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CHIRAL WARD IDENTITIES, EFFECTIVE ACTIONS
AND MESON-GLUEBALL MIXING *)

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A B S T R A C T

The low energy spectrum of 1^+ and 0^- mesons in one-flavour QCD is investigated. It is shown that there exist two light pseudoscalar mesons, which are linear combinations of pure quark and pure glueball states. Simple relations implied by the axial anomaly, for their decay constants are confirmed. The mass generation mechanism for the lightest pseudoscalar meson is identified as due to the mixing of the operators $i\bar{\psi}\gamma_5\psi$ and $(-g^2/16\pi^2)*F_{\mu\nu}F_{\mu\nu}$, and the U(1) problem in this model is resolved. To establish these results, new functional methods are developed to determine the effective action for the appropriate fields in a low energy approximation. A novel feature of our work is the introduction of a "symmetric" effective action, defined using a non-standard Legendre transform. The anomalous chiral Ward identities imply a simple invariance property for this functional, enabling it to be constructed in a particularly systematic way. Finally, the relation of our general results to the literature on effective Lagrangians in the $1/N$ approximation is discussed.

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1. - INTRODUCTION

Despite considerable recent progress¹⁾ in the development of non-perturbative methods, an analytical derivation of the low energy spectrum of quantum chromodynamics from first principles remains beyond the reach of current techniques. Nevertheless, a great deal of understanding of the spectrum can be obtained by indirect methods which focus on the symmetry properties of the underlying theory. Amongst these the method of phenomenological or effective Lagrangians²⁾ has had considerable success, notably in summarizing and extending the results of current algebra.

The basic idea of this approach is to construct a Lagrangian in which the fundamental fields correspond directly to the physical particle states, and which incorporates as fully as possible the symmetries which the underlying theory is expected to exhibit. Supplementing these symmetry constraints with certain comparatively simple dynamical assumptions then determines the form of an effective Lagrangian which may be used to explore the low energy dynamics of the theory.

In this paper, we follow the philosophy outlined above to study the implication of spontaneously broken chiral symmetry for the spectrum of light bound states, especially $J^P = 0^-$ mesons, in quantum chromodynamics (QCD). We are particularly concerned with clarifying the way in which the existence of the axial anomaly affects the spectrum. For simplicity, we only consider here a gauge theory coupled to one flavour of massless quarks, since this case is sufficient to expose most of the novel physics while remaining technically simple.

The low energy bound state spectrum is deduced from the two-point Green functions of appropriate field operators, the particle states appearing as poles in the propagator matrix. We must therefore decide on physical grounds which operators are likely to be relevant, and are guided by the following two fundamental assumptions about the dynamics of QCD. First we assume that QCD exhibits colour confinement, i.e., that only colour singlet states appear in the physical spectrum. This allows us to restrict our attention to gauge invariant operators. Second we assume that due to the strong colour dynamics, the global chiral symmetry [here simply $U_L(1) \times U_R(1)$] is spontaneously broken [in this case to $U(1)$]. Now, in the absence of an anomaly the axial vector current $j_\mu^5 = i\bar{\psi}\gamma_\mu\gamma_5\psi$ would be conserved and the spontaneous breaking of the $U_5(1)$ symmetry would imply the existence of a massless pseudoscalar Goldstone boson coupled to the operator $\phi_5 = i\bar{\psi}\gamma_5\psi$. However, because of the axial anomaly^{3),4)} j_μ^5 is not conserved, and in fact $\partial_\mu j_\mu^5 = (g^2/16\pi^2) *F_{\mu\nu}^a F_{\mu\nu}^a$. This result allows us to evade the consequences of Goldstone's theorem. Nonetheless, we still make the hypothesis that such a 0^- meson exists and is light, acquiring mass due to the anomaly alone. The mechanism for this mass generation, essentially a mixing phenomenon, is made clear in the paper.

This line of reasoning leads us to consider the Green functions of the operators j_μ^5 , $q = (-g^2/16\pi^2)*F_{\mu\nu}^a F_{\mu\nu}^a$, ϕ_5 and $\phi = \bar{\psi}\psi$, associated with the spontaneous breaking of chiral symmetry and with the anomaly. We may reasonably expect that the study of these operators will uncover at least part of the low energy spectrum of QCD.

To implement these ideas formally, we define the generating functional W for connected Green functions of these operators, and using functional methods derive the appropriate anomalous Ward identity for broken chiral symmetry. The integrated form of this Ward identity reveals a simple invariance property of W . We cannot, however, formulate a low energy approximation to the dynamics based directly on W , since it does not admit a small momentum expansion. For this, we need the corresponding effective action Γ , the generating functional for one-particle-irreducible vertices. The propagator matrix, i.e., the matrix of two point Green functions of the operators of interest, is then simply given as the inverse of the matrix of second functional derivatives of Γ .

In fact we find it extremely helpful to define an intermediate functional we call the symmetric effective action $\hat{\Gamma}$, which is obtained from W by a non-standard form of Legendre transform. This novel transform is designed so that $\hat{\Gamma}$ satisfies a particularly simple Ward identity, closely analogous to that for W . Integrating this identity, we find that $\hat{\Gamma}$, just like W , possesses a remarkably simple invariance property, which encapsulates all the information contained in the anomalous Ward identity. It follows directly that $\hat{\Gamma}$ must be constructed from invariants of this symmetry transformation.

We therefore have a clear and well-defined prescription for constructing $\hat{\Gamma}$. The choice of invariants which must be included depends of course on certain dynamical assumptions. Here we make a low energy approximation by restricting $\hat{\Gamma}$ to include only terms with two or fewer derivatives acting on the fields. This completely determines $\hat{\Gamma}$. It is then a relatively straightforward task to relate the second functional derivatives of $\hat{\Gamma}$ to those for the true effective action Γ , and thus determine the propagator matrix.

The low energy approximation we make allows the inclusion of kinetic terms for all the fields in the effective action, and it seems natural to us to do so. As a consequence we determine the low mass spectrum in the sector described here to comprise one 1^+ and one 0^+ meson, but two 0^- mesons. These two 0^- states are linear combinations of pure quark and pure glueball states. In a realistic model along the same lines, our effective action would therefore predict the existence of a new pseudoscalar meson similar to the η' , and would interpret both these mesons as different mixtures of pure quark and glueball states. In addition, the mass generation mechanism for the η' provides a resolution of the original $U(1)$ problem⁵⁾⁻⁸⁾.

From the propagator matrix we can deduce not only the spectrum and the masses of the physical bound states but also their decay constants. We find that the chiral symmetry enforces two relations amongst these.

All this analysis involves only minimal dynamical assumptions, and in particular no use is made of detailed dynamical mechanisms involving instantons⁹⁾, or of the $1/N$ expansion¹⁰⁾. It is remarkable that we can learn so much by a systematic exploitation of the chiral symmetry properties of the theory, and illustrates the power of the effective action method. Nevertheless, in order to make contact with the existing literature in which extensive use is made of the $1/N$ expansion, we discuss in Section 6 the leading order $1/N$ approximations to our results, and in particular recover Witten's formula for the mass of the η' .

There has been considerable literature¹¹⁾⁻²²⁾ in the last two or three years on the subject of effective Lagrangians and the $U(1)$ problem, notably by Arnowitt and Nath¹⁸⁾, di Vecchia and Veneziano¹⁴⁾, Rosenzweig, Schechter and Trahern¹⁵⁾ and Witten^{11),16)}. Most of this work has centred on the ad-hoc construction of effective Lagrangians which satisfy the counting rules of the $1/N$ expansion, or alternatively, in the work of Arnowitt and Nath, satisfy certain properties extracted from current algebra, and which incorporate the effects of the axial anomaly. The systematic methods given here for constructing effective actions which exploit fully the constraints of chiral symmetry seem to us to be superior to these previous approaches. Further, our discussion of the dynamics of the extra 0^- meson and the mixing of the glueball and pure quark states is new. However, unlike these authors, we have not as yet extended our methods to include more than one quark flavour. Nor have we included quark mass terms, which in addition to breaking the flavour symmetry, give the theory a non-trivial dependence on the QCD vacuum angle θ .

The paper is organized as follows. In Section 2, we derive the anomalous chiral Ward identity for the generating functional W . In Section 3 we define, and derive Ward identities for, the symmetric effective action $\hat{\Gamma}$. A self-contained general discussion of the novel Legendre transform is given in Appendix A. Section 4 discusses the construction and interpretation of the propagator matrix under very general assumptions. Then in Section 5 the precise form of the symmetric effective action is written down. The $1/N$ expansion is introduced in Section 6 and Appendix B. Finally, in Section 7, we review our results, assess the value of the methods developed here, and discuss the ways in which these may be generalized.

2. - CHIRAL WARD IDENTITIES FOR W

We are concerned in this paper with the implications of broken chiral symmetry in an $SU(N)$ colour gauge theory coupled to one flavour of massless quarks. In particular we are interested in the masses and decay constants of the lowest mass pseudoscalar and pseudovector bound states in the theory.

The effects of chiral symmetry are expressed in field theory through appropriate Ward identities. In this section we derive and explore some immediate consequences of, the chiral Ward identity for the generating functional W of the connected Green functions of the operators which couple to these bound states.

The Lagrangian of the theory is,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \bar{\psi} \gamma_\mu D_\mu \psi - \theta \frac{g^2}{32\pi^2} {}^*F_{\mu\nu}^a F_{\mu\nu}^a \quad (2.1)$$

where the covariant derivative $D_\mu = \partial_\mu - ig t^a A_\mu^a$, where t^a are the fundamental representation matrices of $SU(N)$, and ${}^*F_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}^a$ where $\epsilon_{0123} = 1$ and the metric has signature (++++).

Under a chiral transformation,

$$\begin{aligned} \psi &\rightarrow e^{i\alpha\gamma_5} \psi \\ \bar{\psi} &\rightarrow \bar{\psi} e^{i\alpha\gamma_5} \end{aligned} \quad (2.2)$$

the Lagrangian changes by an amount,

$$\delta\mathcal{L} = -(\partial_\mu \alpha) j_\mu^5 \quad (2.3)$$

where the (chiral invariant) pseudovector current $j_\mu^5 = i\bar{\psi}\gamma_\mu\gamma_5\psi$. We are also concerned with the operators $j_\pm = \bar{\psi}(1 \pm \gamma_5)\psi$ which transform as

$$j_\pm \rightarrow e^{\pm 2i\alpha} j_\pm \quad (2.4)$$

or equivalently the linear combinations ϕ and ϕ_5 defined by $j_\pm = \phi \mp i\phi_5$, i.e., $\phi_5 = i\bar{\psi}\gamma_5\psi$ and $\phi = \bar{\psi}\psi$. In addition, the operator $q = (-g^2/16\pi^2) {}^*F_{\mu\nu}^a F_{\mu\nu}^a$ will play a central role in our analysis.

The masses and decay constants of the bound states will eventually be deduced from the Green functions of the gauge invariant operators $j_\mu^5(x)$, $q(x)$, $j_+(x)$, $j_-(x)$. To study these Green functions we introduce sources $S_\mu^5(x)$, $\theta(x)$, $S_+(x)$, $S_-(x)$, and consider the generating functional,

$$\begin{aligned} \exp W &= Z \\ &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp \int dx \left(\mathcal{L}(\theta=0) + \bar{S}_\mu^5 j_\mu^5 + \theta q + S_+ j_+ + S_- j_- \right) \end{aligned} \quad (2.5)$$

We omit details of the regularization and gauge fixing in Eq. (2.5) for simplicity. The various functional derivatives of $W[S_\mu^5, \theta, S_+, S_-]$, evaluated at the physical values of the sources, viz. $j_\mu^5(x) = 0$, $\theta(x) = \theta/2$ (the constant vacuum angle), $S_+(x) = 0$, $S_-(x) = 0$, yield the connected Green functions of the operators. This technical device of regarding the QCD vacuum angle as the physical value of a space-time dependent source coupling to $q(x)$, although slightly unusual, is extremely convenient in this analysis.

The Ward identity for W is derived by demanding invariance of the functional integral under the change of variables (2.2). This gives

$$\begin{aligned} 0 &= \frac{\delta W}{\delta \alpha(x)} \\ &= Z^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \left(\partial_\mu j_\mu^5 + q + 2i S_+ j_+ - 2i S_- j_- \right) \\ &\quad \times \exp \int dx \left(\mathcal{L}(\theta=0) + \bar{S}_\mu^5 j_\mu^5 + \theta q + S_+ j_+ + S_- j_- \right) \end{aligned} \quad (2.6)$$

The extra term in Eq. (2.6), proportional to $q(x)$, is the familiar axial anomaly^{3),4)}, which arises through regularization effects in the quantum theory. We emphasize that its presence is independent of the inclusion of the θ term in the Lagrangian (2.1) - it arises even in the theory with $\theta = 0$. [In passing, it is interesting to note that a recent discussion of the anomaly by Fujikawa²³⁾ interprets this term as arising even in the unregularized theory directly from the non-invariance of the functional measure under chiral transformations. He asserts that in the unregularized theory, under the transformation (2.2),

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} \rightarrow \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \int dx \propto \left(-\frac{g^2}{16\pi^2} \right) {}^*F_{\mu\nu}^a F^{a\mu\nu} \quad (2.7)$$

while in the regularized theory $\mathcal{D}\psi \mathcal{D}\bar{\psi}$ is invariant. However, this interpretation is not required to establish Eq. (2.6).]

From Eq. (2.6) we now deduce that $W[S_\mu^5, \theta, S_+, S_-]$ satisfies the following chiral Ward identity,

$$\left(\partial_\mu \frac{\delta}{\delta S_\mu^5} + \frac{\delta}{\delta \theta} + 2i S_+ \frac{\delta}{\delta S_+} - 2i S_- \frac{\delta}{\delta S_-} \right) W = 0 \quad (2.8)$$

It follows immediately that W has the following symmetry property,

$$W[S_\mu^5, \theta, S_+, S_-] = W[S_\mu^5 - \partial_\mu \alpha, \theta + \alpha, S_+ e^{2i\alpha}, S_- e^{-2i\alpha}] \quad (2.9)$$

This important equation is of course simply the integrated form of the differential Ward identity (2.8).

In the remainder of this section we discuss some immediate consequences of Eq. (2.8). First we derive in this functional formalism the well-known Ward identity relation for $\langle 0 | T \phi_5(x) j_\mu^5(y) | 0 \rangle$. Differentiating Eq. (2.8) with respect to S_+ (S_-), and evaluating at the physical source values, yields immediately the relations,

$$\frac{\partial}{\partial y_\mu} \frac{\delta^2 W}{\delta S_+(x) \delta S_\mu^5(y)} + \frac{\delta^2 W}{\delta S_+(x) \delta \theta(y)} \pm 2i \frac{\delta W}{\delta S_+(x)} \delta(x-y) = 0 \quad (2.10)$$

i.e., in canonical notation,

$$\begin{aligned} \frac{\partial}{\partial y_\mu} \langle 0 | T j_\pm(x) j_\mu^5(y) | 0 \rangle + \langle 0 | T j_\pm(x) q(y) | 0 \rangle \\ \pm 2i \langle 0 | j_\pm(x) | 0 \rangle \delta(x-y) = 0 \end{aligned} \quad (2.11)$$

In particular, taking an appropriate linear combination of the identities for j_+ and j_- , and Fourier transforming, gives,

$$-ik_\mu \langle 0 | T \phi_5 j_\mu^5 | 0 \rangle + \langle 0 | T \phi_5 q | 0 \rangle = 2 \langle 0 | \phi | 0 \rangle \quad (2.12)$$

In the absence of the anomalous term, this relation could be used to infer the existence of a massless pseudoscalar Goldstone boson in the ϕ_5 channel if chiral symmetry is spontaneously broken, as indicated by a non-zero vacuum expectation value of ϕ . The existence of the anomaly enables this consequence of Goldstone's theorem to be evaded.

Next, we investigate the dependence of the theory on the vacuum angle θ . Integrating Eq. (2.8) over space-time and evaluating at the physical source values, we obtain

$$\int dx \frac{\delta W}{\delta \theta(x)} = 0 \quad (2.13)$$

This states that the generating functional W is independent of the value of θ . [It is easy to see that Eq. (2.13) may be written in more conventional notation as $2(\partial W / \partial \theta) = 0$, where here θ is the constant vacuum angle.] This result, that in a theory with massless quarks physical quantities are independent of θ , is of course well known. In fact, quite generally, the physical parameter is not θ but $\bar{\theta} = \theta - \arg \det M$, where M is the quark mass matrix.

Finally, differentiating Eq. (2.8) with respect to $\theta(x)$, and again evaluating at physical source values and integrating over space-time gives,

$$\int dx \partial_\mu \frac{\delta^2 W}{\delta S_\mu^5(x) \delta \theta(y)} + \int dx \frac{\delta^2 W}{\delta \theta(x) \delta \theta(y)} = 0 \quad (2.14)$$

Now if, and only if, there is no massless particle coupling to j_μ^5 or q , then the first term in Eq. (2.14) vanishes. In this case the Ward identity requires,

$$\text{F.T.} \quad \left. \frac{\delta^2 W}{\delta \theta \delta \theta} \right|_{k^2=0} = 0 \quad (2.15)$$

that is, in the theory with massless quarks, the correlation function $\int dx \langle 0 | T q(x) q(0) | 0 \rangle$ vanishes. If the Ward identity is satisfied in this way, the $U(1)$ problem is solved - there is no massless pseudoscalar boson coupling to j_μ^5 .

The Ward identity (2.14) may also be satisfied if there does exist a massless pseudoscalar boson coupling to j_μ^5 . In this case we would require,

$$\text{F.T.} \quad \left. \frac{\delta^2 W}{\delta \theta \delta \theta} \right|_{k^2=0} \neq 0 \quad (2.16)$$

Which of these two possibilities is actually realized is determined not by the kinematics of Ward identities but by the detailed dynamics of the particular model considered. In Sections 4 and 5, we construct an effective Lagrangian consistent with all the Ward identities in which the first possibility, i.e., resolution of the $U(1)$ problem through Eq. (2.15), is realized in a natural way.

3. - CHIRAL WARD IDENTITIES AND THE EFFECTIVE ACTION

The effective action $\Gamma[j_\mu^5, q, j_+, j_-]$, that is, the generating functional for one-particle-irreducible vertices with respect to the propagators for the fields $j_\mu^5(x)$, $q(x)$, $j_+(x)$, $j_-(x)$, is defined from W by a standard Legendre transform,

$$\Gamma[j_\mu^5, q, j_+, j_-] = W[s_\mu^5, \theta, s_+, s_-] - \int dx \left(s_\mu^5 j_\mu^5 + \theta q + s_+ j_+ + s_- j_- \right) \quad (3.1)$$

The fields appearing in Eq. (3.1) are defined in the standard way as,

$$j_\mu^5 = \frac{\delta W}{\delta s_\mu^5}, \quad q = \frac{\delta W}{\delta \theta}, \quad j_\pm = \frac{\delta W}{\delta s_\pm} \quad (3.2)$$

and as usual,

$$\frac{\delta \Gamma}{\delta j_\mu^5} = -s_\mu^5, \quad \frac{\delta \Gamma}{\delta q} = -\theta, \quad \frac{\delta \Gamma}{\delta j_\pm} = -s_\pm \quad (3.3)$$

However, for reasons that will become transparent, the constraints of chiral symmetry are more clearly exhibited by defining a different functional $\hat{\Gamma}[\hat{j}_\mu^5, \hat{q}, \hat{j}_+, \hat{j}_-]$, which we call the symmetric effective action, by means of a modified Legendre transform. The motivation for this non-standard transform, first introduced by Zumino in Ref. 2), and a self-contained general discussion in which some important results required in the text are derived, are given in Appendix A.

The idea is to define an effective action which will obey the simplest possible chiral Ward identity, analogous to Eq. (2.8) for W . The integrated form of this identity then implies a symmetry of $\hat{\Gamma}$, just as in Eq. (2.9). $\hat{\Gamma}$ must then be constructed from combinations of the fields which, in addition to being Lorentz invariant, must be invariant under this new symmetry. Finally, having exploited the chiral symmetry to construct $\hat{\Gamma}$, we must then relate the physical quantities of interest, which are expressed in terms of functional derivatives of the true effective action Γ , to functional derivatives of $\hat{\Gamma}$.

We define the symmetric effective action as follows:

$$\hat{\Gamma}[\hat{j}_\mu^5, \hat{q}, \hat{j}_+, \hat{j}_-] = W[s_\mu^5, \theta, s_+, s_-] \quad (3.4)$$

$$- \int d^4x \left\{ \frac{1}{2} a (\hat{j}_\mu^S + S_\mu^S)^2 + \frac{1}{2} b (\hat{q} + \frac{1}{b} \theta)^2 + S_+ \hat{j}_+ + S_- \hat{j}_- \right\} \quad (3.4)$$

cont.

where a and b are arbitrary dimensional constants. Their physical interpretation will be clarified in Section 5. The new fields appearing in Eq. (3.4) are defined as,

$$\hat{j}_\mu^S = \frac{\delta W}{\delta S_\mu^S} - \frac{1}{a} S_\mu^S, \quad \hat{q} = \frac{\delta W}{\delta \theta} - \frac{1}{b} \theta, \quad \hat{j}_\pm = \frac{\delta W}{\delta S_\pm} \quad (3.5)$$

Of course, $\hat{j}_\pm \equiv j_\pm$. It also follows that,

$$\frac{\delta \hat{\Gamma}}{\delta \hat{j}_\mu^S} + a \hat{j}_\mu^S = - S_\mu^S, \quad \frac{\delta \hat{\Gamma}}{\delta \hat{q}} + b \hat{q} = - \theta, \quad \frac{\delta \hat{\Gamma}}{\delta \hat{j}_\pm} = - S_\pm \quad (3.6)$$

In particular, note that

$$\frac{\delta \hat{\Gamma}}{\delta \hat{j}_\mu^S} = - a \frac{\delta W}{\delta S_\mu^S}, \quad \frac{\delta \hat{\Gamma}}{\delta \hat{q}} = - b \frac{\delta W}{\delta \theta} \quad (3.7)$$

while in the standard way,

$$\hat{j}_\pm \frac{\delta \hat{\Gamma}}{\delta \hat{j}_\pm} = - S_\pm \frac{\delta W}{\delta S_\pm} \quad (3.8)$$

This is the property which ensures that $\hat{\Gamma}$ satisfies a simple Ward identity.

The chiral Ward identity for $\hat{\Gamma}$ now follows immediately from that for W , given Eqs. (3.7) and (3.8). It is,

$$\left(\frac{1}{a} \partial_\mu \frac{\delta}{\delta \hat{j}_\mu^S} + \frac{1}{b} \frac{\delta}{\delta \hat{q}} + 2i \hat{j}_+ \frac{\delta}{\delta \hat{j}_+} - 2i \hat{j}_- \frac{\delta}{\delta \hat{j}_-} \right) \hat{\Gamma} = 0 \quad (3.9)$$

This implies that $\hat{\Gamma}$ has the following symmetry property:

$$\hat{\Gamma}[\hat{j}_\mu^5, \hat{q}, \hat{j}_+, \hat{j}_-] = \hat{\Gamma}\left[\hat{j}_\mu^5 - \frac{1}{a} \partial_\mu \alpha, \hat{q} + \frac{1}{b} \alpha, \hat{j}_+ e^{2i\alpha}, \hat{j}_- e^{-2i\alpha}\right] \quad (3.10)$$

and must therefore be constructed from invariants under the transformation (3.10).

The actual construction of $\hat{\Gamma}$ depends of course on important dynamical assumptions as well as symmetry relations, and will be postponed until Section 5, after we have discussed the relation of $\hat{\Gamma}$ to the true effective action Γ .

At this point, however, we can make an interesting observation about the form of $\hat{\Gamma}$ directly from the relation (3.10). Making the specific choice $\alpha = -b\hat{q}$ for the transformation parameter, we find

$$\hat{\Gamma}[\hat{j}_\mu^5, \hat{q}, \hat{j}_+, \hat{j}_-] = \hat{\Gamma}\left[\hat{j}_\mu^5 + \frac{b}{a} \partial_\mu \hat{q}, 0, \hat{j}_+ e^{-2ib\hat{q}}, \hat{j}_- e^{2ib\hat{q}}\right] \quad (3.11)$$

That is, the symmetric effective action is a functional only of the invariant combinations $(\hat{j}_\mu^5 + b/a(\partial_\mu \hat{q}))$ and $\hat{j}_\pm e^{\mp 2ib\hat{q}}$. This observation summarizes in the most concise form possible the implications of the anomalous chiral Ward identities.

The relation between Γ and $\hat{\Gamma}$ is given by the following formulae,

$$\hat{\Gamma} = \Gamma - \int d^4x \left\{ \frac{1}{2} a (j_\mu^5)^2 + \frac{1}{2} b q^2 + j_\mu^5 \frac{\delta \Gamma}{\delta j_\mu^5} + q \frac{\delta \Gamma}{\delta q} \right\} \quad (3.12)$$

and

$$\Gamma = \hat{\Gamma} - \int d^4x \left\{ \frac{1}{2a} \left(\frac{\delta \hat{\Gamma}}{\delta j_\mu^5} \right)^2 + \frac{1}{2b} \left(\frac{\delta \hat{\Gamma}}{\delta q} \right)^2 + j_\mu^5 \frac{\delta \hat{\Gamma}}{\delta j_\mu^5} + q \frac{\delta \hat{\Gamma}}{\delta q} \right\} \quad (3.13)$$

The relative complexity of these formulae indicates how much more transparent the chiral symmetry is in the symmetric effective action $\hat{\Gamma}$. The advantage of this approach lies in the ease with which the constraints of chiral symmetry can be implemented provided the non-standard transform (3.4) is used.

In addition, to deduce the physical predictions of this theory it is not necessary to know Γ explicitly. All that is required in order to construct the propagator matrix, and thus determine the masses and decay constants

of the bound states, is the matrix of second functional derivatives of Γ . As shown in Appendix A, this is quite simply related to the corresponding matrix of second functional derivatives of $\hat{\Gamma}$.

4. - THE PROPAGATOR MATRIX

So far we have defined the symmetric effective action $\hat{\Gamma}$ and have constructed chiral Ward identities for both $\hat{\Gamma}$ and the generating functional W . We are now in a position to use these results to deduce the general form for the propagator matrix of the bound state operators j_μ^5, q, j_+, j_- that is consistent with broken chiral symmetry and the anomaly.

We assume that chiral symmetry is spontaneously broken due to the strong colour interaction, i.e., that the operator $\phi = \bar{\psi}\psi$ acquires a non-zero vacuum expectation value $\langle\phi\rangle$. (It is convenient in this part of the discussion to use the operators ϕ_5, ϕ in preference to j_+, j_- .) Further, we assume that parity is not spontaneously broken but remains a good symmetry. This requires a word of explanation. The original QCD Lagrangian (2.1) does contain an explicit parity-violating term proportional to θ . However, as we have shown in Section 2, in the theory with massless quarks the generating functional is independent of θ , and so in this case there is no explicit parity violation. That parity is not spontaneously broken, i.e., that $\langle 0|q|0\rangle = 0$, is a dynamical assumption.

Rewriting the Ward identity for $\hat{\Gamma}$ in terms of $\hat{\phi}$ and $\hat{\phi}_5$ gives,

$$\left(\frac{1}{a} \partial_\mu \frac{\delta}{\delta j_\mu^5} + \frac{1}{b} \frac{\delta}{\delta \hat{q}} + 2 \left(\hat{\phi}_5 \frac{\delta}{\delta \hat{\phi}} - \hat{\phi} \frac{\delta}{\delta \hat{\phi}_5} \right) \right) \hat{\Gamma} = 0 \quad (4.1)$$

Differentiating with respect to each of the fields in turn, evaluating at the vacuum expectation values (VEV) $\hat{j}_\mu^5 = 0, \hat{q} = \text{const.}, \hat{\phi}_5 = 0, \hat{\phi} = \langle\phi\rangle = \text{const.}$, and taking the Fourier transform then gives the following relation amongst the second functional derivatives of $\hat{\Gamma}$:

$$-\frac{1}{a} i k_\mu \frac{\delta^2 \hat{\Gamma}}{\delta j_\mu^5 \delta j_\mu^5} + \frac{1}{b} \frac{\delta^2 \hat{\Gamma}}{\delta j_\mu^5 \delta \hat{q}} - 2\langle\phi\rangle \frac{\delta^2 \hat{\Gamma}}{\delta j_\mu^5 \delta \hat{\phi}_5} = 0 \quad (4.2)$$

$$-\frac{1}{a} i k_\mu \frac{\delta^2 \hat{\Gamma}}{\delta \hat{q} \delta j_\mu^5} + \frac{1}{b} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{q} \delta \hat{q}} - 2\langle\phi\rangle \frac{\delta^2 \hat{\Gamma}}{\delta \hat{q} \delta \hat{\phi}_5} = 0 \quad (4.3)$$

$$-\frac{1}{a} i k_\mu \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5^\mu \delta \hat{j}_\mu^5} + \frac{1}{b} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5^\mu \delta \hat{q}^\mu} - 2\langle \phi \rangle \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5^\mu \delta \hat{q}_5^\mu} = 0 \quad (4.4)$$

$$-\frac{1}{a} i k_\mu \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}^\mu \delta \hat{j}_\mu^5} + \frac{1}{b} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}^\mu \delta \hat{q}^\mu} - 2\langle \phi \rangle \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}^\mu \delta \hat{q}_5^\mu} = 0 \quad (4.5)$$

In the final two relations the equations of motion $\delta \hat{\Gamma} / \delta \hat{\phi} = 0$, $\delta \hat{\Gamma} / \delta \hat{\phi}_5 = 0$ [i.e. Eq. (3.6) with the sources $S_\pm = 0$] have been used.

Notice the restrictions imposed on the form of the allowed VEVs. $\hat{\phi}$ and $\hat{\phi}_5$ are identical to the physical fields ϕ and ϕ_5 , and we may use the freedom of making a chiral rotation to set $\langle 0 | \phi_5 | 0 \rangle = 0$. However, since $\hat{q}$ differs from the physical field q it may acquire a VEV. In fact, from Eqs. (3.2) and (3.5), $\hat{q}(x) = q(x) - \theta(x)/b$, and so $\langle 0 | \hat{q} | 0 \rangle = \langle 0 | q | 0 \rangle - \theta/2b$. Since we require $\langle 0 | q | 0 \rangle = 0$ by parity conservation, we must allow $\langle 0 | \hat{q} | 0 \rangle = \text{const.}$ The VEV of $\hat{j}_\mu^5$ is zero by Lorentz invariance.

It should also be observed that in general second functional derivatives are not symmetric. The asymmetries are in practice of two kinds: (i) terms depending on derivatives of field VEVs, which therefore vanish, and (ii) terms involving derivatives of delta functions, which change sign. This is required in order for the matrix of second functional derivatives to be Hermitian.

We now introduce the notation,

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{j}_\mu^5 \delta \hat{j}_\nu^5} = \tilde{P}(k^2) \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + P(k^2) \frac{k_\mu k_\nu}{k^2} \quad (4.6)$$

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{j}_\mu^5 \delta \hat{q}^\mu} = i Q(k^2) k_\mu = - \frac{\delta^2 \hat{\Gamma}}{\delta \hat{q}^\mu \delta \hat{j}_\mu^5} \quad (4.7)$$

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{q}^\mu \delta \hat{q}^\mu} = R(k^2) \quad (4.8)$$

The Ward identity relations then imply that,

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{j}_\mu^5 \delta \hat{\phi}_5} = - i k_\mu \frac{1}{2\langle\phi\rangle} \left(\frac{1}{a} P - \frac{1}{b} Q \right) = - \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5 \delta \hat{j}_\mu^5} \quad (4.9)$$

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{q} \delta \hat{\phi}_5} = \frac{1}{2\langle\phi\rangle} \left(-\frac{1}{a} Q k^2 + \frac{1}{b} R \right) = \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5 \delta \hat{q}} \quad (4.10)$$

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5 \delta \hat{\phi}_5} = \frac{1}{4\langle\phi\rangle^2} \left(\frac{1}{a} P k^2 - \frac{2}{ab} Q k^2 + \frac{1}{b^2} R \right) \quad (4.11)$$

Parity conservation implies that each term in Eq. (4.5) vanishes. (Strictly speaking, this argument applies directly only to the derivatives of Γ . However, it can easily be checked that the conditions are equivalent.) The axial part of the second functional derivative matrix therefore decouples from the remaining term $\delta^2 \hat{\Gamma} / \delta \hat{\phi} \delta \hat{\phi}$.

Summarizing, and introducing the generic notation $\hat{F} = \{j_\mu^5, \hat{q}, \hat{\phi}_5\}$, we have found

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{F} \delta \hat{F}} = \begin{bmatrix} \tilde{P} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + P \frac{k_\mu k_\nu}{k^2} & i Q k_\mu & - \frac{i k_\mu}{2\langle\phi\rangle} \left(\frac{1}{a} P - \frac{1}{b} Q \right) \\ & R & \frac{1}{2\langle\phi\rangle} \left(-\frac{1}{a} Q k^2 + \frac{1}{b} R \right) \\ \text{h.c.} & & \frac{1}{4\langle\phi\rangle^2} \left(\frac{1}{a^2} P k^2 - \frac{2}{ab} Q k^2 + \frac{1}{b^2} R \right) \end{bmatrix} \quad (4.12)$$

We now require the corresponding matrix of derivatives of the true effective action Γ . The required relation is derived in Appendix A, and is most simply expressed in symbolic notation (omitting the factors of a, b) as follows. Writing,

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{F} \delta \hat{F}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (4.13)$$

where A is a 2×2 matrix, etc., then

$$\frac{\delta^2 \Gamma}{\delta F \delta F} = \begin{bmatrix} -1 - A^{-1} & -A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{bmatrix} \quad (4.14)$$

The precise form including the correct factors of a, b is written out in full in Eqs. (A.27) to (A.30). We find

$$\frac{\delta^2 \Gamma}{\delta F \delta F} = \begin{bmatrix} -\left(a + \frac{a^2}{P}\right)\left(g_{\mu 0} - \frac{k_\mu k_0}{k^2}\right) - \left(a + \frac{a^2 R}{(-Q^2 k^2 + PR)}\right) \frac{k_\mu k_\nu}{k^2} & \frac{abQ}{(-Q^2 k^2 + PR)} i k_\mu & \frac{1}{2\langle\phi\rangle} i k_\mu \\ & -\left(b + \frac{b^2 P}{(-Q^2 k^2 + PR)}\right) & -\frac{1}{2\langle\phi\rangle} \\ \text{h.c.} & & 0 \end{bmatrix} \quad (4.15)$$

Finally, the propagator matrix is given simply by inverting this. Denoting the sources generically by $S = \{S_\mu^5, \theta, S_5\}$, we obtain

$$\frac{\delta^2 W}{\delta S \delta S} = - \left(\frac{\delta^2 \Gamma}{\delta F \delta F} \right)^{-1} \quad (4.16)$$

$$= \left[\begin{array}{ccc} \frac{1}{\alpha} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) - \frac{1}{\Delta} \frac{k_\mu k_\nu}{k^2} & -\frac{1}{\Delta} \frac{i k_\mu}{k^2} & -\frac{2\langle\phi\rangle}{\Delta} (\gamma + \epsilon) \frac{i k_\mu}{k^2} \\ & -\frac{1}{\Delta} & \frac{2\langle\phi\rangle}{\Delta} \left(\frac{\beta}{k^2} + \gamma \right) \\ \text{h.c.} & & \frac{4\langle\phi\rangle^2}{\Delta} \left(\frac{\beta\epsilon}{k^2} - \gamma^2 \right) \end{array} \right] \quad (4.17)$$

where

$$\begin{aligned} \alpha &= a + a^2 \tilde{P}^{-1} \\ \beta &= -a - a^2 R (-Q^2 k^2 + PR)^{-1} \\ \gamma &= abQ (-Q^2 k^2 + PR)^{-1} \\ \epsilon &= -b - b^2 P (-Q^2 k^2 + PR)^{-1} \end{aligned} \quad (4.18)$$

and,

$$\Delta = \frac{\beta}{k^2} + \epsilon + 2\gamma \quad (4.19)$$

This form for $\delta^2 W / \delta S \delta S$ is easily seen to be consistent with the following non-trivial relations, which are derived from the chiral Ward identity (2.8) for W by differentiating with respect to the sources:

$$-i k_\mu \frac{\delta^2 W}{\delta S_\mu^5 \delta S_\mu^5} + \frac{\delta^2 W}{\delta S_\mu^5 \delta \theta} = 0 \quad (4.20)$$

$$-i k_\mu \frac{\delta^2 W}{\delta \theta \delta S_\mu^5} + \frac{\delta^2 W}{\delta \theta \delta \theta} = 0 \quad (4.21)$$

$$-ik_\mu \frac{\delta^2 W}{\delta S_5 \delta S_5^\mu} + \frac{\delta^2 W}{\delta S_5 \delta \theta} - 2\langle \phi \rangle = 0 \quad (4.22)$$

To investigate the structure of the propagator matrix (4.17) further, we require some knowledge about the unknown functions $\tilde{P}$, P , Q , R which are derived from the symmetric effective action $\hat{\Gamma}$. In the next section we construct the low energy approximation to $\hat{\Gamma}$ consistent with the symmetry (3.10), and discuss in detail the dynamical assumptions which must be made.

We can however obtain a great deal of information from the propagator matrix with only a very limited knowledge of the form of these functions. For the remainder of this section we make the assumptions, to be justified later by the explicit form of $\hat{\Gamma}$, that $\tilde{P}$, P , Q , R are power series in k^2 , and that in the low energy approximation P , Q are constants while $\tilde{P}$, R are of $O(k^2)$.

The masses of the physical bound states in the theory are given by the poles in the propagator matrix. We therefore begin by identifying these poles.

First, consider the zero-momentum limit of the Ward identity relation (4.21). From Eq. (4.17) we find,

$$\left. \frac{\delta^2 W}{\delta \theta \delta \theta} \right|_{k^2=0} = -\Delta^{-1}(0) \quad (4.23)$$

Given the above assumptions on $\tilde{P}$, P , Q , R the function $\Delta(k^2)$ has a pole at $k^2 = 0$, displayed explicitly in its definition (4.19), the residue being simply $\beta(0)$. For $\beta(0) \neq 0$ therefore, we find $\Delta^{-1}(0) = 0$. Referring to the discussion at the end of Section 2, we see that in this model the $U(1)$ problem is resolved. There is no massless pseudoscalar boson corresponding to the spontaneous breaking of the $U(1)$ chiral symmetry.

From the propagator matrix (4.17) we may check explicitly that there are indeed no $k^2 = 0$ poles in the O^- sector. For example,

$$\left. \frac{\delta^2 W}{\delta S_5 \delta \theta} \right|_{k^2=0} = \frac{2\langle \phi \rangle}{k^2 \Delta(k^2)} \left(\beta + \gamma k^2 \right) \Big|_{k^2=0} = 2\langle \phi \rangle \quad (4.24)$$

as required by the relation (4.22).

The only remaining place where a $k^2 = 0$ pole could arise is in the propagator $\delta^2 W / \delta S_\mu^5 \delta S_\nu^5$. The residue of this possible pole is simply

$$\alpha^{-1}(0) + \beta^{-1}(0) = (\hat{P}(0) - P(0)) (a + \hat{P}(0))^{-1} (a + P(0))^{-1} \quad (4.25)$$

which vanishes if $\hat{P}(0) = P(0)$. As we shall see, this is indeed satisfied by the explicit form for $\hat{P}$.

Now consider $\Delta(k^2)$. Explicitly,

$$\Delta(k^2) = (k^2)^{-1} (-Q^2 k^2 + PR)^{-1} \left\{ -(a + b k^2) (-Q^2 k^2 + PR) - b^2 P^2 k^2 + 2abQk^2 - a^2 R \right\} \quad (4.26)$$

The numerator is quadratic in k^2 , and in general has two zeros, say at $k^2 = m_G^2, m_\eta^2$. Thus $\Delta^{-1}(k^2)$ has two poles, at $k^2 = m_G^2$ and $k^2 = m_\eta^2$.

As for the remaining terms in $\delta^2 W / \delta S \delta S$, clearly $\Delta^{-1}(\gamma + \epsilon)/k^2$ and $\Delta^{-1}((\beta/k^2) + \gamma)$ also only have poles at m_G^2 and m_η^2 , the possible poles at $k^2 = 0$ and $(-Q^2 k^2 + PR) = 0$ cancelling against the denominator of $\Delta(k^2)$. Also, since

$$\beta\epsilon - \gamma^2 k^2 = ab(-Q^2 k^2 + PR)^{-1} \left\{ -Q^2 k^2 + PR + bP + aR + ab \right\} \quad (4.27)$$

we see that the term $\Delta^{-1}((\beta\epsilon/k^2) - \gamma^2)$ also only exhibits poles at m_G^2, m_η^2 .

Finally, there is a further pole in the $\delta^2 W / \delta S_\mu^5 \delta S_\nu^5$ entry, arising from the term $\alpha^{-1}(k^2)$. We denote the position of this pole as m_D^2 .

Summarizing, the propagator matrix has three poles: $k^2 = m_D^2$, the pole in $\alpha^{-1}(k^2)$, and $k^2 = m_G^2, m_\eta^2$, the poles in $\Delta^{-1}(k^2)$. If we include the scalar propagator, we find a further pole, say at $k^2 = m_S^2$. We therefore deduce that, in this low energy approximation, the sector of the theory described here has four bound state particles, a pseudovector 1^+ meson, which we call the D with mass m_D , two pseudoscalar 0^- mesons, the η and G with masses m_η and m_G , and a scalar 0^+ meson, the S with mass m_S .

We now discuss the mixing and decay constants of these particles. To understand this, we write a spectral decomposition^{*)} of the propagators isolating the pole terms. That is,

$$\begin{aligned} \langle 0 | T q(x) q(y) | 0 \rangle_{F.T.} &= K \Delta_F(m_G^2) + L \Delta_F(m_\eta^2) \\ &+ \int_{m_-^2}^{\infty} d\sigma^2 \rho(\sigma^2) \Delta_F(\sigma^2) \end{aligned} \quad (4.28)$$

where

$$\Delta_F(m^2) = \frac{-1}{k^2 - m^2}$$

is the Feynman propagator, $\rho(\sigma^2)$ is a spectral function, and with a relativistic normalization of the one-particle states, the decay constants K, L are given by

$$\begin{aligned} K^{1/2} &= \langle 0 | q(0) | G \rangle \\ L^{1/2} &= \langle 0 | q(0) | \eta \rangle \end{aligned} \quad (4.29)$$

From the propagator matrix (4.17) we find,

$$\begin{aligned} K^{-1} &= \left. \frac{d}{ds} \Delta(s) \right|_{s=m_G^2} \\ L^{-1} &= \left. \frac{d}{ds} \Delta(s) \right|_{s=m_\eta^2} \end{aligned} \quad (4.30)$$

Similarly we find

$$\langle 0 | T \phi_S(x) \phi_S(y) | 0 \rangle_{F.T.} = M \Delta_F(m_G^2) + N \Delta_F(m_\eta^2) + \text{spectral terms} \quad (4.31)$$

^{*)} A remark about the positivity of the spectral decomposition (4.28), and the following equations, is in order. This expansion is for the time ordered product: as a sum of positive terms it is necessarily positive at $k^2 = 0$. This is not inconsistent with $(\delta^2 W / \delta \theta \delta \theta) |_{k^2=0} = 0$. The functional approach yields the covariant time ordered product: this differs by the inclusion of contact, or Schwinger, terms, which would appear as constants on the right-hand side of Eq. (4.28), allowing the expression to vanish at $k^2 = 0$.

where

$$M = -4 \langle \phi \rangle^2 \left(\frac{\beta \epsilon}{m_G^2} - \gamma^2 \right) (m_G^2) \quad K \quad (4.32)$$

$$N = -4 \langle \phi \rangle^2 \left(\frac{\beta \epsilon}{m_\eta^2} - \gamma^2 \right) (m_\eta^2) \quad L \quad ,$$

$$\langle 0 | T q(x) \phi_S(y) | 0 \rangle_{F.T.} = G \Delta_F(m_G^2) + H \Delta_F(m_\eta^2) + \text{spectral terms} \quad (4.33)$$

where

$$G = -2 \langle \phi \rangle \left(\frac{\beta}{m_G^2} + \gamma \right) (m_G^2) \quad K \quad (4.34)$$

$$H = -2 \langle \phi \rangle \left(\frac{\beta}{m_\eta^2} + \gamma \right) (m_\eta^2) \quad L \quad ,$$

$$\langle 0 | T j_\mu^S(x) q(y) | 0 \rangle_{F.T.} = \frac{K}{m_G^2} i k_\mu \Delta_F(m_G^2) + \frac{L}{m_\eta^2} i k_\mu \Delta_F(m_\eta^2) + \text{spectral terms}, \quad (4.35)$$

$$\langle 0 | T j_\mu^S(x) \phi_S(y) | 0 \rangle_{F.T.} = \frac{E}{m_G^2} i k_\mu \Delta_F(m_G^2) + \frac{F}{m_\eta^2} i k_\mu \Delta_F(m_\eta^2) + \text{spectral terms} \quad (4.36)$$

where

$$E = 2 \langle \phi \rangle (\gamma + \epsilon) (m_G^2) \quad K \quad (4.37)$$

$$F = 2 \langle \phi \rangle (\gamma + \epsilon) (m_\eta^2) \quad L \quad ,$$

$$\begin{aligned} \langle 0 | T j_\mu^S(x) j_\nu^S(y) | 0 \rangle_{F.T.} &= \frac{K}{m_G^4} k_\mu k_\nu \Delta_F(m_G^2) + \frac{L}{m_\eta^4} k_\mu k_\nu \Delta_F(m_\eta^2) \\ &\quad - Z \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_D^2} \right) \Delta_F(m_D^2) + \text{spectral terms} \end{aligned} \quad (4.38)$$

where

$$Z^{-1} = \frac{d}{ds} \alpha(s) \Big|_{s=m_\rho^2} \quad (4.39)$$

The following relations may be easily established:

$$\begin{aligned} E - G &= 0 & F - H &= 0 \\ G^2 - KM &= 0 & H^2 - LN &= 0 \end{aligned} \quad (4.40)$$

Using these, the consistency of the factorization of the decay constants may be readily checked. From the spectral decomposition we deduce,

$$\langle 0 | \eta(0) | G \rangle \equiv F_{\eta G} = K^{1/2} \quad (4.41)$$

$$\langle 0 | \eta(0) | \eta \rangle \equiv F_{\eta \eta} = L^{1/2} \quad (4.42)$$

$$\langle 0 | \phi_S(0) | G \rangle \equiv F_{\phi_S G} = E K^{-1/2} \quad (4.43)$$

$$\langle 0 | \phi_S(0) | \eta \rangle \equiv F_{\phi_S \eta} = F L^{-1/2} \quad (4.44)$$

$$\langle 0 | j_\mu^5(0) | G \rangle \equiv i k_\mu \frac{F_{\eta G}}{m_G} = i k_\mu \frac{K^{1/2}}{m_G^2} \quad (4.45)$$

$$\langle 0 | j_\mu^5(0) | \eta \rangle \equiv i k_\mu \frac{F_{\eta \eta}}{m_\eta} = i k_\mu \frac{L^{1/2}}{m_\eta^2} \quad (4.46)$$

$$\langle 0 | j_\mu^5(0) | \eta \rangle \equiv \epsilon_\mu^{(\lambda)}(k) F_{\eta \eta} = \epsilon_\mu^{(\lambda)}(k) Z^{1/2} \quad (4.47)$$

where $\epsilon_{\mu}^{(\lambda)}(k)$ is the polarization vector for the state with polarization λ , and satisfies

$$\sum_{\lambda=1}^3 \epsilon_{\mu}^{(\lambda)}(k) \epsilon_{\nu}^{(\lambda)}(k) = - \left(g_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{m_D^2} \right)$$

Two relations amongst the decay constants follow directly, viz.

$$\langle 0 | j_{\mu}^5(0) | G \rangle = \frac{i k_{\mu}}{m_G^2} \langle 0 | q(0) | G \rangle \quad (4.48)$$

and

$$\langle 0 | j_{\mu}^5(0) | \eta \rangle = \frac{i k_{\mu}}{m_{\eta}^2} \langle 0 | q(0) | \eta \rangle \quad (4.49)$$

Of course these results follow directly from the anomaly equation,

$$\partial_{\mu} j_{\mu}^5 = -q \equiv \frac{g^2}{16\pi^2} {}^*F_{\mu\nu}^a F_{\mu\nu}^a \quad (4.50)$$

These results, giving the masses and decay constants of the bound state particles D , η and G , essentially exhaust the physical information to be deduced from the propagator matrix under the given assumptions on the form of the functions $\tilde{F}$, P , Q , R . It is of some interest however to find the linear combinations of the operators j_{μ}^5 , q and ϕ_5 which create the mass eigenstates. After some calculation we find the following results,

$$\langle 0 | T j_{\mu}^5(x) j_{\nu}^5(y) | 0 \rangle_{F.T.} = - \left(g_{\mu\nu} - \frac{k_{\mu} k_{\nu}}{m_D^2} \right) \Delta_F(m_D^2) + \text{spectral terms} \quad (4.51)$$

$$\langle 0 | T Q(x) Q(y) | 0 \rangle_{F.T.} = \Delta_F(m_G^2) + \text{spectral terms} \quad (4.52)$$

$$\langle 0 | T \bar{\Xi}_5(x) \Xi_5(y) | 0 \rangle_{F.T.} = \Delta_F(m_\eta^2) + \text{spectral terms} \quad (4.53)$$

where

$$J_\mu^5 = F_{ij}^{-1} (j_\mu^5 + x \partial_\mu q + y \partial_\mu \phi_5) \quad (4.54)$$

with

$$x = (F_{qG} F_{\phi_5 3} - F_{q3} F_{\phi_5 G})^{-1} \left(\frac{F_{qG} F_{\phi_5 3}}{m_G^2} - \frac{F_{q3} F_{\phi_5 G}}{m_\eta^2} \right) \quad (4.55)$$

$$y = (F_{qG} F_{\phi_5 3} - F_{q3} F_{\phi_5 G})^{-1} \left(\frac{1}{m_\eta^2} - \frac{1}{m_G^2} \right) F_{q3} F_{\phi_5 G}$$

and

$$Q = (F_{qG} F_{\phi_5 3} - F_{q3} F_{\phi_5 G})^{-1} (F_{\phi_5 3} q - F_{q3} \phi_5) \quad (4.56)$$

$$\bar{\Xi}_5 = (F_{qG} F_{\phi_5 3} - F_{q3} F_{\phi_5 G})^{-1} (-F_{\phi_5 G} q + F_{qG} \phi_5) \quad (4.57)$$

To summarize, we have constructed the propagator matrix for the pseudoscalar and pseudovector operators j_μ^5 , q and ϕ_5 , subject to the constraints imposed by the anomalous Ward identities for broken chiral symmetry. Making certain very simple dynamical assumptions, we have deduced that in a low energy approximation the spectrum in this sector of the theory comprises one 1^+ pseudovector meson and two 0^- pseudoscalar mesons. These physical particles are created by the linear combination of the operators j_μ^5 , q and ϕ_5 indicated in Eqs. (4.54), (4.56) and (4.57).

Notice in particular that there are two 0^- states, created by linear combinations of q and ϕ_5 . These physical 0^- states are thus simply different linear combinations of pure quark and pure glueball states, and are thus essentially similar in character. Of course, in a realistic phenomenological model it may well be that one of the physical states is mostly a pure quark state

while the other is mostly glueball. We emphasize however that this theory predicts that there are two such particles, and that in principle they need not be qualitatively distinct.

We have also shown in a very simple and direct way how in this dynamical model the axial anomaly enforces, through the anomalous chiral Ward identities, the absence of a pseudoscalar Goldstone boson pole at $k^2 = 0$. This observation constitutes a solution to the $U(1)$ problem in this one-flavour theory.

5. - THE SYMMETRIC EFFECTIVE ACTION, MASSES AND DECAY CONSTANTS

We now return to the problem left open in Section 4, namely the explicit construction of the symmetric effective action $\hat{\Gamma}[\hat{j}_\mu^5, \hat{q}, \hat{j}_+, \hat{j}_-]$ in a low energy approximation. We give explicit expressions for the masses and decay constants of the physical particles in terms of the parameters introduced in $\hat{\Gamma}$. We find that the propagator matrix, enlarged to include the scalar propagator, depends on ten parameters. Since it describes a total of twelve physical quantities, viz. four masses and eight decay constants, there exist two relations. These have already been exhibited, in Eqs. (4.48) and (4.49). We also show that the parameters a and b occurring in the non-standard Legendre transform have a simple interpretation in terms of the masses and decay constants.

In the course of deriving these results we clarify the mechanism by which the η acquires a mass, and note an interesting connection between the pseudovector (D) meson mass and the kinetic terms for the field $\hat{q}(x)$.

The symmetric effective action is constructed from invariants under the Lorentz and gauge groups. As explained in Section 3 it must also involve only invariants under the transformation,

$$\begin{aligned} \hat{j}_\mu^5 &\rightarrow \hat{j}_\mu^5 - \frac{1}{a} \partial_\mu \alpha \\ \hat{q} &\rightarrow \hat{q} + \frac{1}{b} \alpha \\ \hat{j}_\pm &\rightarrow \hat{j}_\pm e^{\pm 2i\alpha} \end{aligned} \tag{5.1}$$

We may list here some of these invariants. Beginning with terms involving no derivatives, it is clear that any function of $(\hat{j}_+ \hat{j}_-)$ is invariant. To form an invariant from $\hat{j}_\pm$ and $\hat{q}$ which is linear in $\hat{q}$, it is necessary to use a logarithm. In fact, any function of $(\hat{q} + (i/4b) \log \hat{j}_+ / \hat{j}_-)$ is invariant under (5.1), and is dimensionally consistent and Hermitian. Such a term has been introduced before in the literature on effective Lagrangians^{14),15)}. Previously, however, it has been motivated on the grounds of the $1/N$ expansion. Here we

see clearly how it arises as a straightforward consequence of the constraints of chiral symmetry, as expressed through the Ward identities (3.9), (3.10), entirely independent of $1/N$ considerations.

The symmetry (5.1) is analogous to a local Abelian gauge symmetry, and we see immediately that $(\partial_\mu \hat{j}_\nu^5 - \partial_\nu \hat{j}_\mu^5)$ is an invariant. Defining the covariant derivatives

$$\mathcal{D}_\mu \hat{j}_\pm \equiv (\partial_\mu \pm 2ia \hat{j}_\mu^5) \hat{j}_\pm \rightarrow (\mathcal{D}_\mu \hat{j}_\pm) e^{\pm 2ia} \quad (5.2)$$

under (5.1), we find that $(\mathcal{D}_\mu \hat{j}_+)(\mathcal{D}_\mu \hat{j}_-)$ is another invariant. Further, $(\partial_\mu \hat{q} + (a/b) \hat{j}_\mu^5)$ is invariant, and we can form additional covariant derivatives similar to (5.2) but with $\hat{j}_\mu^5$ replaced by $((-b/a) \partial_\mu \hat{q})$.

Now in order to obtain the kind of non-trivial dynamical results obtained in Section 4, we must in addition make some detailed dynamical assumptions as to the form of $\hat{\Gamma}$.

First, we assume that Γ can be written in position space as an expansion in the number of space-time derivatives acting on the field. This excludes, for example, terms which would imply at the tree-level a $\log k^2/m^2$ behaviour of the propagators, characteristic of a cut. This assumption is therefore one of pole dominance for the Green functions. (It can be justified in terms of the $1/N$ expansion.)

Second, we assume a low energy approximation by including only terms involving two or fewer derivatives. Higher derivative terms are suppressed in the low energy regime by powers of k^2/Λ^2 , where Λ is the characteristic mass scale of the theory. Further, we include kinetic (second derivative) terms for all the fields in $\hat{\Gamma}$. In view of the mixing which occurs amongst these operators, this seems to be a natural thing to do. Of course, should it be the case phenomenologically that one of the physical particles has a mass $m \gg \Lambda$, then since the low energy analysis would be valid only for $k^2 \ll m^2$ the corresponding kinetic term should be omitted. We do not however expect this to be the case.

Finally, we reiterate the assumptions that have been made already. By considering only gauge invariant operators, we implicitly assume confinement. We assume that chiral symmetry is broken spontaneously. In principle, since the θ dependent term in the original QCD Lagrangian is P and T violating, we must allow for possible explicit parity violation proportional to θ ; however, since in the massless theory the generating functional is independent of θ , we may conclude that there is no parity violation.

The most general symmetric effective action satisfying the above condition is,

$$\begin{aligned} \hat{\Gamma}[\hat{j}_\mu^5, \hat{q}, \hat{j}_+, \hat{j}_-] = \int d^4x \left\{ U\left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-}, \hat{j}_+ \hat{j}_-\right) \right. \\ + \frac{1}{2} f_1 (D_\mu \hat{j}_+) (D_\mu \hat{j}_-) - \frac{1}{4} f_2 (\partial_\mu \hat{j}_+^5 - \partial_\mu \hat{j}_-^5)^2 \\ + \frac{1}{2} f_3 (\partial_\mu \hat{q} + \frac{a}{b} \hat{j}_\mu^5)^2 \\ \left. + \frac{1}{2} f_4 \left(\partial_\mu \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right) \right)^2 + \frac{1}{2} f_5 (\partial_\mu (\hat{j}_+ \hat{j}_-))^2 \right\} \end{aligned} \quad (5.3)$$

where $D_\mu \hat{j}_\pm = (\partial_\mu \pm 2ia \hat{j}_\mu^5) \hat{j}_\pm$, and like U the f_i ($i=1, \dots, 5$) are arbitrary functions of the two invariants $(\hat{q} + (i/4b) \log \hat{j}_+/\hat{j}_-)$ and $(\hat{j}_+ \hat{j}_-)$. It is of course possible to write down other invariants besides those included above in $\hat{\Gamma}$. Such terms may however all be re-expressed in the form of those appearing in Eq. (5.3).

At first sight, it may seem remarkable that a Lagrangian such as (5.3) involving arbitrary functions can give any information about the theory. The way this comes about is however straightforward, as we now show.

We first determine the vacuum expectation values of the fields from the equations of motion [see Eq. (3.6)].

$$\left(\frac{\delta \hat{\Gamma}}{\delta \hat{j}_\mu^5} + a \hat{j}_\mu^5 \right) \Big|_{\text{VEV}} = 0 \quad (5.4)$$

$$\left(\frac{\delta \hat{\Gamma}}{\delta \hat{q}} + b \hat{q} \right) \Big|_{\text{VEV}} = -\theta/2 \quad (5.5)$$

$$\frac{\delta \hat{\Gamma}}{\delta \hat{j}_\pm} \Big|_{\text{VEV}} = 0 \quad (5.6)$$

Evaluating the functional derivatives, and looking for solutions $\hat{j}_\mu^5 = 0$, $\hat{q} = \text{const.}$, $\hat{\phi}_5 = 0$, $\hat{\phi} = \text{const.}$, we find the equations,

$$\left(\frac{\partial U}{\partial x} + b \hat{q} \right) \Big|_{\text{VEV}} + \theta/2 = 0 \quad (5.7)$$

$$\left(2\phi \frac{\partial U}{\partial \eta} \right) \Big|_{\text{VEV}} = 0 \quad (5.8)$$

$$\left(\frac{1}{2b\phi} \frac{\partial U}{\partial x} \right) \Big|_{\text{VEV}} = 0 \quad (5.9)$$

The equation for $\hat{j}_\mu^5$ becomes trivial. For simplicity, we use here the notation $U(x,y)$ for $U(\hat{q} + (i/4b)\log \hat{j}_+/\hat{j}_-, \hat{j}_+\hat{j}_-)$. These equations imply,

$$\langle \hat{q} \rangle = -\theta/2b \quad (5.10)$$

$$\frac{\partial U}{\partial \eta} (-\theta/2b, \langle \phi \rangle^2) = 0 \quad (5.11)$$

$$\frac{\partial U}{\partial x} (-\theta/2b, \langle \phi \rangle^2) = 0 \quad (5.12)$$

The first of these gives the VEV for $\hat{q}$. Recalling that the physical field $q = \hat{q} + \theta/2b$ we find as expected that $\langle 0|q|0 \rangle = 0$. The second equation determines the VEV $\langle \phi \rangle$, (recall $\phi \equiv \hat{\phi}$), by minimizing the potential term with respect to $(\hat{j}_+\hat{j}_-)$. We demand a chiral symmetry breaking solution $\langle \phi \rangle \neq 0$. The final equation expresses the θ independence of the theory. To see this, recall from Eq. (2.13) that,

$$0 = \int dx \frac{\delta W}{\delta \theta(x)} \Big|_{\text{physical sources}} = -\frac{1}{b} \int dx \frac{\delta \hat{r}}{\delta \hat{q}} \Big|_{\text{VEV}} \quad (5.13)$$

$$= -\frac{1}{b} \int dx \left. \frac{\partial \mathcal{U}}{\partial x} \right|_{\text{VEV}} \quad (5.13)$$

cont.

Alternatively, the result follows immediately from the $\hat{\Gamma}$ Ward identity (4.1), using the equation of motion $(\delta \hat{\Gamma} / \delta \hat{\phi}_5) |_{\text{VEV}} = 0$.

Next, we require the second functional derivatives of $\hat{\Gamma}$, evaluated at these VEVs. We find

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{j}_\mu^5(x) \delta \hat{j}_\nu^5(y)} = f_2 (\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) \delta(x-y) + \left(f_3 \frac{a^2}{b^2} + 4a^2 \langle \phi^2 \rangle f_1 \right) g_{\mu\nu} \delta(x-y) \quad (5.14)$$

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{j}_\mu^5(x) \delta \hat{q}_f(y)} = f_3 \frac{a}{b} \partial_\mu \delta(x-y) \quad (5.15)$$

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{q}_f(x) \delta \hat{q}_f(y)} = -(f_3 + f_4) \partial^2 \delta(x-y) + \frac{\partial^2 \mathcal{U}(-\theta/2b, \langle \phi^2 \rangle)}{\partial x^2} \delta(x-y) \quad (5.16)$$

$$\begin{aligned} \frac{\delta^2 \hat{\Gamma}}{\delta \phi(x) \delta \phi(y)} &= -\left(f_1 + 4 \langle \phi^2 \rangle f_5 \right) \partial^2 \delta(x-y) \\ &\quad + 4 \langle \phi^2 \rangle \frac{\partial^2 \mathcal{U}(-\theta/2b, \langle \phi^2 \rangle)}{\partial y^2} \delta(x-y) \end{aligned} \quad (5.17)$$

The remaining ones can be deduced from these using the Ward identity relations (4.2) to (4.4). In addition, we find

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{q}_f \delta \hat{\phi}} = 2b \langle \phi \rangle \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_5 \delta \hat{\phi}} = 2 \langle \phi \rangle \frac{\partial^2 \mathcal{U}(-\theta/2b, \langle \phi^2 \rangle)}{\partial x \partial y} \quad (5.18)$$

This would imply parity-violating terms in the derivatives of the true effective action, and must therefore be absent for the reasons already discussed. We are therefore forced to impose the condition,

$$\frac{\partial^2 U}{\partial x \partial y} (-\theta/2b, \langle \phi \rangle^2) = 0 \quad (5.19)$$

on the otherwise arbitrary function U .

Taking the Fourier transform of Eqs. (5.14) to (5.17), we determine the functions $\tilde{P}$, P , Q , R introduced in Section 4:

$$\tilde{P}(k^2) = -f_2 k^2 + 4a^2 \langle \phi \rangle^2 f_1 + f_3 \frac{a^2}{b^2} \quad (5.20)$$

$$P(k^2) = 4a^2 \langle \phi \rangle^2 f_1 + f_3 \frac{a^2}{b^2} \quad (5.21)$$

$$Q(k^2) = f_3 \frac{a}{b} \quad (5.22)$$

$$R(k^2) = (f_3 + f_4) k^2 + u \quad (5.23)$$

where here we have used the abbreviated notation f_i for $f_i(-\theta/2b, \langle \phi \rangle^2)$ and u for $(\partial^2 U / \partial x^2)(-\theta/2b, \langle \phi \rangle^2)$. Also,

$$\begin{aligned} \text{F.T.} \left(\frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right) &= \text{F.T.} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \phi \delta \phi} \right) = (f_1 + 4\langle \phi \rangle^2 f_5) k^2 \\ &\quad + 4\langle \phi \rangle^2 \frac{\partial^2 U}{\partial y^2} (-\theta/2b, \langle \phi \rangle^2) \end{aligned} \quad (5.24)$$

Notice that these second functional derivatives, evaluated at the VEVs involve just ten unknown parameters, viz. a , b , f_i ($i=1, \dots, 5$), u , $\langle \phi^2 \rangle$ and $(\partial^2 U / \partial y^2)(-\theta/2b, \langle \phi \rangle^2)$. The dependence on θ has been eliminated.

These forms for $\tilde{P}$, P , Q , R are consistent with those assumed in Section 4. We therefore retain all of the results derived there. It remains only to express them in terms of these ten parameters.

First consider the mass terms. From Eqs. (5.20) and (5.21) we verify that $\tilde{P}(0) = P(0)$, the final observation required for us to conclude that there is no $k^2 = 0$ pole anywhere in the propagator matrix. The squared mass of the 1^+ meson D is given by the pole in $\alpha^{-1}(k^2)$, so we find

$$m_D^2 = f_2^{-1} \left(a + 4a^2 \langle \phi \rangle^2 f_1 + f_3 \frac{a^2}{b^2} \right) \quad (5.25)$$

Notice here that in an extension of the model to (say) three quark flavours, the analogues of the first two terms would be present for all members of the 1^+ nonet, whereas the final term $(a^2/b^2)(f_3/f_2)$ would appear for the singlet only. In a realistic model this term therefore describes the mass splitting of the octet and singlet pseudovector mesons. It is interesting to note that it enters the effective action as part of the kinetic term for $\hat{q}$, whose inclusion leads to the existence of the second 0^- meson.

The squared masses of the two 0^- mesons η and G are given by the zeros of the function $\Delta(k^2)$. From Eq. (4.26), we see that they are the roots of the quadratic equation,

$$-(a + bk^2) (-Q^2 k^2 + PR) - b^2 P^2 k^2 + 2abQk^2 - a^2 R = 0 \quad (5.26)$$

We will not need to solve this explicitly. Notice however the crucial point that the term in Eq. (5.26) independent of k^2 , i.e., $-a(a + P(0)) R(0)$, is proportional to the parameter u . Setting $u = 0$ we would obtain an equation for which one solution is $m^2 = 0$. In such a case, the η would become a true, massless Goldstone boson. Recalling that $u = (\partial^2 U / \partial x^2) (-\theta/2b, \langle \phi \rangle^2)$ is given by the term in the action which is a function of $(\hat{q} + (1/4b) \log \hat{j}_+ / \hat{j}_-)$, we observe that it is solely the mixing term of the operators $\hat{q}$ and $\hat{\phi}_5$ which gives a mass to the η . The precise form of this term in $\hat{\Gamma}$ is determined by the anomalous chiral Ward identity. We have therefore isolated the mechanism by which the anomaly, by enforcing the operator mixing of q and ϕ_5 , is responsible for the mass generation for the η .

Finally, there is the 0^+ meson S , with squared mass,

$$m_S^2 = - \left(f_1 + 4 \langle \phi \rangle^2 f_5 \right)^{-1} 4 \langle \phi \rangle^2 \frac{\partial^2 K}{\partial y^2} (-\theta/2b, \langle \phi \rangle^2) \quad (5.27)$$

determined from Eq. (5.24).

The complete expression for the important function $\Delta(k^2)$ is now most conveniently written as,

$$\Delta(k^2) = -b (k^2 - m_G^2) (k^2 - m_\gamma^2) (k^2)^{-1} \left(k^2 + \frac{u(4a^2 \langle \phi \rangle^2 f_1 + f_3 a^2 / b^2)}{4a^2 \langle \phi \rangle^2 f_1 (f_3 + f_4) + f_3 f_4 a^2 / b^2} \right)^{-1} \quad (5.28)$$

and we observe explicitly that since

$$\text{F.T.} \left(\frac{\delta^2 W}{\delta \theta \delta \theta} \right) = -\Delta^{-1}(k^2) \quad (5.29)$$

in the zero-momentum limit,

$$\left. \frac{\delta^2 W}{\delta \theta \delta \theta} \right|_{k^2=0} = 0 \quad (5.30)$$

The decay constants are given in terms of the constants Z, E, F, K, L in Eqs. (4.41) to (4.47). In addition the 0^+ meson has a decay constant

$$\langle 0 | \phi(0) | S \rangle \equiv F_S = (f_1 + 4 \langle \phi \rangle^2 f_5)^{-1/2} \quad (5.31)$$

The decay constant of the 1^+ meson is,

$$F_{1^+} = Z^{1/2} = f_2^{-1/2} \quad (5.32)$$

From their definition in Eqs. (4.30) and (4.37) as pole residues involving the function $\Delta(k^2)$, we can simply determine that

$$E = - \frac{m_G^2}{m_G^2 - m_\gamma^2} 2 \langle \phi \rangle (m_\gamma^2 + \sigma) \quad (5.33)$$

$$F = - \frac{m_\gamma^2}{m_\gamma^2 - m_G^2} 2 \langle \phi \rangle (m_G^2 + \sigma) \quad (5.34)$$

where,

$$g = \frac{a}{b} + \frac{f_4}{4b\langle\phi\rangle^2 f_1(t_3+t_4) + \frac{1}{b} f_3 f_4} \quad (5.35)$$

and,

$$K = \frac{\rho(m_G^2)}{m_G^2 - m_\eta^2} \quad (5.36)$$

$$L = \frac{\rho(m_\eta^2)}{m_\eta^2 - m_G^2} \quad (5.37)$$

where

$$\rho(m^2) = -\frac{1}{b} m^2 \left(m^2 + \frac{u(4a^2\langle\phi\rangle^2 f_1 + f_3 a^2/b^2)}{4a^2\langle\phi\rangle^2 f_1(t_3+t_4) + f_3 f_4 a^2/b^2} \right) \quad (5.38)$$

We have now collected the complete set of expressions for the decay constants and masses in terms of the ten parameters arising from $\hat{\Gamma}$. Inverting these expressions, we can express the parameters a and b , which were introduced for dimensional consistency in the non-standard Legendre transform, as follows:

$$a = \left(\frac{F_{jD}^2}{m_D^2} + \frac{F_{jG}^2}{m_G^2} + \frac{F_{j\eta}^2}{m_\eta^2} \right)^{-1} \quad (5.39)$$

and

$$b = - \left(\frac{F_{qG}^2}{m_G^2} + \frac{F_{q\eta}^2}{m_\eta^2} \right)^{-1} \quad (5.40)$$

This completes our discussion of the mass spectrum and decay constants in the one quark flavour model. To summarize, we have constructed a symmetric effective action $\hat{\Gamma}$, which incorporates all the symmetry constraints of the anomalous chiral Ward identities, in a low-energy approximation in which only second derivatives of the fields are retained. The decay constants and masses of the physical bound states have been expressed in terms of the parameters appearing in $\hat{\Gamma}$. In addition, the Legendre transform parameters a and b have been identified with the sum of the ratio of decay constants to masses of the particles to which the appropriate field couples.

The low mass particle spectrum of the theory has been deduced, and the field operators which create these mass eigenstates have been determined. In particular, it has been shown that the operator mixing of ϕ_5 and q gives a mass to the η . To emphasize this point, we refer again to the remarks following Eq. (5.26), where it was noted that in the absence of the mixing parameter u the η mass would vanish.

Finally, the symmetry constraints enabled just two relations amongst the decay constants to be derived. These relations, which are given in Eqs. (4.48) and (4.49), follow directly from the anomaly equation. It is important to realize that these simple relations are in fact the only ones that can be found using just the chiral symmetry properties of the theory. The systematic construction we have presented means that the effective action $\hat{\Gamma}$ in Eq. (5.3) summarizes all the information contained in the anomalous chiral Ward identities. Of course, should any additional symmetry or dynamical principle now be applied which would constrain the parameters appearing in $\hat{\Gamma}$, further relations amongst the decay constants would follow immediately.

6. - 1/N EXPANSION

There have recently been a number of investigations of the $U(1)$ problem and related issues within the context of the $1/N$ expansion. [N is the number of colours in the gauge group $SU(N)$.] Notable amongst these was Witten's derivation¹¹⁾ of a formula for the η' mass in terms of the vacuum energy of the pure gauge theory with no quarks. Effective Lagrangians incorporating the constraints of broken chiral symmetry to lowest order in $1/N$ have been constructed by several authors, including Arnowitt and Nath¹⁸⁾, di Vecchia and Veneziano¹⁴⁾, Rosenzweig, Schechter and Trahern¹⁵⁾, and Witten¹⁶⁾.

In order to make contact with this body of work, we present in this section the leading $1/N$ approximation to the results already derived in full generality in the rest of the text. In particular we recover Witten's mass formula.

However, even working to lowest order in $1/N$, the analysis presented here represents a generalization of the work of these authors, and the results concerning the second pseudoscalar boson are new.

It is worth emphasizing, however, that the use of the $1/N$ expansion here serves merely to give an approximation to the results already obtained, and adds nothing of substance to the analysis already given.

To implement the $1/N$ expansion in this theory, it is convenient first to normalize the operators such that the current operators create mesons with an amplitude of $O(1)$. We therefore define meson operators,

$$M_{\pm} = N^{-1/2} j_{\pm} \quad M_{\mu}^5 = N^{-1/2} j_{\mu}^5 \quad (6.1)$$

With this normalization, the Green functions for products of meson operators M and glueball operators G have the following N dependence:

$$\langle 0 | T M_1 \dots M_n | 0 \rangle \sim N^{1-n/2} \quad (6.2)$$

$$\langle 0 | T M_1 \dots M_n G_1 \dots G_m | 0 \rangle \sim N^{1-n/2-m} \quad (6.3)$$

$$\langle 0 | T G_1 \dots G_m | 0 \rangle \sim N^{2-m} \quad (6.4)$$

For a clear review of the derivation of these results, see Coleman's 1979 Erice lectures¹⁰⁾.

Since all meson and glueball propagators are now of $O(1)$, it follows that the orders of the LPI vertices are the same as for the corresponding Green functions. So by rewriting the effective action with this new normalization of meson operators and using the results (6.2) to (6.4), it is possible to assign by inspection an order in $1/N$ to all the coefficients in the expansion of the functions f_i , ($i=1, \dots, 5$) and U in terms of the fields. Some details of this procedure are discussed in Appendix B. The end result is the following assignment to the ten parameters introduced in Section 5:

$$\begin{aligned}
 a &\sim O(N^{-1}) & b &\sim O(1) \\
 f_1 &\sim O(N^{-1}) & f_2 &\sim O(N^{-1}) \\
 f_3 &\sim O(1) & f_4 &\sim O(1) \\
 f_5 &\sim O(N^{-3}) & u &\sim O(1) \\
 \langle \phi \rangle^2 &\sim O(N^2) & \frac{\partial^2 u}{\partial y^2}(-\theta/2b, \langle \phi \rangle^2) &\sim O(N^{-3})
 \end{aligned} \tag{6.5}$$

The derivation of the masses and decay constants of the bound states given in Section 5 can now be repeated, this time working to lowest order in an expansion in $1/N$.

The following results are straightforwardly obtained:

$$m_S^2 = - (f_1 + 4\langle \phi \rangle^2 f_5)^{-1} 4\langle \phi \rangle^2 \frac{\partial^2 u}{\partial y^2}(-\theta/2b, \langle \phi \rangle^2) \sim O(1) \tag{6.6}$$

$$m_D^2 = f_2^{-1} (a + 4a^2 \langle \phi \rangle^2 f_1) \sim O(1) \tag{6.7}$$

$$m_G^2 = - \left(\frac{u+b}{f_3 + f_4} \right) \sim O(1) \tag{6.8}$$

$$m_\eta^2 = - \frac{u}{4ab \langle \phi \rangle^2 f_1 (u+b)} (a + 4a^2 \langle \phi \rangle^2 f_1) \sim O(N^{-1}) \tag{6.9}$$

Notice that the squared mass of the η vanishes to $O(1)$. The leading contribution, as obtained from Eq. (5.26), is of $O(N^{-1})$. Notice also that the term in the D mass equal to $(f_3/f_2)(a^2/b^2)$ drops out of Eq. (6.7) as it is of $O(N^{-1})$. Referring to the remarks following Eq. (5.25), we observe that in a realistic three-flavour model, this implies that the octet-singlet mass splitting for the pseudovector mesons is of $O(N^{-1})$.

The decay constants are similarly simplified,

$$\langle 0 | q(0) | G \rangle \equiv F_{qG} = (f_3 + f_4)^{-1/2} \sim O(1) \quad (6.10)$$

$$\langle 0 | q(0) | \gamma \rangle \equiv F_{q\gamma} = m_\gamma \left(-\frac{u}{b(u+b)} \right)^{1/2} \sim O(N^{-1/2}) \quad (6.11)$$

$$\begin{aligned} \langle 0 | M_5(0) | G \rangle \equiv F_{M_5G} &= -2N^{-1/2} \langle \phi \rangle (f_3 + f_4)^{1/2} \\ &\times \left(m_\gamma^2 + \frac{a}{b} + \frac{f_4}{4b\langle \phi \rangle^2 f_1(f_3 + f_4)} \right) \sim O(N^{-1/2}) \quad (6.12) \end{aligned}$$

$$\langle 0 | M_5(0) | \gamma \rangle \equiv F_{M_5\gamma} = 2N^{-1/2} \langle \phi \rangle m_\gamma \left(-\frac{u}{b(u+b)} \right)^{-1/2} \sim O(1) \quad (6.13)$$

$$\langle 0 | M_\mu^5(0) | G \rangle \equiv i k_\mu \frac{F_{M_5G}}{m_G} = \frac{i k_\mu}{m_G^2} N^{-1/2} \langle 0 | q(0) | G \rangle \sim O(N^{1/2}) \quad (6.14)$$

$$\langle 0 | M_\mu^5(0) | \gamma \rangle \equiv i k_\mu \frac{F_{M_5\gamma}}{m_\gamma} = \frac{i k_\mu}{m_\gamma^2} N^{-1/2} \langle 0 | q(0) | \gamma \rangle \sim O(1) \quad (6.15)$$

$$\langle 0 | M_\mu^5(0) | D \rangle \equiv \epsilon_\mu^{(\lambda)}(k) F_{M_5D} = \epsilon_\mu^{(\lambda)}(k) N^{-1/2} f_2^{-1/2} \sim O(1) \quad (6.16)$$

$$\begin{aligned} \langle 0 | M(0) | S \rangle \equiv F_S &= N^{-1/2} (f_1 + 4\langle \phi \rangle^2 f_5)^{-1/2} \quad (6.17) \\ &\sim O(1) \end{aligned}$$

All the decay constants are therefore of $O(1)$, except those describing the coupling of a meson operator to the G , and the glueball operator q to the η , which are suppressed by a factor $N^{-1/2}$. Recall that in the context of the $1/N$ expansion, glueballs and mesons are distinct objects, and we may here identify the G as a O^- glueball and the η as a O^- meson.

The Legendre transform parameters are identified to leading order as,

$$Na = \left(\frac{F_{M\eta}^2}{m_\eta^2} + \frac{F_{M\eta}^2}{m_\eta^2} \right)^{-1} \quad (6.18)$$

$$b = - \left(\frac{F_{qG}^2}{m_G^2} + \frac{F_{q\eta}^2}{m_\eta^2} \right)^{-1} \quad (6.19)$$

Finally, we can rederive Witten's formula. In the notation of Ref. 11) this states, for a one-flavour theory,

$$m_\eta^2 = \frac{4}{f_\eta^2} \left(\frac{d^2 E}{d\theta^2} \right)_{\text{no quarks}} \quad (6.20)$$

Here, E is the vacuum energy in the corresponding pure gauge theory, i.e., in the $N \rightarrow \infty$ limit. In the notation used here, this relation simply reads

$$\left(\frac{\delta^2 W}{\delta\theta \delta\theta} \right)_{N \rightarrow \infty, k^2=0} = - \frac{1}{m_\eta^2} | \langle 0 | q(0) | \eta \rangle |^2 \quad (6.21)$$

Notice that at $k^2 = 0$ the exact expression $\delta^2 W / \delta\theta \delta\theta$ is zero. However, if the limit $N \rightarrow \infty$ is taken first, before setting $k^2 = 0$, this is not necessarily so. This is the limit required for Witten's formula to hold.

The exact expression for $\delta^2 W / \delta\theta \delta\theta$ is given by Eqs. (5.28) and (5.29). It is,

$$\frac{\delta^2 W}{\delta \theta \delta \theta} = \frac{1}{b} (k^2 - m_G^2)^{-1} (k^2 - m_3^2)^{-1} k^2 \left(k^2 + \frac{u (4a^2 \langle \phi \rangle^2 f_1 + f_3 a^2/b^2)}{4a^2 \langle \phi \rangle^2 f_1 (f_3 + f_4) + f_3 f_4 a^2/b^2} \right)^{-1} \quad (6.22)$$

Recalling that m_η^2 is of $O(N^{-1})$, we find that in the limit $N \rightarrow \infty$ with k^2 fixed, independent of N ,

$$\left(\frac{\delta^2 W}{\delta \theta \delta \theta} \right)_{N \rightarrow \infty} = \frac{1}{b} (k^2 - m_G^2)^{-1} \left(k^2 + \frac{u}{f_3 + f_4} \right) \quad (6.23)$$

and so,

$$\left(\frac{\delta^2 W}{\delta \theta \delta \theta} \right)_{N \rightarrow \infty, k^2 \neq 0} = \frac{u}{b(u+b)} \quad (6.24)$$

using Eq. (6.8) for m_G^2 . Referring to Eq. (6.11), we see that this is indeed equal to $(-1/m_\eta^2) |\langle 0 | q(0) | \eta \rangle|^2$, and we confirm Eq. (6.21). Witten's formula therefore appears as a simple consequence of the complete set of results for the decay constants and masses of the light bound states in the $1/N$ expansion.

To complete this comparison of the $1/N$ approximation to our general formalism with the work of previous authors¹⁴⁾⁻¹⁶⁾, we can show how to reduce the true effective action to the form used by them to obtain results to leading order in $1/N$. Of course, our results are for one flavour only. In Appendix B, we derive a reduced form of the symmetric effective action which reproduces the leading $1/N$ approximation to the two-point Green functions. This

$$\begin{aligned} \hat{\Gamma} = \int d^4x \left\{ \frac{1}{2} f_1 (\hat{j}_+ \hat{j}_-) (\partial_\mu \hat{j}_+)(\partial_\mu \hat{j}_-) - \frac{1}{4} f_2 (\hat{j}_+ \hat{j}_-) (\partial_\mu \hat{j}_+^5 - \partial_\mu \hat{j}_-^5)^2 \right. \\ + \frac{1}{2} f_3 \left(\partial_\mu \hat{q} + \frac{a}{b} \hat{j}_+^5 \right)^2 + \frac{1}{2} f_4 \left(\partial_\mu \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right) \right)^2 \\ + \frac{1}{2} f_5 (\hat{j}_+ \hat{j}_-) (\partial_\mu (\hat{j}_+ \hat{j}_-))^2 - V(\hat{j}_+ \hat{j}_-) \\ \left. + u' \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right) + \frac{u}{2} \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right)^2 \right\} \quad (6.25) \end{aligned}$$

where the equations of motion require that $V'(\langle\phi\rangle^2) = 0$ and $u' = \theta u/2b$. Furthermore, these authors omit the pseudovector current altogether, and do not include a kinetic term for the operator q . This means that their effective action describes only a single 0^- bound state (together of course with the 0^+ meson). Making these changes in Eq. (6.25) gives,

$$\hat{\Gamma} = \int d^4x \left\{ \frac{1}{2} f_1 (\partial_\mu \hat{j}_+)(\partial_\mu \hat{j}_-) + \frac{1}{2} f_5 (\partial_\mu (\hat{j}_+ \hat{j}_-))^2 - V(\hat{j}_+ \hat{j}_-) + \frac{u}{2} \left(\hat{q} + \frac{\theta}{2b} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right)^2 - \frac{u\theta^2}{8b^2} \right\} \quad (6.26)$$

Referring now to the definitions (3.1) and (3.4), and setting the source $\theta(x)$ to its physical value since $q(x)$ is no longer a dynamical field, we can write the true effective action as,

$$\Gamma = \int d^4x \left\{ \frac{1}{2} f_1 (\partial_\mu j_+)(\partial_\mu j_-) + \frac{1}{2} f_5 (\partial_\mu (j_+ j_-))^2 - V(j_+ j_-) + \frac{1}{2} u \left(q + \frac{i}{4b} \log \frac{j_+}{j_-} \right)^2 - \frac{u\theta^2}{8b^2} + \frac{1}{2} b q^2 - \frac{\theta}{2} q \right\} \quad (6.27)$$

since $q = \hat{q} + \theta/2b$. Since there is no kinetic term for q , it may be eliminated using the equation of motion,

$$\frac{\delta \Gamma}{\delta q} = -\frac{\theta}{2} \quad (6.28)$$

which implies,

$$q = -\frac{u}{u+b} \left(\frac{i}{4b} \log \frac{j_+}{j_-} \right) \quad (6.29)$$

Substituting back into Eq. (6.27) we find,

$$\Gamma = \int d^4x \left\{ \frac{1}{2} f_1 (\partial_\mu j_+)(\partial_\mu j_-) + \frac{1}{2} f_5 (\partial_\mu (j_+ j_-))^2 - V(j_+ j_-) + \frac{1}{8} \frac{u}{b(u+b)} \left(\theta + \frac{i}{2} \log \frac{j_+}{j_-} \right)^2 - \frac{u(u+2b)}{8b^2(u+b)} \theta^2 \right\} \quad (6.30)$$

The final term proportional to θ^2 is independent of the fields and may be dropped.

Apart from the f_5 term, which only contributes to the scalar meson propagator, this is now the effective action previously used^{14),16)} in analyzing θ dependence and the $U(1)$ problem in the context of the $1/N$ expansion. Notice that the coefficient of the novel term $-\frac{1}{2}(\theta + (i/2)\log j_+/j_-)^2$ is $-u/4b(u+b)$. Referring to Eq. (6.24) we see that this is equal to the expression $(d^2E/d\theta^2)$ appearing in Witten's mass formula, i.e., the second derivative with respect to θ of the vacuum energy in the corresponding pure gauge theory with no quarks.

7. - DISCUSSION

In this paper we have used functional methods to construct an effective action which describes one sector of the low energy dynamics of quantum chromodynamics with one quark flavour. By constructing the propagator matrix for the fields j_μ^5 , q and ϕ_5 associated with the spontaneous breaking of chiral symmetry and with the axial anomaly, we have determined the mass spectrum and decay constants of the low-mass 1^+ and 0^- mesons, together with those of the associated 0^+ meson. The symmetry implied by the anomalous chiral Ward identities has been carefully implemented in the effective action, and we have shown how this symmetry determines the mixing of the field operators and consequently the mass generation mechanism for the light 0^- meson.

One of the new features of our effective action is the inclusion of the kinetic term for the field q . This implies directly the existence of the second 0^- meson. It is natural in the framework of the low energy approximation we are using to include this term. We should emphasize, however, that the operator mixing mechanism for the mass generation for the original pseudoscalar meson, and thus the resolution of the $U(1)$ problem, is independent of the inclusion of this kinetic term.

Using the methods developed here, this simple model may be readily extended to a more realistic theory. The inclusion of more quark flavours is straightforward, the chiral Ward identities being generalized to the enlarged chiral group. Including non-zero quark masses has two effects. First, they provide an explicit breaking of the flavour symmetries. Second, they introduce a non-trivial dependence on the vacuum angle θ into the theory, the relation $\int dx (\delta W/\delta \theta(x)) = 0$, which expresses the θ independence of the physics in the massless theory, being lost when all the quark masses are non-zero. An extension of the results of this paper to a realistic QCD theory should prove interesting.

On the basis of the results obtained here, however, we can already make some informed speculation about the spectrum in the realistic case. Since we have only derived relations amongst the decay constants of the bound states,

and not their masses, we cannot predict the mass ratio of the two expected pseudoscalar mesons. Nevertheless we would expect an extra 0^- state to exist in the QCD spectrum with a reasonably light mass. This view is supported by phenomenological estimates^{24)-26),18)} which indicate that the gluonic component of the η' wave function is large. The mass of the complementary pseudoscalar state is therefore unlikely to differ too greatly. It is possible that this particle is to be identified with the recently discovered i or G (1440 MeV). For reviews of the phenomenology of this state, see Refs. 27) and 28).

We also noted that, just as the mass splitting of the singlet (η') and octet pseudoscalar mesons is determined by the potential term involving $q(x)$ in the effective action, so the mass splitting of the singlet and octet pseudovector mesons is determined by part of the kinetic term for $q(x)$. Formally, both mass splittings are of $O(1/N)$. This provides a possibly interesting connection between the conventional meson spectrum and the existence of the extra mixed glueball state.

Finally we comment briefly on the formalism we have developed in this paper. This general approach of using functional Ward identities to establish the invariants from which an effective action can be constructed has a very general validity. Effective actions can be constructed to describe the dynamics of any set of fields which are considered on physical grounds to be relevant for a particular problem. The symmetry constraints can always be expressed in terms of Ward identities for the generating functional W . One of the main results of this paper is the demonstration of how, by choosing an appropriately generalized Legendre transform, it is then possible to define an effective action for which the Ward identities imply a simple symmetry property. This effective action is constructed from invariants of this symmetry transformation, subject to certain physically motivated dynamical assumptions. This method is systematic and powerful, and we hope that it will find wide application in quantum field theory.

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APPENDIX A - LEGENDRE TRANSFORMS

We discuss here the motivations for the use of the non-standard Legendre transform in Section 3, and derive general relations enabling functional derivatives of the symmetric effective action defined with this new transform to be compared to derivatives of the true effective action.

Consider first the simple example of one field, and define as usual,

$$\Gamma[\phi] = W[J] - \int dx J \phi \quad (A.1)$$

where,

$$\phi = \frac{\delta W}{\delta J} \quad \frac{\delta \Gamma}{\delta \phi} = -J \quad (A.2)$$

Compare the non-standard Legendre transform,

$$\hat{\Gamma}[\hat{\phi}] = W[J] - \int dx \frac{1}{2} a (\hat{\phi} + \frac{1}{a} J)^2 \quad (A.3)$$

The parameter a is required for dimensional consistency. Here,

$$\hat{\phi} = \frac{\delta W}{\delta J} - \frac{1}{a} J \quad \frac{\delta \hat{\Gamma}}{\delta \hat{\phi}} + a \hat{\phi} = -J \quad (A.4)$$

Now observe that with the original Legendre transform (A.1)

$$J \frac{\delta W}{\delta J} = -\phi \frac{\delta \Gamma}{\delta \phi} \quad (A.5)$$

With the new transform (A.3) this result is lost, but is replaced by

$$\frac{\delta W}{\delta J} = -\frac{1}{a} \frac{\delta \hat{\Gamma}}{\delta \hat{\phi}} \quad (A.6)$$

Notice that this result remains true for any Legendre transform of the form

$$\hat{\Gamma}[\hat{\phi}] = W[J] - F[\hat{\phi} + \frac{1}{a} J] \quad (A.7)$$

As explained in Section 3, we wish to define an effective action $\hat{\Gamma}$ which obeys a simple Ward identity analogous to that for W , and which will therefore have a correspondingly simple invariance property. This will greatly simplify the problem of actually constructing this effective action $\hat{\Gamma}$ in a suitable dynamical approximation. The price that has to be paid for this simplicity is that it is then necessary to relate the symmetric $\hat{\Gamma}$ to the true effective action Γ defined by the standard Legendre transform, which is the generating functional for 1PI vertices.

Referring now to the Ward identity (2.8) for W , which is quite typical of such identities, we see that there are two types of term, one involving just derivatives with respect to the source, $\delta W/\delta J$, as in Eq. (A.6), and the other involving products $J(\delta W/\delta J)$, as in Eq. (A.5). In order to construct a $\hat{\Gamma}$ which satisfies a Ward identity with the same structure as that for W , we therefore make a Legendre transform of the original kind for sources entering the Ward identity in the form (A.5), and make the new transform for sources entering in the form (A.6).

To elaborate a little more, in the case of a gauge-like symmetry involving a source J_μ transforming as

$$J_\mu \rightarrow J_\mu + a \partial_\mu \omega + J_\mu \times \omega \quad (A.8)$$

then if we require the corresponding field to transform similarly, i.e.,

$$\hat{\phi}_\mu \rightarrow \hat{\phi}_\mu - \partial_\mu \omega + \hat{\phi}_\mu \times \omega \quad (A.9)$$

we find that the combination

$$\left(\hat{\phi}_\mu + \frac{1}{a} J_\mu \right) \rightarrow \left(\hat{\phi}_\mu + \frac{1}{a} J_\mu \right) + \left(\hat{\phi}_\mu + \frac{1}{a} J_\mu \right) \times \omega \quad (A.10)$$

The non-linear term $\partial_\mu \omega$ has cancelled. This provides some additional insight into why the Legendre transforms of type (A.7) are useful here.

Having thus motivated the use of the non-standard transform, and indeed of mixed transforms as actually used in Section 3, we must now discuss the relation of $\hat{\Gamma}$ to Γ . We do this first for the simple one-field example above, then in full generality for mixed transforms.

First notice that the fields ϕ and $\hat{\phi}$ are not identical. In fact,

$$\phi = \hat{\phi} + \frac{1}{a} J = -\frac{1}{a} \frac{\delta \hat{\Gamma}}{\delta \hat{\phi}} \quad (\text{A.11})$$

The different effective actions are related by the formula,

$$\Gamma[\phi] = \hat{\Gamma}[\hat{\phi}] - \int dx \left\{ \frac{1}{2a} \left(\frac{\delta \hat{\Gamma}}{\delta \hat{\phi}} \right)^2 + \hat{\phi} \frac{\delta \hat{\Gamma}}{\delta \hat{\phi}} \right\} \quad (\text{A.12})$$

More useful are the relations amongst the functional derivatives of Γ and $\hat{\Gamma}$. We find (suppressing the space-time arguments of ϕ and $\hat{\phi}$ for simplicity),

$$\frac{\delta \Gamma}{\delta \phi} = \frac{\delta \hat{\Gamma}}{\delta \hat{\phi}} + a \hat{\phi} \quad (\text{A.13})$$

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} &= -\frac{\delta J}{\delta \phi} = a \left(-1 + \frac{\delta \hat{\phi}}{\delta \phi} \right) = a \left(-1 + \left(\frac{\delta \phi}{\delta \hat{\phi}} \right)^{-1} \right) \\ &= -a - a^2 \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi} \delta \hat{\phi}} \right)^{-1} \end{aligned} \quad (\text{A.14})$$

etc.

We now generalize the discussion to the case of an arbitrary number of fields $\hat{\phi}_\alpha$ defined with the original Legendre transform, and an arbitrary number of fields $\hat{A}_i$ defined with the new transform, i.e.,

$$\hat{\Gamma}[\hat{A}_i, \hat{\phi}_\alpha] = W[\kappa_i, J_\alpha] - \int dx \left\{ \frac{1}{2} a_i \left(A_i + \frac{1}{a_i} \kappa_i \right)^2 + J_\alpha \phi_\alpha \right\} \quad (\text{A.15})$$

The usual convention of summation over repeated indices is made, but is not applied to the indices on the parameters a_i . The fields $\hat{A}_i, \hat{\phi}_\alpha$ are defined by,

$$\hat{A}_i = \frac{\delta W}{\delta \kappa_i} - \frac{1}{a} \kappa_i, \quad \hat{\phi}_\alpha = \frac{\delta W}{\delta J_\alpha} \quad (\text{A.16})$$

and the following relations hold,

$$\frac{\delta \hat{\Gamma}}{\delta \hat{A}_i} + a_i \hat{A}_i = -K_i \quad \frac{\delta \hat{\Gamma}}{\delta \hat{\phi}_\alpha} = -J_\alpha \quad (A.17)$$

This symmetric effective action is to be compared to the standard generating functional of 1PI vertices defined by,

$$\Gamma[A_i, \phi_\alpha] = W[K_i, J_\alpha] - \int dx \{ K_i A_i + J_\alpha \phi_\alpha \} \quad (A.18)$$

Here the fields are,

$$A_i = \frac{\delta W}{\delta K_i} \quad \phi_\alpha = \frac{\delta W}{\delta J_\alpha} \quad (A.19)$$

and,

$$\frac{\delta \Gamma}{\delta A_i} = -K_i \quad \frac{\delta \Gamma}{\delta \phi_\alpha} = -J_\alpha \quad (A.20)$$

Notice that of course $\hat{\phi}_\alpha \equiv \phi_\alpha$.

We now derive the relation between the second functional derivatives of Γ and $\hat{\Gamma}$. Using Eqs. (A.16), (A.17), (A.19) and (A.20) we find,

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta A_i \delta A_j} &= - \frac{\delta K_j}{\delta A_i} = a_j \frac{\delta}{\delta A_i} (\hat{A}_j - A_j) \\ &= -a_j \delta_{ij} + a_j \frac{\delta \hat{A}_j}{\delta A_i} \end{aligned} \quad (A.21)$$

$$\begin{aligned} \frac{\delta^2 \Gamma}{\delta A_i \delta \phi_\rho} &= - \frac{\delta J_\rho}{\delta A_i} = \frac{\delta}{\delta A_i} \left(\frac{\delta \hat{\Gamma}}{\delta \hat{\phi}_\rho} \right) \\ &= \frac{\delta \hat{A}_k}{\delta A_i} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A}_k \delta \hat{\phi}_\rho} \right) + \frac{\delta \hat{\phi}_\gamma}{\delta A_i} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\gamma \delta \hat{\phi}_\rho} \right) \end{aligned} \quad (A.22)$$

$$\frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta A_j} = a_j \frac{\delta}{\delta \phi_\alpha} (\hat{A}_j - A_j) = a_j \frac{\delta \hat{A}_j}{\delta \phi_\alpha} \quad (\text{A.23})$$

$$\frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta \phi_\rho} = \frac{\delta \hat{A}_k}{\delta \phi_\alpha} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A}_k \delta \hat{\phi}_\rho} \right) + \frac{\delta \hat{\phi}_\sigma}{\delta \phi_\alpha} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\sigma \delta \hat{\phi}_\rho} \right) \quad (\text{A.24})$$

Again we have suppressed the space-time arguments of the fields. These follow the pattern of the discrete indices with integration replacing summation over repeated labels.

Next we need the matrix $\delta(\hat{A}, \hat{\phi})/\delta(A, \phi)$ which we evaluate as the inverse of $\delta(A, \phi)/\delta(\hat{A}, \hat{\phi})$. We have,

$$\left[\frac{\delta(A, \phi)}{\delta(\hat{A}, \hat{\phi})} \right]_{i\alpha, j\beta} = \begin{bmatrix} -\frac{1}{a_i} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{A}_j \delta \hat{A}_i} & -\frac{1}{a_i} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\rho \delta \hat{A}_i} \\ 0 & \delta_{\alpha\rho} \end{bmatrix} \quad (\text{A.25})$$

$$\Rightarrow \left[\frac{\delta(\hat{A}, \hat{\phi})}{\delta(A, \phi)} \right]_{i\alpha, j\beta} = \begin{bmatrix} -a_j \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A} \delta \hat{A}} \right)^{-1}_{ji} & -\frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\rho \delta \hat{A}_k} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A} \delta \hat{A}} \right)^{-1}_{ki} \\ 0 & \delta_{\alpha\beta} \end{bmatrix} \quad (\text{A.26})$$

Finally, substituting back into Eqs. (A.21) to (A.24) we find,

$$\frac{\delta^2 \Gamma}{\delta A_i \delta A_j} = -a_i \delta_{ij} - a_i a_j \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A} \delta \hat{A}} \right)^{-1}_{ij} \quad (\text{A.27})$$

$$\frac{\delta^2 \Gamma}{\delta A_i \delta \phi_p} = -a_i \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A} \delta \hat{A}} \right)^{-1}_{ik} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{A}_k \delta \phi_p} \quad (\text{A.28})$$

$$\frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta A_j} = -a_j \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\alpha \delta \hat{A}_k} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A} \delta \hat{A}} \right)^{-1}_{kj} \quad (\text{A.29})$$

$$\frac{\delta^2 \Gamma}{\delta \phi_\alpha \delta \phi_p} = \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\alpha \delta \hat{\phi}_p} - \frac{\delta^2 \hat{\Gamma}}{\delta \hat{\phi}_\alpha \delta \hat{A}_k} \left(\frac{\delta^2 \hat{\Gamma}}{\delta \hat{A} \delta \hat{A}} \right)^{-1}_{kl} \frac{\delta^2 \hat{\Gamma}}{\delta \hat{A}_l \delta \phi_p} \quad (\text{A.30})$$

These are the results required to derive Eq. (4.15) of Section 4 in the example in the text, where there are just two fields of each type.

Note the simple matrix structure of these equations. If we denote the fields by $F = \{A_i, \phi_\alpha\}$ and neglect the parameters a_i [which have to be put in correctly using Eqs. (A.27) to (A.30)], we can summarize our results in the following symbolic form. Letting,

$$\frac{\delta^2 \hat{\Gamma}}{\delta \hat{F} \delta \hat{F}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (\text{A.31})$$

so that A is the matrix $\delta^2 \hat{\Gamma} / \delta \hat{A}_i \delta \hat{A}_j$, etc. we find

$$\frac{\delta^2 \Gamma}{\delta F \delta F} = \begin{bmatrix} -1 - A^{-1} & -A^{-1} B \\ -C A^{-1} & D - C A^{-1} B \end{bmatrix} \quad (\text{A.32})$$

Relations amongst the higher derivatives of $\hat{\Gamma}$ and Γ can easily be obtained using the same methods, but these are not required in this paper.

APPENDIX B - 1/N POWER COUNTING

In this Appendix we give some details of the assignment of orders in $1/N$ to the parameters arising from the effective action. This completes the discussion of the $1/N$ expansion given in Section 5.

As indicated there, we rescale the quark bilinears to give correctly normalized meson operators as follows,

$$M_{\pm} = N^{-1/2} j_{\pm} \quad M_{\pm}^5 = N^{-1/2} j_{\pm}^5 \quad (B.1)$$

The connected Green functions for products of meson (M) or glueball (G) operators have the following orders¹⁰⁾,

$$\langle 0 | T M_1 \dots M_n | 0 \rangle \sim N^{1-n/2} \quad (B.2)$$

$$\langle 0 | T M_1 \dots M_n G_1 \dots G_m | 0 \rangle \sim N^{1-n/2-m} \quad (B.3)$$

$$\langle 0 | T G_1 \dots G_m | 0 \rangle \sim N^{2-m} \quad (B.4)$$

The 1PI vertices have the same order as the corresponding Green functions, since all propagators are of $O(1)$.

In terms of the operators M , the symmetric effective action becomes,

$$\begin{aligned} \hat{\Gamma} = \int d^4x \left\{ U \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{M}_+}{\hat{M}_-} \right), N \hat{M}_+ \hat{M}_- \right. \\ + \frac{N}{2} f_1 (D_{\mu} \hat{M}_+) (D_{\mu} \hat{M}_-) - \frac{N}{4} f_2 (\partial_{\mu} M_{\mu}^5 - \partial_{\nu} M_{\nu}^5)^2 \\ + \frac{1}{2} f_3 \left(\partial_{\mu} \hat{q} + \frac{a}{b} N^{1/2} \hat{M}_{\mu}^5 \right)^2 \\ \left. + \frac{1}{2} f_4 \left(\partial_{\mu} \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{M}_+}{\hat{M}_-} \right) \right)^2 + \frac{N^2}{2} f_5 (\partial_{\mu} (\hat{M}_+ \hat{M}_-))^2 \right\} \end{aligned} \quad (B.5)$$

where $D_{\mu} \hat{M}_{\pm} = (\partial_{\mu} \pm 2ia N^{1/2} \hat{M}_{\mu}^5) \hat{M}_{\pm}$, and the $f_i(x,y)$ are functions of the two variables $x = \hat{q} + (i/4b) \log \hat{M}_+ / \hat{M}_-$ and $y = N \hat{M}_+ \hat{M}_-$.

To assign powers of $1/N$ we argue as follows. Write,

$$f_1(x, y) = \sum_{r,s} f_1^{rs} x^r y^s \quad (B.6)$$

First we observe that $N f_1^{00}$ is the coefficient of a term proportional to $(\partial_{\mu+} \hat{M}_+)(\partial_{\mu-} \hat{M}_-)$ in $\hat{\Gamma}$. Since from the general counting rules (B.2) to (B.4) this is of $O(1)$, we conclude that f_1^{00} is of $O(N^{-1})$. Similarly, $N^{3/2} f_1^{00} a$ is the coefficient of an $(\hat{M}_+ \hat{M}_+^5 \partial_{\mu+} \hat{M}_-)$ vertex, which is of $O(N^{-1/2})$, from which we conclude that a is of $O(N^{-1})$. In the same way, for example, $N^2 f_1^{11}$ is the coefficient of a term proportional to $(\hat{M}_+ \hat{M}_- q \partial_{\mu+} \hat{M}_+ \partial_{\mu-} \hat{M}_-)$ which is of $O(N^{-2})$, and so f_1^{11} is of $O(N^{-4})$. Collecting such results, we deduce

$$f_1^{rs} \sim O(N^{-1-r-2s}) \quad (B.7)$$

Similarly we find $f_2^{rs} \sim O(N^{-1-r-2s})$. The situation for f_3 is slightly different, however, because of the additional power of N in the Green function for a string of purely gluonic operators [see Eq. (B.4)]. Here we find,

$$f_3^{rs} \sim O(N^{-r}) \quad \left. f_3^{rs} \right|_{s \neq 0} \sim O(N^{-1-r-2s}) \quad (B.8)$$

There is one additional subtlety here. $N^{1/2} f_3^{00} a/b$ is the coefficient of $(\partial_{\mu+} \hat{q} \hat{M}_+^5)$, from which we deduce that b is of $O(1)$. With this assignment, however, the coefficient of the $(\hat{M}_+^5)^2$ term, viz. $N f_3^{00} a^2/b^2$, is only of $O(N^{-1})$ whereas from the counting rules we would expect it to be of $O(1)$. The explanation is that this term is suppressed in the symmetric effective action because of the chiral invariance property of $\hat{\Gamma}$. In fact, the true effective action Γ contains an extra term proportional to $N a (\hat{M}_+^5)^2$ which is indeed of $O(1)$ and unsuppressed. Similar suppressions also appear for some other terms, and care has to be used in assigning orders in $1/N$ correctly to the coefficients.

Now consider the potential term $U(x, y)$. Similar analysis gives,

$$U^{10} \sim O(1) \quad U^{r0} \sim O(N^{2-r}) \quad (B.9)$$

$$\left. U^{rs} \right|_{s \neq 0} \sim O(N^{-1-r-2s})$$

The VEV $\langle \phi \rangle^2 = N \langle M \rangle^2$ is found from the equation of motion,

$$\frac{\partial U}{\partial y}(-\theta/2b, \langle \phi \rangle^2) = 0 \quad (B.10)$$

and from the above formula for U^{rs} we conclude that $\langle \phi \rangle^2$ is of $O(N^2)$.

We are now able to write down the orders of all the parameters appearing in the second functional derivatives of $\hat{\Gamma}$. Knowing $a \sim O(N^{-1})$, $b \sim O(1)$, $\langle \phi \rangle^2 \sim O(N^2)$, we find

$$f_1 \equiv f_1(-\theta/2b, \langle \phi \rangle^2) \sim f_1(0, \langle \phi \rangle^2) \sim O(N^{-1}) \quad (B.11)$$

$$f_2 \sim f_2(0, \langle \phi \rangle^2) \sim O(N^{-1}) \quad (B.12)$$

$$f_3 \sim f_3^{00} \sim O(1) \quad (B.13)$$

$$f_4 \sim f_4^{00} \sim O(1) \quad (B.14)$$

$$f_5 \sim f_5(0, \langle \phi \rangle^2) \sim O(N^{-2}) \quad (B.15)$$

Similarly, for the second derivatives of U ,

$$u \equiv \frac{\partial^2 U}{\partial x^2}(-\theta/2b, \langle \phi \rangle^2) \sim 2 U^{20} \sim O(1) \quad (B.16)$$

$$\frac{\partial^2 U}{\partial y^2}(-\theta/2b, \langle \phi \rangle^2) \sim \left. \frac{\partial^2 U}{\partial y^2}(0, y) \right|_{y=\langle \phi \rangle^2} \sim O(N^{-3}) \quad (B.17)$$

These are the results stated in the text in Eq. (6.5), and are the best expression of the $1/N$ dependence of the theory. We can, however, summarize the above discussion by writing an effective action which contains only those terms necessary to generate the lowest order results for the two-point Green functions. (Notice that it is of course meaningless to talk about writing an effective action "to lowest order in $1/N$ ". Only the Green functions and 1PI vertices have a definite order, and the terms in the generating functional which must be kept in order to generate the lowest order approximation depend on the particular Green function or vertex required.) This is given by

$$\begin{aligned}
 \hat{\Gamma} = \int d^2x \left\{ \frac{1}{2} f_1(\hat{j}_+\hat{j}_-) (\partial_\mu \hat{j}_+) (\partial_\mu \hat{j}_-) \right. & - \frac{1}{4} f_2(\hat{j}_+\hat{j}_-) (\partial_\mu \hat{j}_+^5 - \partial_\mu \hat{j}_-^5)^2 \\
 + \frac{1}{2} f_3 \left(\partial_\mu \hat{q} + \frac{a}{b} \hat{j}_+^5 \right)^2 & + \frac{1}{2} f_4 \left(\partial_\mu \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right) \right)^2 \\
 + \frac{1}{2} f_5(\hat{j}_+\hat{j}_-) (\partial_\mu (\hat{j}_+\hat{j}_-))^2 & - v(\hat{j}_+\hat{j}_-) \\
 + u' \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right) & + \frac{u}{2} \left(\hat{q} + \frac{i}{4b} \log \frac{\hat{j}_+}{\hat{j}_-} \right)^2 \left. \right\}
 \end{aligned}
 \tag{B.18}$$

where the equations of motion require that $V'(\langle \phi \rangle^2) = 0$ and that $u' = \theta u / 2b$.

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