how rotational invariance is re-established as one proceeds through the crossover region in the $SU(2)$ lattice gauge theory. The quantity we have studied is the potential $V(\vec{r})$ between two static sources in the fundamental representation, placed at arbitrary space separation vector $\vec{r} = (r_x, r_y, r_z)$ on the lattice. $V(\vec{r})$ is measured considering the expectation value of the product of two Wilson factors

$$W = \text{Tr} (U_{i_1 i_2} U_{i_2 i_3} \dots U_{i_N i_1}) \quad (1)$$

associated with the loop extending through the lattice in the time direction and closed by virtue of the periodic boundary conditions.

$$V(\vec{r}) = -T \ln \langle W(\vec{r}) W(0) \rangle \quad (2)$$

T is the physical temperature of the system, determined by the relation

$$T = 1/n_t \alpha(g, \Lambda) \quad , \quad (3)$$

where n_t is the extension of the system along the time direction (in lattice units), a the lattice spacing, g the bare coupling constant and Λ the lattice scale parameter. V is, strictly speaking, the free energy of the quantum system in the presence of two static sources at separation $\vec{r}$.

In the course of our analysis we have measured $\langle W \rangle$ and $\langle W(\vec{r})W(0) \rangle$ for a wide range of the parameters and thus obtained detailed information on the behaviour of the potential as a function of physical separation and temperature. We also present our results on the potential, although they constitute to some extent a verification of earlier analyses by Kuti, Polonyi and Szlachanyi⁶⁾ and by McLerran and Svetitsky⁷⁾. An investigation of the recovery of rotational invariance in the Abelian $U(1)$ model, based on an approach similar to ours, has been carried out by De Grand and Toussaint¹³⁾.

The model is defined in Euclidean space-time with Wilson's action. The Boltzmann factor is then

$$\exp \left\{ -\beta \sum_{\square} \frac{1}{2} \text{Tr} (\mathbb{1} - U_{\square}) \right\}, \quad (4)$$

where the sum extends over all plaquettes of the lattice, U is the parallel transporter around the plaquette and the coupling parameter β is related to the non-renormalized coupling constant g by $\beta = 4/g^2$. We have considered lattices extending for n_s sites in each of the space directions and n_t sites in the time direction, with periodic boundary conditions. We approximated the full gauge group by its discrete icosahedral subgroup $\tilde{Y}$ to improve the efficiency of the computation. It is well known that the $\tilde{Y}$ model gives a very good approximation to the $SU(2)$ theory throughout the physically relevant region^{14),5)}. The details of the program we used have been published in Ref. 15).

We have made measurements of expectation values of thermal loop factors [i.e., the Wilson factors for transport along the time axis at fixed space co-ordinates, see Eq. (1)] and of products of loop factors at arbitrary space-like lattice separation $\vec{n} = (n_x, n_y, n_z)$. We denote these quantum averages by

$$w_0 = \langle W \rangle \quad (5)$$

and

$$w(\vec{n}) = \langle W(\vec{n}) W(0) \rangle. \quad (6)$$

We studied lattices with $n_t = 4, 6, 8$, $n_s = 8$ and $n_t = 6$, $n_s = 16$. For a proper interpretation of w_0 and $w(\vec{n})$ in terms of a free energy, i.e.,

$$w_0 = \exp\{-V_0/T\} \quad (7)$$

$$w(\vec{n}) = \exp\{-V(\vec{r})/T\} \quad (8)$$

$[\vec{r} = a\vec{n}$ and $T = 1/an_t$, see Eq. (3)], n_s should be much larger than n_t . In practice one observes that the numerical results remain reasonably stable even if n_s becomes comparable to n_t . For the systems with $n_s = 8$ the averages were taken over several hundred (typically 1000) Monte Carlo iterations starting from configurations at approximate equilibrium, which were obtained by adiabatically cooling the system from $\beta = 0$ in a few hundred iterations. For the simulations with $n_s = 16$, eight replicas of the configuration with $n_s = 8$ obtained at the end of the simulation were combined into a single, larger lattice and then 300 further iterations were performed. We have measured the correlations $w(\vec{n})$ at all separations $\vec{n}$ with $|\vec{n}| < 5$, folding the data obtained with separation vectors related by symmetries of the lattice to ameliorate the statistics. However, only those values of $\vec{n}$ for which $w(\vec{n})$ was larger than the error induced by fluctuations were kept for further analysis.

The first question we want to address is the recovery of rotational invariance. To estimate the degree of distortion introduced in the potential by the lattice we proceeded as follows. The values of the potential $V(\vec{r})$ were obtained through Eq. (2). One dimensional arrays of aligned, nearby spatial lattice sites

$$\vec{n}_j = \vec{n}_0 + j \Delta\vec{n} \quad (9)$$

were then considered. By "nearby" we mean that only small steps $\Delta\vec{n}$ [for instance, $\Delta\vec{n} = (1,1,0)$] were taken into account; the lines defined by Eq. (9), however, were not constrained to pass through the origin. The values

$$V_j = V(\vec{r}_j = a \vec{n}_j) \quad (10)$$

were fit in terms of a Coulomb plus linear form

$$V(\vec{r}) = c_0 + c_1/r + c_2 r. \quad (11)$$

separately for each of these lines. The vectors $\vec{r}^{(k)}$ where the fitted potentials took a definite value V_0 were finally determined. Clearly, the points $\vec{r}^{(k)}$ (k labels the different rectilinear arrays of lattice sites) give a representation of the equipotential surfaces of the interpolated potential function.

The results of the above analysis, for $\beta = 2$, $n_s = 8$, $n_t = 4$ and for $\beta = 2.25$, $n_s = 16$, $n_t = 6$, are displayed in Figs. 1a and 1b, respectively. Points corresponding to several different constant values of the potential are shown in the graphs and contours defined by the equation

$$r(\psi) = r_0 + r_1 \cos 4\psi \quad (12)$$

are fitted through them. These lines have been added to the figures both to provide a guide to the eye (otherwise points obtained with different V_0 should be labelled or plotted separately to avoid confusion) and to give a more objective indication of the departure from rotational symmetry. Notice that the crossover from the strong coupling regime to the scaling regime occurs between $\beta \approx 2$ and $\beta \approx 2.2$. The fact that the transition is accompanied by a restoration of rotational invariance is manifest from the figures.

We turn now to the behaviour of the potential as a function of separation and temperature. We have used the results of Refs. 4) and 5) to establish a relation between string tension κ , lattice spacing a and coupling parameter β . The asymptotic form

$$a(\beta) = \left(\frac{11}{6\pi^2\beta} \right)^{-\frac{51}{121}} e^{-\frac{3\pi^2\beta}{11}} \quad (13)$$

has been used down to $\beta = 2.15$ and the functional form deduced directly from the measurement of the string tension has been used for lower values of β . The slight discrepancy between the results of Refs. 4) and 5) ($\Lambda = 0.013 \sqrt{\kappa}$ versus $\Lambda = 0.011 \sqrt{\kappa}$) is probably due to the different methods used to extract the string tension from the expectation value of rectangular loop factors. For definiteness we assume $\Lambda = 0.013 \sqrt{\kappa}$ and, in the following, all dimensional quantities will be thus expressed in units of $\sqrt{\kappa}$.

As the physical temperature increases, the system undergoes a transition from the confining phase to a Debye screened, unconfined phase. In Refs. 6) and 7) the critical temperature for the transition was estimated to be $T_c \approx 0.39$ (in units of $\sqrt{\kappa}$). The occurrence of the phase transition is characterized by

the fact that the free energy of a single source becomes finite. Thus one has (for a lattice with $n_s = \infty$) $w_0 = 0$ for $T < T_c$, w_0 finite for $T > T_c$. (To be precise, w_0 should be defined through the limit $w(\vec{r}) \xrightarrow{|\vec{r}| \rightarrow \infty} |w_0|^2$ or by introducing a source coupled to W which is then sent to zero.) Our measurements of w_0 are reproduced in Fig. 2. The results are consistent with $T_c \approx 0.39$. In Refs. 16) a slightly larger value for T_c is quoted, but the discrepancy may be explained by the different procedure used to obtain T_c . Taking for the critical temperature the point where the curve giving $|w_0|$ has maximal slope in Fig. 2 would indeed produce a higher T_c . The discrepancy should disappear in the infinite volume limit.

Assuming a critical temperature of $0.39 \sqrt{K}$, we fitted out data for the on-axis and off-axis potential $V(\vec{r})$ in terms of the Coulombic plus linear form, i.e.,

$$V(\vec{r}) = c_0 + c_1 V_c(\vec{r}) + c_2 r \quad (14)$$

for $T < T_c$, and in terms of a Debye-screened form, i.e.,

$$V(\vec{r}) = c_0 + c_1 V_c(\vec{r}) e^{-c_2 r} \quad (15)$$

for $T > T_c$. In the continuum limit $V_c(\vec{r})$ would be given by $1/\pi r$, but one cannot expect, even in the scaling region, an amount of rotational symmetry larger than that present in the lattice Coulomb potential.

$$V_c(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\pi/a}^{\pi/a} d^3 k \frac{e^{-i \vec{k} \cdot \vec{r}}}{\sum_i \sin^2(k_i a/2)} \quad (16)$$

Thus we have used the lattice form of Eq. (16) in the fit. [We have verified that replacing the integration in Eq. (16) with a finite sum, as would be appropriate for a finite size lattice, to a high degree of approximation only affects the potential through the addition of a constant term. Since the constant in Eqs. (14) and (15) is not physically meaningful in the continuum limit (it corresponds to a self-energy which must be renormalized), we have disregarded finite volume corrections to $V_c(\vec{r})$.]

Our results for the potential are summarized in the table, where we reproduce the values of β at which the measurements have been made, the size of the lattice, the range r_1 to r_2 over which $V(\vec{r})$ has been evaluated (r_0 is also the lattice spacing), the physical temperature T , and the values found for the parameters c_1 and c_2 . All dimension-full quantities (r_0, r_1, T, c_2) are given in units of corresponding powers of $\sqrt{\kappa}$, the square-root of the physical string tension.

Errors have been included in this table when the statistical fluctuations in the determination of $V(\vec{r})$ appeared to induce an uncertainty in the fit. The discrepancies between the values found for the parameters of the potential at identical or very close values of the temperature may be explained in part by this degree of uncertainty. But they are more likely due to finite size effects, to insufficient thermalization for the larger lattices and, especially, to the fact that the potentials are fit over different ranges of r . This last dependence, of course, does not reflect a deficiency of the computation; one expects the effective charge in the Coulomb potential to vary with r , in agreement with the renormalization group.

Altogether, the pattern which emerges for the behaviour of the potential is rather consistent. In Fig. 3a we display the force between static sources derived from the potentials we found [after replacement of $V_c(\vec{r})$ with its continuum form $1/\pi r$] for a few values of the temperature. The confining nature of the force for $T < T_c$ is evident, as well as the transition to a Debye-screened potential as T becomes larger than T_c . In the confining region the string tension appears to be lower than 1, of the order of 0.5. A decrease in the tension with temperature is of course expected. The curve

$$\kappa(T) = \left(\frac{T_c - T}{T} \right)^{1/2}$$

is also plotted in Fig. 3a. There is no reason, however, why the exponent in the behaviour of $\kappa(T)$ should be 1/2, and we have added that specific curve to the graph only to exemplify a possible temperature dependence.

In Fig. 3b we plot all the confining forces (solid lines), corresponding to the first seven entries of the table. Onto these data we superimpose graphs of the function

$$F(r) = \frac{9\pi}{22} \left\{ \frac{1}{r^2 \ln(1 + \frac{1}{\Lambda_c^2 r^2})} - \Lambda_c^2 \right\} + k \quad (17)$$

with $k = 0.46$ and $\Lambda_c = 0.1, 0.3$ and 1 (broken lines). The expression in Eq. (17) reproduces the expected asymptotic freedom behaviour for small r and the constant behaviour ($F \rightarrow k$) for large r . Apart from that it is not singled out by any theoretical consideration. The figure does show however that the potentials are consistent with the expected asymptotic behaviour and also that the parameter Λ_c determining the strength of the running charge is much larger than the lattice parameter ($\Lambda_{\text{lattice}} \sim 0.013$), in agreement with the results of Ref. 4) [cf. also Ref. 17)].

A refinement of our analysis, based on longer MC simulations and larger lattices, could be used to obtain a rather precise expression for the potential between static sources. This could then serve as an input for phenomenological calculations of the mass spectrum, although it appears today that the spectrum can be computed directly by Monte Carlo simulations¹⁸⁾, bypassing the evaluation of the potential.

It is gratifying, however, to see that the transition to the scaling domain is accompanied by a restoration of rotational symmetry and that the behaviour of the potential with separation and temperature conforms with theoretical expectations. Our analysis, we believe, provides yet another piece of evidence for the correctness of the lattice formulation of quantum gauge theories.

Table

The fits to the potential

	β	$n_s^3 \times n_t$	r_0	r_1	T	c_1	c_2
confining	2.25	$8^3 \times 8$	0.516	1.153	0.242	-0.909	0.447 ± 0.034
	2	$8^3 \times 4$	0.800	2.400	0.312	-1.767	0.500 ± 0.036
	2.25	$8^3 \times 6$	0.516	1.548	0.323	-1.016	0.439 ± 0.056
	2.25	$16^3 \times 6$	0.516	2.248	0.323	-0.923	0.518 ± 0.026
	2.05	$8^3 \times 4$	0.768	2.304	0.326	-1.682	0.395 ± 0.008
	2.10	$8^3 \times 4$	0.721	2.163	0.347	-1.156	0.537 ± 0.035
	2.15	$8^3 \times 4$	0.662	1.986	0.378	-1.218	0.456 ± 0.030
screened	2.375	$8^3 \times 6$	0.377	1.131	0.442	-0.870	0.401 ± 0.501
	2.375	$16^3 \times 6$	0.377	1.846	0.442	-1.295	0.034 ± 0.056
	2.5	$8^3 \times 8$	0.275	0.825	0.454	-0.714	1.116 ± 0.738
	2.5	$8^3 \times 6$	0.275	0.825	0.606	-0.598	2.916 ± 0.073
	2.5	$16^3 \times 6$	0.275	1.348	0.606	-0.558	1.695 ± 0.225
	2.75	$8^3 \times 8$	0.146	0.438	0.856	-0.472	0 ± 8.9
	2.75	$8^3 \times 6$	0.146	0.438	1.141	-0.424 ± 1.061	8.7 ± 7.4
	3	$8^3 \times 6$	0.077	0.231	2.156	-0.263 ± 0.165	12.9 ± 8.8

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FIGURE CAPTIONS

Fig. 1 : Restoration of rotational invariance from (a) $\beta = 2$, $n_s = 8$, $n_t = 4$ to (b) $\beta = 2.25$, $n_s = 16$, $n_t = 6$; the equipotential curves are obtained through fits as described in the text.

Fig. 2 : Determination of the critical temperature.

Fig. 3 : The force between static sources:

(a) for different values of $T/\sqrt{k}$

(b) all confining forces.

We plot the force only in the range between r_0 and r_1 (cf. the table) when our fits apply.

Fig. 1a

Fig. 1b

Fig. 2

Fig. 3a

Fig. 3b