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THE RELAXED HYPERMULTIPLY : AN UNCONSTRAINED

$N = 2$ SUPERFIELD THEORY

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A B S T R A C T

We present an off-shell version of the hypermultiplet, without a central charge or gauge antisymmetric tensors. We give the unconstrained superfield form of the action and its coupling to $N = 2$ super-Yang-Mills theory. This allows an $N = 2$ superspace formulation of the $N = 4$ super-Yang-Mills theory, whose ultra-violet finiteness is then ensured by $N = 2$ non-renormalization theorems.

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1. - INTRODUCTION

Supersymmetric field theories have been shown to have remarkable cancellations of ultra-violet divergences, which are more dramatic the greater the number, N , of supersymmetries. These cancellations are most easily seen in the context of superfield perturbation theory, where supersymmetry is manifest. So far, superfield perturbation theory has been developed only for $N = 1$ supersymmetry. There are two principal difficulties in extending it to $N > 1$. Firstly one needs a complete off-shell representation of supersymmetry including auxiliary fields ; this is the key step towards an unconstrained superfield formulation of the free theory. Secondly, one needs to generalize this to include interactions.

In $N = 2$ supersymmetry an important obstacle has been the lack of an unconstrained superfield formulation for the simplest representation, which is known as the "hypermultiplet" ¹⁾, with four physical scalar and two physical spinor fields. The problem derives from the fact that off-shell multiplets of $N = 2$ supersymmetry either have components of spin $> \frac{1}{2}$ or have a central charge. A representation of the hypermultiplet with a central charge has been known for some time ²⁾, but this does not allow an unconstrained superfield formulation. Another representation, known as the $N = 2$ "linear multiplet" ³⁾, has maximum spin one off-shell but can propagate just spins zero and $\frac{1}{2}$ if one of the spin zero states is represented by a gauge antisymmetric tensor ⁴⁾. The disadvantage of the linear multiplet is that its couplings are very restrictive. In particular it does not allow self-coupling or coupling to spin 1 gauge fields. A way of avoiding these difficulties would be to let the fields of spin $> \frac{1}{2}$ be auxiliary. In this paper we present such a solution, give the unconstrained superfield action, and couple the system to $N = 2$ super-Yang-Mills theory. By putting this new version of the hypermultiplet in the adjoint representation of the Yang-Mills gauge group we obtain an $N = 2$ superspace formulation of the $N = 4$ Yang-Mills theory.

The (self-interacting) $N = 2$ Yang-Mills theory has not yet been formulated in unconstrained superfields, but it is likely that the unconstrained formulation of the free theory ⁵⁾ can be generalized. In this case we would have a complete $N = 2$ superfield formulation of $N = 4$ Yang-Mills theory, so that the $N = 2$ non-renormalization theorems ⁶⁾ can be used. As the authors of Ref. 6) have pointed out, if one had an off-shell $N = 4$ (or 3) superfield version of super-Yang-Mills, ultra-violet finiteness would follow immediately. However, as we shall explain, it is sufficient to have an $N = 2$ superspace formulation.

The conventions used in this paper are summarized, for convenience, in the Appendix.

2. - THE MULTIPLIETS AND THE COMPONENT ACTION

The starting point of our construction is the linear multiplet which is described by a real SU(2) triplet superfield, L^{ij} , satisfying the constraint

$$D_\alpha^{(i} L^{jk)} = 0 \quad (2.1)$$

Its components are L^{ij} , $\lambda_\alpha^i \equiv D_{\alpha j} L^{ij}$, $N = D_{ij} L^{ij}$, and $G_{\alpha\dot{\alpha}} = -i[D_\alpha^i, \bar{D}_{\dot{\alpha}}^j] L_{ij}$, $G_{\alpha\dot{\alpha}}$ being conserved. Since the presence of a conserved vector in the multiplet is what restricts the interactions our strategy will be to relax the constraint (2.1) such that $G_{\alpha\dot{\alpha}}$ is no longer conserved^{*)}. This step requires the introduction of new multiplets which, in general, will also contain conserved tensors. A suitable multiplet⁸⁾ that does not have conserved tensors is described by an SU(2) $\mathbb{5}$ superfield L^{ijkl} , satisfying the similar constraint,

$$D_\alpha^{(i} L^{jklm)} = 0 \quad (2.2)$$

A quick method of finding the components is to use the "tableau calculus"⁹⁾. For the linear multiplet we have

$$L^{ij} = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \quad (2.3)$$

with the constraint

$$D_\alpha^i L^{jh} \sim \begin{array}{|c|} \hline \cdot \\ \hline \end{array} \times \begin{array}{|c|c|} \hline & \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \cdot \\ \hline \end{array} \quad (2.4)$$

where D_α^i is represented by a box with a dot (and $\bar{D}_{\dot{\alpha}i}$ by a box with a cross). The independent components of L^{ij} are therefore seen to be

$$\begin{array}{c} \begin{array}{|c|c|} \hline & \\ \hline \end{array} \\ L^{ij} \end{array}, \begin{array}{c} \begin{array}{|c|c|} \hline & \cdot \\ \hline \end{array} \\ \lambda_\alpha^i \end{array}, \begin{array}{c} \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array} \\ N \end{array}, \begin{array}{c} \begin{array}{|c|c|} \hline \cdot & \times \\ \hline \end{array} \\ G_{\alpha\dot{\alpha}} \end{array} \quad (2.5)$$

Since the singlet scalar is complex it follows that the vector represented by the last diagram must be conserved in order that the number of boson and fermion components match. The superfield L^{ijkl} is represented by

^{*)} This is similar in spirit to the approach advocated in Ref. 7).

$$L^{ijkl} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \quad (2.6)$$

with the constraint

$$\begin{array}{|c|} \hline \cdot \\ \hline \end{array} \times \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \cdot & & & \\ \hline \end{array} \quad (2.7)$$

so that the independent components are

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \cdot & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \cdot & \cdot & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \cdot & \cdot & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \cdot & \cdot & \cdot & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \\ L^{ijk}, \quad \psi_a^{ijk}, \quad M_{ij}, \quad V_{\alpha i}, \quad \bar{\chi}_{\alpha i}, \quad C \quad (2.8)$$

A count of the components now shows that the triplet of vectors represented by the fourth diagram are not conserved. Furthermore the last diagram represents a singlet real scalar that is of the right dimension to become the divergence of the linear multiplet vector if L^{ijkl} has dimension one. L^{ij} is a superspin zero, superisospin zero multiplet with $8+8$ components, while L^{ijkl} is a superspin zero, superisospin one, multiplet with $24+24$ components and is a rearrangement of a triplet of linear multiplets.

The constraint (2.1) implies the absence of a $\underline{4}$ of spinors in the L^{ij} multiplet. Relaxing this constraint will allow a $\underline{4}$ to appear in the multiplet, and we can identify this with the $\underline{4}$ of spinors in the L^{ijk} multiplet. In this way we arrive at a combined multiplet, with $32+32$ components, and no conserved vectors, which we call the "relaxed hypermultiplet". It is described by the superfields L^{ij}, L^{ijkl} subject to the constraints

$$\begin{aligned} D_{\alpha}^{(i} L^{jk)} &= D_{\alpha k} L^{ijkl} \\ D_{\alpha}^{(i} L^{jklm)} &= 0 \end{aligned} \quad (2.9)$$

The components and transformation rules ^{*)} of this multiplet are as follows

*) The transformations of a superfield are given by Q_{α}^i acting on the superfield. If after applying Q_{α}^i we set $\theta = \bar{\theta} = 0$, we get the same result for the variation of the lowest component as if we had applied D_{α}^i . Thus, we define the components of the multiplet to be $(L^{ij})|_{\theta=0}$, $(D_{\alpha j} L^{ij})|_{\theta=0}$, etc., and the transformations are given by (2.10).

$$L^{ij}, L^{ijkl}$$

$$D_{\alpha}^i L^{jk} = \frac{2}{3} \varepsilon^{i(j} \lambda_{\alpha}^{k)} + \psi_{\alpha}^{ijk}; \quad \lambda_{\alpha}^j = D_{\alpha k} L^{jk}, \quad \psi_{\alpha}^{ijk} = D_{\alpha l} L^{ijkl}$$

$$D_{\alpha}^i L^{ijkl} = \frac{4}{5} \varepsilon^{i(j} \psi_{\alpha}^{klm)}$$

$$D_{\alpha}^i \psi_{\beta}^{jkl} = -\frac{3}{8} \varepsilon_{\alpha\beta} \varepsilon^{i(j} M^{kl)}; \quad M^{ij} = -2 D_{kl} L^{ijkl}$$

$$D_{\alpha}^i \lambda_{\beta}^j = -\frac{3}{4} \varepsilon_{\alpha\beta} M^{ij} + \frac{1}{4} \varepsilon_{\alpha\beta} \varepsilon^{ij} N; \quad N = -2 D_{ij} L^{ij}$$

$$\bar{D}_{\alpha\dot{\alpha}} \psi_{\alpha}^{ijk} = -\frac{3i}{8} \delta_{\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}}^{ijk} - \frac{5i}{4} \partial_{\alpha\dot{\alpha}} L_{\alpha}^{ijk};$$

$$V_{\alpha\dot{\alpha}}^{ij} = i [\bar{D}_{\alpha k}, D_{\alpha\dot{\alpha}}] L^{ijkl}$$

$$\bar{D}_{\alpha\dot{\alpha}}^i \lambda_{\alpha}^j = -\frac{3i}{4} V_{\alpha\dot{\alpha}}^{ij} - \frac{i}{4} \varepsilon^{ij} G_{\alpha\dot{\alpha}} - \frac{3i}{2} \partial_{\alpha\dot{\alpha}} L^{ij}; \quad G_{\alpha\dot{\alpha}} = i [\bar{D}_{\alpha j}, D_{\alpha\dot{\alpha}}] L^{ij}$$

$$D_{\alpha}^i M^{jk} = D_{\alpha}^i N = 0$$

$$\bar{D}_{\alpha\dot{\alpha}}^i M^{jk} = \frac{2i}{3} \varepsilon^{i(j} \bar{\xi}_{\alpha\dot{\alpha}}^{k)} + 8i \frac{1}{3} \partial_{\alpha\dot{\alpha}} \psi_{\alpha}^{ijk}; \quad \bar{\xi}_{\alpha\dot{\alpha}}^k = \bar{D}_{\alpha\dot{\alpha}} M^{kl}$$

$$\bar{D}_{\alpha\dot{\alpha}}^i N = 3 \bar{\xi}_{\alpha\dot{\alpha}}^i + 4i \partial_{\alpha\dot{\alpha}} \lambda^{\alpha i}$$

$$D_{\alpha}^i V_{\beta\dot{\beta}}^{jk} = \frac{2}{3} \varepsilon_{\alpha\beta} \varepsilon^{i(j} \bar{\xi}_{\beta\dot{\beta}}^{k)} - \frac{5}{3} \varepsilon_{\alpha\beta} \partial_{\beta\dot{\beta}} \psi_{\alpha}^{ijk} - 2 \partial_{\beta\dot{\beta}} (\varepsilon_{\alpha} \psi_{\beta}^{ijk})$$

$$D_{\alpha}^i G_{\beta\dot{\beta}} = 3i \varepsilon_{\alpha\beta} \bar{\xi}_{\beta\dot{\beta}}^i - 3 \varepsilon_{\alpha\beta} \partial_{\beta\dot{\beta}} \lambda^{\alpha i} - 2 \partial_{\beta\dot{\beta}} (\varepsilon_{\alpha} \lambda_{\beta}^i)$$

$$D_{\alpha}^i \bar{\xi}_{\alpha\dot{\alpha}}^j = i \partial_{\alpha\dot{\alpha}} M^{ij}$$

$$\bar{D}_{\alpha\dot{\alpha}}^i \bar{\xi}_{\beta\dot{\beta}}^j = -\varepsilon_{\alpha\beta} \partial_{\beta\dot{\beta}} V_{\alpha\dot{\alpha}}^{ij} + \frac{1}{6} \varepsilon_{\alpha\beta} \varepsilon^{ij} \partial_{\beta\dot{\beta}} G_{\alpha\dot{\alpha}} + \partial_{\beta\dot{\beta}} (\varepsilon_{\alpha} V_{\beta\dot{\beta}}^{ij})$$

(2.10)

It is clear, e.g., from the presence of the dimension-one component L^{ijkl} , that this multiplet alone cannot be used to write an action for the hypermultiplet. We will need another multiplet with $24+24$ components described by a real scalar superfield V satisfying the constraints

$$D_{\alpha\beta} V = [D_{\alpha}^i, \bar{D}_{\beta i}] V = 0 \quad (2.11)$$

This multiplet is also a superspin zero, superisospin 1 multiplet and is the "contragredient" multiplet to L^{ijkl} in the sense that the components are the same except for the inversion of their dimensions. The components and transformation rules are :

$$\begin{aligned} V \\ D_{\alpha}^i V &= \psi_{\alpha}^i \\ D_{\alpha}^i \psi_{\beta}^j &= -\frac{1}{2} \varepsilon_{\alpha\beta} K^{ij} \quad ; \quad K^{ij} = -2 D^{ij} V \\ \bar{D}_{\alpha}^i \psi_{\beta}^j &= -\frac{i}{2} A_{\beta\alpha}^{ij} + \frac{i}{2} \varepsilon^{ij} \partial_{\beta\alpha} V \quad ; \quad A_{\alpha\alpha}^{ij} \equiv i [\bar{D}_{\alpha}^i, D_{\alpha}^j] V \\ D_{\alpha}^i K^{jk} &= 0 \\ \bar{D}_{\alpha}^i K^{jk} &= \bar{\xi}_{\alpha}^{ijk} - 4\frac{i}{3} \varepsilon^{i(j} \partial_{\alpha} \psi^{\alpha j)} \quad ; \quad \bar{\xi}_{\alpha}^{ijk} = \bar{D}_{\alpha}^i K^{jk} \\ D_{\alpha}^i A_{\beta\beta}^{jk} &= i \varepsilon_{\alpha\beta} \bar{\xi}_{\beta}^{ijk} + \frac{1}{3} \varepsilon_{\alpha\beta} \varepsilon^{i(j} \partial_{\beta} \psi^{\alpha j)} + 2 \partial_{\beta(\alpha} \varepsilon^{i(j} \psi_{\beta}^{k)} \\ D_{\alpha}^i \bar{\xi}_{\alpha}^{jkl} &= -i \partial_{\alpha\alpha} \varepsilon^{i(j} K^{kl)} \\ \bar{D}_{\alpha}^i \bar{\xi}_{\beta}^{jkl} &= \varepsilon^{i(j} \partial_{\alpha} (\bar{\xi}_{\beta}^{\alpha k l)} + \frac{1}{4} \varepsilon_{\alpha\beta} \varepsilon^{i(j} \partial^{\alpha} \bar{\xi}^{kl)} \\ &\quad + \frac{1}{2} \varepsilon_{\alpha\beta} C^{ijkl} \quad ; \quad C^{ijkl} = \bar{D}_{\alpha}^i \bar{\xi}_{\alpha}^{jkl} \\ D_{\alpha}^i C^{jklm} &= -2i \varepsilon^{i(j} \partial_{\alpha\beta} \bar{\xi}_{\beta}^{klm)} \end{aligned} \quad (2.12)$$

The action for the off-shell hypermultiplet may now be written as the sum of two separate super-invariants,

$$I = I_1 + I_2 \quad (2.13)$$

where

$$\begin{aligned} I_1 = \int d^4x \{ & L_{ij} \square L^{ij} - \frac{25}{6} L_{ijkl} \square L^{ijkl} + 8 \frac{1}{4} \lambda^{\alpha i} \partial_{\alpha i} \bar{\lambda}^{\dot{i}}_{\dot{i}} \\ & - 16 \frac{1}{3} \psi^{\alpha ijk} \partial_{\alpha i} \bar{\psi}^{\dot{i}}_{\dot{i}jk} - \frac{3}{2} L_{ij} \partial^{\alpha i} V_{\alpha i}^{\dot{j}} - \frac{2}{3} (\lambda^{\alpha i} \xi_{\alpha i} + \bar{\lambda}_{\alpha i} \bar{\xi}^{\alpha i}) \\ & - \frac{3}{4} M^{ij} \bar{M}_{ij} + \frac{1}{19} N \bar{N} + \frac{3}{8} V^{\alpha i j} V_{\alpha i j} - \frac{1}{36} G^{\alpha i} G_{\alpha i} \} \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} I_2 = \int d^4x \{ & \frac{1}{6} V \partial^{\alpha i} G_{\alpha i} + (\psi^{\alpha i} \xi_{\alpha i} + \bar{\psi}_{\alpha i} \bar{\xi}^{\alpha i}) \\ & + \frac{3}{4} (K^{ij} \bar{M}_{ij} + \bar{K}_{ij} M^{ij}) - \frac{3}{4} A_{\alpha i}^{\dot{j}} V^{\alpha i}_{\dot{j}} \\ & - 2 (\xi^{\alpha ijk} \psi_{\alpha ijk} + \bar{\xi}_{\alpha ijk} \bar{\psi}^{\alpha ijk}) + \frac{5}{4} (ijk) L_{ijk} \} \end{aligned} \quad (2.15)$$

The equations of motion resulting from I_1 alone propagate L_{ij} , L^{ijkl} and ψ_{α}^{ijk} , i.e., twice the number of states required for the hypermultiplet. With the addition of I_2 one sees that L^{ijkl} and ψ_{α}^{ijk} are suppressed while λ_{α}^i is set equal to ψ_{α}^i and propagates and V acquires a kinetic term on eliminating $G_{\alpha\dot{\alpha}}$. The propagating fields are therefore just those of the hypermultiplet. Notice that the final result recalls the supersymmetric Lagrange multiplier approach ⁴⁾ in which, however, there was an off-shell central charge.

3. - UNCONSTRAINED SUPERFIELDS

In order to write the model of (2.13) in terms of unconstrained superfields we must solve the constraints on L^{ij} , L^{ijkl} , and V . The highest component of the multiplet (2.10) is the dimension 5/2 spinor $\xi_{\alpha i}$, which must be the last component of an unconstrained prepotential $\rho_{\alpha i}$, which therefore has dimension -3/2. In terms of $\rho_{\alpha i}$ the solutions to the constraints of (2.9) are

$$L^{ijkl} = \frac{2}{5} D^{(ij} \bar{D}^{kl)} X, \quad X = D_\alpha^i \rho_i^\alpha + \bar{D}_{\dot{\alpha}}^i \bar{\rho}_{\dot{\alpha}}^i$$

$$L^{ij} = \frac{3}{2} \bar{D}^{ij} D_\alpha^k \rho_k^\alpha + \frac{1}{2} \bar{D}_k^{(i} D^{j)k} D_\alpha^l \rho_l^\alpha + c.c.$$

(3.1)

The solution to the constraint V is

$$V = D^{ij} \bar{D}^{kl} X_{ijkl}, \quad X_{ijkl} = X_{(ijkl)}, \text{ real}$$

(3.2)

In terms of $\rho_{\alpha i}$ and X_{ijkl} the actions I_1 and I_2 of Section 2 are (up to an over-all factor)

$$I_1 = \int d^4x d^8\theta \{ \lambda^{\alpha i} \rho_{\alpha i} + c.c. \}, \quad \lambda_{\dot{\alpha}}^i = D_{\dot{\alpha} j} L^{ij}(\rho)$$

(3.3)

$$I_2 = \int d^4x d^8\theta L^{ijkl}(\rho) X_{ijkl}$$

(3.4)

The superspace equations of motion are

$$L^{ijkl} = 0$$

$$\lambda_{\alpha i} = D_{\alpha i} V$$

(3.5)

from which the component equations immediately follow using (2.10) and (2.12).

The action admits a large number of gauge transformations. These can be obtained in two steps. Firstly, the constraints on $V(L^{ijkl})$ determine the gauge transformations of $X(X_{ijkl})$ by demanding that I_2 be left invariant. These are

$$\delta X = D_{\alpha\beta} T^{\alpha\beta} \Rightarrow \delta \rho_i^\alpha = D_{\beta i} T^{\alpha\beta}, \quad T_{\alpha\beta} = T_{\beta\alpha}$$

(3.6)

$$\delta X = [D_{\alpha}^i, \bar{D}_{\dot{\alpha} i}] Z^{\alpha\dot{\alpha}} \Rightarrow \delta \rho_i^\alpha = \bar{D}_{\dot{\alpha} i} Z^{\alpha\dot{\alpha}}, \quad Z_{\alpha\dot{\alpha}} \text{ real}$$

(3.7)

$$\delta X_{ijkl} = D_\alpha^m \Lambda_{ijklm}^{\alpha} + c.c. \quad \Lambda_{ijklm}^{\alpha} = \Lambda_{ijklm}^{\alpha} \quad (3.8)$$

Secondly there are those transformations of $\rho_{\alpha i}$ that leave $X(\rho)$ invariant. These are

$$\delta \rho_{\alpha i} = D_{\alpha i} T$$

$$\delta \rho_{\alpha i} = D^{\beta j} T_{\alpha \beta ij} ; T_{\alpha \beta ij} = T_{\alpha \beta ij}$$

$$\delta \rho_{\alpha i} = \bar{D}_{\dot{\alpha}}^j Z^{\alpha}{}_{\dot{\alpha} ij} ; Z^{\alpha}{}_{\dot{\alpha} ij} = Z^{\alpha}{}_{\dot{\alpha} ij}, \text{ real}$$

$$\delta \rho_{\alpha i} = i \bar{D}_{\dot{\alpha}}^j Y_{\alpha \dot{\alpha} ij} ; \partial^{\mu \nu} Y_{\alpha \dot{\alpha} ij} = 0, Y_{\alpha \dot{\alpha} ij} \text{ real} \quad (3.9)$$

However, not all of these are invariances of I_1 . Demanding that I_1 be invariant further constrains the parameters T and $Y_{\alpha \dot{\alpha}}$ as follows

$$Y_{\alpha \dot{\alpha}} = 0$$

$$\bar{D}^{\dot{\alpha} j} D^{\beta k} T + c.c. = 0 \quad (3.10)$$

The remaining independent piece of T is $T = i \bar{D}^{k\ell} S_{k\ell}$, with $S_{k\ell}$ real, so the full list of invariances of the action $I_1 + I_2$ is

$$\delta \rho_i^{\alpha} = D_{\beta i} T^{\alpha \beta}$$

$$\delta \rho_i^{\alpha} = \bar{D}_{\dot{\alpha} i} Z^{\alpha \dot{\alpha}}$$

$$\delta \rho_i^{\alpha} = D_{\beta}^j T_{ij}^{\alpha \beta}$$

$$\delta \rho_i^{\alpha} = \bar{D}_{\dot{\alpha} j} Z^{\alpha \dot{\alpha} ij}$$

$$\delta \rho_i^{\alpha} = i D_{\beta}^j \bar{D}^{\dot{\beta} k} S_{jk}, \delta X_{ijkl} = D_{\alpha}^m \Lambda_{ijklm}^{\alpha} + c.c. \quad (3.11)$$

Notice that the constraint on the parameter T in (3.10) is precisely of the form of a linear multiplet as to be expected. Also, since the last component of X is a divergence $\partial^{\alpha \dot{\alpha}} G_{\alpha \dot{\alpha}}$, the L^{ijkl} multiplet has an invariance that projects out the transverse part of $G_{\alpha \dot{\alpha}}$. This corresponds to the transverse $Y_{\alpha \dot{\alpha}}$ parameter of (3.9) which must be absent in the full theory.

The action I_2 of (3.4) can be written as

$$I_2 = \int d^4x d^4\theta V X \quad (3.12)$$

In the version of the hypermultiplet presented above V is constrained, but we now show that this is not essential. If V is unconstrained the field equations become

$$\begin{aligned} X &= 0 \\ \lambda_{\dot{\alpha}}^i &= D_{\dot{\alpha}}^i V \end{aligned} \quad (3.13)$$

which lead to the same x space equations. This version of the model has more components because the gauge transformations of $\rho_{\alpha i}$ are only those that leave $X(\rho)$ invariant, i.e., those of (3.11) with $T_{\alpha\beta} = Z_{\alpha\dot{\alpha}} = 0$. This form of the action is reminiscent of the superspace action of linearized ten-dimensional supergravity¹⁰⁾.

Finally we note that there is an $N = 1$ analogue of our $N = 2$ model. Suppose we relax the constraint $D^2 L = 0$ that defines an $N = 1$ linear multiplet by writing

$$D^2 L = i D^2 G \quad (3.14)$$

where G is a general scalar multiplet. While the constraint on L is relaxed, G is constrained such that its last component is the divergence of the vector in L just as for the $N = 2$ case. The solution to the constraint (3.14) is

$$\begin{aligned} L &= -i (D^\alpha \rho_\alpha - \bar{D}_{\dot{\alpha}} \bar{\rho}^{\dot{\alpha}}) \\ G &= D^\alpha \rho_\alpha + \bar{D}_{\dot{\alpha}} \bar{\rho}^{\dot{\alpha}} \end{aligned} \quad (3.15)$$

By analogy with our $N = 2$ case we take the action to be

$$I = i \int d^4x d^4\theta (\lambda^\alpha(\rho) \rho_\alpha + \text{c.c.}) + \int d^4x d^4\theta G(\rho) V \quad (3.16)$$

where $\lambda^\alpha = D^\alpha L$ and V is an unconstrained real Lagrange multiplier field. The field equations are

$$G = 0, \quad i D_\alpha L = D_\alpha V \quad (3.17)$$

These equations together with the constraint (3.14) imply that $\bar{D}^2 \bar{\phi} = 0$ where $\phi = V - iL$ is chiral. This shows that (3.16) is equivalent to the usual Wess-Zumino model for a chiral field. Notice that the constraint (3.14) can be written as $D^2(L - iG) = 0$ which makes it clear that the multiplet in question is a "complex linear" one.

4. - YANG-MILLS COUPLING

$N = 2$ super-Yang-Mills theory is described in superspace ¹¹⁾ by a potential $\mathcal{A}_A = \{\mathcal{A}_\alpha^i, \bar{\mathcal{A}}_{\dot{\alpha}i}, \mathcal{A}_{\alpha\dot{\alpha}}\}$ in the adjoint representation of the gauge group G . The field strengths F_{AB} are constrained by

$$\begin{aligned} F_{\alpha\beta}^{ij} &= \varepsilon_{\alpha\beta} \varepsilon^{ij} \bar{W} \\ F_{\alpha\dot{\beta}}^{ij} &= 0 \end{aligned} \quad (4.1)$$

The Bianchi identities imply that $\bar{W}$ is covariantly chiral,

$$D_\alpha^i \bar{W} = 0, \quad D_\alpha^i = D_\alpha^i - i[\mathcal{A}_\alpha^i, \quad] \quad (4.2)$$

and satisfies the higher order constraint

$$D_\alpha^i W - \bar{D}^{\dot{\alpha}i} \bar{W} = 0 \quad (4.3)$$

W is the only possible covariantly chiral superfield (in the absence of central charges). For example for ϕ in the adjoint representation of G ,

$$D_\alpha^i \bar{\phi} = 0 \Rightarrow [\bar{W}, \bar{\phi}] = 0 \quad (4.4)$$

On the other hand the integrability condition for the $N = 2$ linear multiplet L^{ij} satisfying the gauge covariant constraint

$$D_\alpha^{(i} L^{jk)} = 0 \quad (4.5)$$

is that $F_{(\alpha\beta)}^{(ij)} = 0$, which is satisfied. Similarly the covariant constraints on L^{ijkl} and V are consistent with Yang-Mills coupling.

We should point out here that although the linear multiplet constraint is consistent with the Yang-Mills coupling it cannot be solved locally. This is because the conserved vector $G_{\alpha\dot{\alpha}}$ of the free multiplet is now covariantly conserved. This is not a problem of the L^{ijkl} or V multiplets or of the "relaxed hypermultiplet" whose gauge covariant constraints are now

$$\begin{aligned} \mathcal{D}_{\alpha}^{(i} L^{jk)} &= \mathcal{D}_{\alpha\dot{\alpha}} L^{ijk\dot{\alpha}} \\ \mathcal{D}_{\alpha}^{(i} L^{jklm)} &= 0 \end{aligned} \quad (4.6)$$

Since we wish to couple the relaxed hypermultiplet in such a way as to obtain an $N = 2$ formulation of the $N = 4$ Yang-Mills theory we shall choose the multiplets to be in the adjoint representation. The solutions to the constraints (4.6) are

$$\begin{aligned} L^{ijkl} &= \frac{2}{5} \mathcal{D}^{(ij} \bar{\mathcal{D}}^{kl)} X, \quad X = \mathcal{D}_{\alpha}^i \rho_i^{\alpha} + c.c. \\ L^{ij} &= \frac{3}{2} \bar{\mathcal{D}}^{ij} \mathcal{D}_{\alpha}^k \rho_k^{\alpha} + \frac{1}{2} \bar{\mathcal{D}}_k^{(i} \mathcal{D}^{j)k} \mathcal{D}_{\alpha}^m \rho_m^{\alpha} \\ &\quad - \frac{3}{4} \bar{\mathcal{D}}^{ij} [\bar{W}, \mathcal{D}_{\alpha}^m \rho_m^{\alpha}] + c.c. \end{aligned} \quad (4.7)$$

Similarly for V , we have the gauge covariant constraints

$$\mathcal{D}_{\alpha\dot{\beta}} V = [\mathcal{D}_{\alpha}^i, \bar{\mathcal{D}}_{\dot{\alpha}i}] V = 0 \quad (4.8)$$

with the solution

$$V = \mathcal{D}^{(ij} \bar{\mathcal{D}}^{kl)} X_{ijkl} \quad (4.9)$$

The coupled action is

$$I = \int d^4x d^8\theta \{ \lambda^{\alpha i} p_{\alpha i} + \text{c.c.} + L^{ijkl} \chi_{ijkl} \} \quad (4.10)$$

with $\lambda^{\alpha i} = D^\alpha_j L^{ij}$.

The gauge invariances of (4.10) for $p_{\alpha i}$ and X_{ijkl} are just the covariantizations of (3.11), with the exception of the S_{kl} transformation which now takes the form

$$\delta p_{\alpha i} = i D_{\alpha i} \bar{D}^{jk} S_{jk} - [W, D_{\alpha}^k S_{ik}] + [D_{\alpha}^k W, S_{ik}] \quad (4.11)$$

To obtain the $N = 4$ Yang-Mills action we must add the $N = 2$ Yang-Mills action which is

$$I = \left[\int d^4x d^4\theta W^2 \right] + \text{c.c.} \quad (4.12)$$

In the linearized limit W may be written in terms of a prepotential U_{ij} ⁵⁾

$$W = \bar{D}^k D^j U_{ij} \quad (4.13)$$

Although the generalization of (4.13) to the interacting case is not yet known it is likely that it exists. Given that this is so, we can take over the results of Grisaru and Siegel ⁶⁾ that the only counter terms beyond one loop are integrals over the full superspace of local products of quantities covariant with respect to the constrained background fields, i.e., $\mathcal{A}_\alpha^i$, L^{ij} , L^{ijkl} , V . On dimensional grounds there are no such counter terms. Hence the only primitive ultra-violet divergences occur at one loop. However, we know from component and $N = 1$ superfield calculations that the one-loop β function vanishes. Hence we conclude that it vanishes to all orders. Notice that this use of non-renormalization theorems is an improvement with respect to the $N = 1$ argument ¹²⁾ where it is necessary to assume that $O(4)$ invariance is preserved and that it implies a single renormalization constant.

5. - COMMENTS

The problem of the off-shell hypermultiplet without central charges was one of the apparently intractable auxiliary field problems of extended supersymmetric field theories. The fact that a solution exists is another encouraging sign that auxiliary fields may indeed exist for all theories of interest. The main significance of this result is that it enables one to consider extended superspace perturbation theory. We have not discussed Feynman rules for $N = 2$ superfields in this paper. Indeed, apart from the fact that we do not yet have a completely satisfactory formulation of the $N = 2$ Yang-Mills theory, the hypermultiplet sector will have a complicated ghost structure that would make actual calculations very cumbersome. However, as we have seen, the power of the $N = 2$ non-renormalization theorems can make calculations unnecessary, at least with regard to ultra-violet divergences.

One peculiarity of the relaxed hypermultiplet model is that it is not possible to add a mass term, without extending the multiplet, because a mass would require either the presence of a central charge or that higher spins propagate. We do not expect this to be a problem for infra-red regularization, however, for which one may use dimensional regularization. There is one way of introducing a massive parameter within the $(56+56)$ component multiplet which is to add an action with a D'Alembertian inserted into each term. The resulting higher derivative action then propagates many massive ghosts in addition to the massless ones. If one analyzes these states for the combined hypermultiplet $N = 2$ Yang-Mills system one finds that all the states, both massless and massive fall into $N = 4$ representations. The massive ghost states belong to representations with a central charge proportional to the mass. This would normally be considered a signal that the field representation is off-shell $N = 4$ supersymmetric with a central charge. However, it does not appear to be possible to define locally $N = 4$ transformations on the $64+64$ multiplet of fields.

In this paper we have restricted our attention to Yang-Mills coupling in the adjoint representation, which means that the Abelian limit is a free theory. This problem can be overcome by complexifying the relaxed hypermultiplet and we do not anticipate any inconsistencies of the coupling. Such an Abelian model is of interest because in the absence of the Maxwell kinetic term we expect to obtain an $N = 2$ superfield formulation of a hyper-Kähler σ model. This is because we expect to be able to eliminate the Maxwell fields, at least in the component model, to obtain an interacting $N = 2$ supersymmetric (in four dimensions) model of scalars and spinors. All such models are hyper-Kähler σ models¹³⁾. In two

dimensions, these models have been shown to be finite on-shell because of the absence of (x space) counter terms ¹³⁾. The advantage of a superfield formulation is that supersymmetry is manifest ; finiteness should follow from the non-renormalization theorems.

Finally, we may consider coupling the relaxed hypermultiplet to $N = 2$ supergravity. The constraints are consistent with the $N = 2$ superspace geometry ¹⁴⁾, so we anticipate no problems. We point out that we do not expect to be able to use the relaxed hypermultiplet (as a compensating multiplet) to obtain an essentially new off-shell $N = 2$ supergravity. This is because the supergravity coupling will require the term DV in the action, where D is the dimension three scalar of the $32+32$ minimal multiplet ¹⁴⁾. Hence the V field equation will no longer set $\partial^{\alpha\dot{\alpha}}G_{\alpha\dot{\alpha}}$ to zero, but rather equate it to D . One can therefore eliminate both of the $24+24$ multiplets and regain the standard $40+40$ multiplet ¹⁵⁾.

APPENDIX

Our conventions for two component spinors are

$$\psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \quad \psi_\alpha = \psi^\beta \varepsilon_{\beta\alpha}, \quad \varepsilon^{\alpha\beta} \varepsilon_{\alpha\gamma} = \delta^\beta_\gamma \quad (\text{A.1})$$

and similarly for dotted indices. Complex conjugation acts as

$$(\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}}, \quad (\varepsilon^{\alpha\beta})^* = \varepsilon^{\dot{\alpha}\dot{\beta}} \quad (\text{A.2})$$

For SU(2) indices we have

$$V^i = \varepsilon^{ij} V_j, \quad V_i = V^j \varepsilon_{ji}, \quad \varepsilon^{ij} \varepsilon_{ik} = \delta^j_k \quad (\text{A.3})$$

Complex conjugation raises or lowers SU(2) indices, but note that indices on a field must be assigned a "natural" position. For example, if

$$(V^i)^* = \bar{V}_i \quad (\text{A.4})$$

then

$$(V_i)^* = -\bar{V}^i = -\varepsilon^{ij} \bar{V}_j \quad (\text{A.5})$$

Note that by "reality" for a fieldlike L^{ij} we mean

$$\bar{L}_{ij} = L^{kl} \varepsilon_{ki} \varepsilon_{lj}$$

The basic anticommutators for the Yang-Mills gauge covariant spinor derivatives are

$$\begin{aligned} \{D_\alpha^i, \bar{D}_{\dot{\beta}j}\} &= i \delta_j^i D_{\alpha\dot{\beta}} \\ \{D_\alpha^i, D_{\dot{\beta}j}\} &= -i \varepsilon_{\alpha\beta} \varepsilon^{ij} [\bar{W}_{\beta\dot{\beta}}] \end{aligned} \quad (\text{A.6})$$

The products of $\mathbb{D}$'s are given by

$$\mathbb{D}_\mu^i \mathbb{D}_\rho^j = \varepsilon_{\alpha\beta} \mathbb{D}^{ij} + \varepsilon^{ij} \mathbb{D}_{\alpha\beta} - i/2 \varepsilon_{\alpha\beta} \varepsilon^{ij} [\bar{W}, \cdot]$$

$$\mathbb{D}_\mu^i \mathbb{D}^{jk} = 2 \varepsilon^{ij} \mathbb{D}^{3k}$$

$$\mathbb{D}_\alpha^i \mathbb{D}_{\beta\gamma} = -2 \varepsilon_{\alpha(\beta} \mathbb{D}^{3\gamma)} + i \varepsilon_{\alpha(\beta} [\bar{W}, \mathbb{D}_{\gamma)}^i]$$

$$\mathbb{D}_\mu^i \mathbb{D}_\rho^{3j} = \varepsilon_{\alpha\beta} \varepsilon^{ij} \mathbb{D}^4 + i/2 \varepsilon_{\alpha\beta} [\bar{W}, \mathbb{D}^{ij}]$$

$$\mathbb{D}_\alpha^{3i} \mathbb{D}_\rho^j = \varepsilon_{\alpha\rho} \varepsilon^{ij} \mathbb{D}^4 + i/2 \varepsilon^{ij} [\bar{W}, \mathbb{D}_{\alpha\rho}]$$

$$\mathbb{D}_\mu^i \mathbb{D}^4 = -i/2 [\bar{W}, \mathbb{D}_\mu^{3i}]$$

(A.7)

The $\bar{\mathbb{D}}$ products follow by complex conjugation.

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