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MODULO 8 PERIODICITY OF REAL CLIFFORD ALGEBRAS
AND PARTICLE PHYSICS

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ABSTRACT

After a review of the properties of real Clifford algebras, we discuss the isomorphism existing between these algebras and matrix algebras over the real, complex or quaternion field. This is done for all dimensions and all possible signatures of the metric. The modulo 8 periodicity theorem is discussed and extended. A comment is made about the appearance of "hidden" symmetries in supergravity theories.

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INTRODUCTION

In the first section of this paper we review the basic definitions and elementary properties of Clifford algebras. Of course, everybody knows that if the dimension of space-time is $2r$, the complexification of the Clifford algebra is isomorphic to the full $2^r \times 2^r$ matrix algebra over the complex numbers, Dirac spinors being then 2^r component objects with complex entries. However, here we are interested in the structure of the real Clifford algebra (prior to complexification), associated with a metric of signature (p,q) . The results exhibited in the first section, although standard in mathematics, are not very often quoted in physical papers. Their use, however, could very often clarify the exposure of physical theories such as Kaluza-Klein theories, extended supergravities, supersymmetries or even string models. It is known that a real Clifford algebra associated with a given quadratic form is isomorphic to a matrix algebra over the real, complex or quaternion field. The precise structure of this matrix algebra is given in the second part of this paper, first for metrics of signature $(n,0)$ or $(0,n)$, but for $1 \leq n \leq 8$. The periodicity theorem¹⁾, to be discussed in the same section, tells us, roughly speaking, that once you know what happens for $n \leq 8$ you also know what happens for $n > 8$. In the same section, we generalize this theorem to the case of an arbitrary metric of signature (p,q) ; this is the main result of this section. In the third and final section, we look at the isometry group of the "natural" metric defined on the vector space associated with the defining representation of these matrix algebras and show a connection between this group and the local "hidden" symmetries appearing in maximal supergravity theories for a number of extradimensions ranging from 1 to 8.

1. - SOME PROPERTIES OF CLIFFORD ALGEBRAS

V being a real vector space of dimension n endowed with a non-degenerate quadratic form of signature (p,q) , i.e., a "symmetric metric" g with p + signs and q - signs ($p+q = n$), we select an orthonormal basis $(e_1, e_2, \dots, e_n)$ with

$$\begin{aligned} g(e_i, e_i) &= +1 & \text{if } i \in \{1, 2, \dots, p\} \\ g(e_i, e_i) &= -1 & \text{if } i \in \{p+1, \dots, n\} \\ g(e_i, e_j) &= 0 & \text{if } i \neq j \end{aligned} \quad (1)$$

The Clifford algebra $\mathcal{C}(p,q)$ may be defined as the universal algebra generated over the real field $\mathbb{R}$ by a unit, 1, and the n symbols e_i subject to the relations (2)

$$\begin{aligned} e_i^2 &= +1 & \text{if } i \in \{1, 2, \dots, p\} \\ e_j^2 &= -1 & \text{if } j \in \{p+1, \dots, n\} \\ e_i e_j + e_j e_i &= 0 & \text{if } i \neq j \end{aligned} \quad (2)$$

the vector space V is then automatically embedded in the Clifford algebra and, if v, w are two vectors in V , one has

$$Vw + wv = 2g(v, w)$$

It has been known for a long time that this algebra (whose dimension is 2^n over $\mathbb{R}$) can be realized in term of matrices; the precise structure of this matrix algebra (structure depending strongly on the dimensionality of space and on the signature) will be investigated in the next paragraph but it should be noticed that a realization in term of matrices is not always the more convenient^{*)}. Let us now give some useful information about these algebras. Most of what follows are standard results and, for further details, for example for a generalization of these results when the basic field is not $\mathbb{R}$ but any field of finite characteristic, see Ref. 2). Let us call $\mathcal{C}_+$ the subalgebra generated by linear combinations of products of an even number of generators e_i , and $\mathcal{C}_-$ the vectorial subspace (not a subalgebra) generated by linear combinations of products of an odd number of generators e_i . Let us denote by $\varepsilon = e_1 e_2 \dots e_n$ the product of all the generators (every possible confusion with the Levi Civita symbol or with the volume element is deliberate, see footnote below). We shall discuss separately the cases n even, n odd.

First case: $n = p + q$ is even. $\mathcal{C}$ is then a simple algebra (it does not contain an invariant algebra besides $\{0\}$ and $\mathcal{C}$ itself) - its centre, generated by 1, is isomorphic with the real numbers $\mathbb{R}$. The centre of $\mathcal{C}_+$, however, is

^{*)} When studying vectors and spinors over a Riemannian manifold M of dimension n , it is convenient to proceed to the following construction: above the point $P \in M$, one can construct the tangent space T_P and the cotangent space T_P^* . Now, given two one-forms $(\alpha, \beta, \text{elements of } T_P^*)$ one knows the exterior product $\alpha \wedge \beta$. One then defines the Clifford product of α and β by the following relation $\alpha\beta = \alpha \wedge \beta + g(\alpha, \beta)$ where g is the metric on T_P^* . $\alpha\beta$ is then an element of the exterior algebra at P , Ext_P , which is also a vector space of dimension 2^n . Then $\alpha\beta + \beta\alpha = 2g(\alpha, \beta)$. It is easy to generalize this definition and to give a meaning to the Clifford product of a p -form by a q -form. Thus one can identify Ext_P and $\mathcal{C}_P(n)$ and think of them as vector spaces endowed with two inequivalent algebra structures; further details on this construction can be found for example in Refs 2)-4). Notice that if we choose locally an orthonormal moving coframe $\{\theta^a\}$, then $\theta^a \theta^b = \theta^a \wedge \theta^b$ if $a \neq b$ since $g(\theta^a, \theta^b) = 0$. The volume element $\varepsilon = \sqrt{\det \|g_{\alpha\beta}\|} dx^1 \dots dx^n = \varepsilon_{\mu\nu\rho\sigma} \dots$ can also be written as $\varepsilon = \theta^1 \wedge \theta^2 \wedge \dots \wedge \theta^n$, $\varepsilon = \theta^1 \theta^2 \dots \theta^n$.

generated by 1 and ϵ . ϵ is indeed an element of $\mathbb{C}_+$ which commutes with every element in $\mathbb{C}_+$ (it also anticommutes with every element in $\mathbb{C}_-$). One then has two subcases:

$$[\mathbb{C}_+ \text{ is simple}] \Leftrightarrow [\epsilon^2 = -1] \Leftrightarrow [p-q \not\equiv 0(4)] \Leftrightarrow [\text{The centre of } \mathbb{C}_+ \text{ is isomorphic with } {}^+\mathbb{C}]$$

or

$$[\mathbb{C}_+ \text{ is simple}] \Leftrightarrow [\epsilon^2 = +1] \Leftrightarrow [p-q \equiv 0(4)] \Leftrightarrow [\text{The centre of } \mathbb{C}_+ \text{ is isomorphic with } \mathbb{R} \oplus \mathbb{R}]$$

In this last subcase, $(\epsilon^2 = +1)$, one can break $\mathbb{C}_+$ into two simple ideals by constructing the idempotent $P = 1 - \epsilon/2$ and $Q = 1 + P$. Indeed $\mathbb{C}_+ = \mathbb{C}_+(1 - \epsilon/2) \oplus \mathbb{C}_+(1 + \epsilon/2)$. The proof of the above equivalences is standard, the result involving congruence modulo 4 rests on the calculation of ϵ^2 , using the defining relations (2)*). From the point of view of representation theory $\mathbb{C}$, being simple, admits only one irreducible representation ρ (up to equivalence) acting on what is called the space of spinors S . This spin representation ρ induces a representation ρ_+ of $\mathbb{C}_+$ which is irreducible in the first subcase ($\mathbb{C}_+$ simple) and reducible in the second subcase ($\mathbb{C}_+$ not simple). Indeed, if $\mathbb{C}_+$ is not simple, ρ breaks down into two irreducible inequivalent representations of $\mathbb{C}_+$ - the two half spin representations.

Second case: $n = p + q$ is odd. $\mathbb{C}_+$ is then a simple algebra and its centre is isomorphic with $\mathbb{R}$. Here ϵ is an element of $\mathbb{C}_-$ commuting with every element in $\mathbb{C}_-$ but also with every element in $\mathbb{C}_+$. The whole centre of $\mathbb{C}$ is therefore generated by 1 and ϵ and one has again to distinguish between two cases:

$$[\mathbb{C} \text{ is simple}] \Leftrightarrow [\epsilon^2 = -1] \Leftrightarrow [p-q \not\equiv 1(4)] \Leftrightarrow [\text{The centre of } \mathbb{C} \text{ is isomorphic with } \mathbb{C}]$$

or

$$[\mathbb{C} \text{ is not simple}] \Leftrightarrow [\epsilon^2 = +1] \Leftrightarrow [p-q \equiv 1(4)] \Leftrightarrow [\text{The centre of } \mathbb{C} \text{ is isomorphic with } \mathbb{R} \oplus \mathbb{R}]$$

In this last subcase $(\epsilon^2 = +1)$, one can break $\mathbb{C}$ into two simple ideals $\mathbb{C} = \mathbb{C}(1 - \epsilon/2) \oplus \mathbb{C}(1 + \epsilon/2)$. In both subcases, $\mathbb{C}$ is isomorphic to $\mathbb{C}_+ \otimes [\text{centre of } \mathbb{C}]$. From the point of view of representation theory, $\mathbb{C}_+$, being simple, admits only one irreducible representation (up to equivalence), ρ_+ , acting on the space of spinors S . This spin representation can be extended to a representation of $\mathbb{C}$ in only one way if $\mathbb{C}$ is simple and in exactly two ways if $\mathbb{C}$ is not simple.

*) Indeed $\epsilon^2 = (-1)^{n/2}(-1)^q = (-1)^{(p+q)/2}$ if n is even.

As is clear from the above, the structure of $\mathbb{C}(p,q)$, for a given dimension $n = p + q$, depends strongly on the signature (p,q) . It may be, in some rare cases, that two different pairs (p,q) lead to isomorphic Clifford algebra but this would be an exception. In particular, $\mathbb{C}(3,1)$ is not isomorphic to $\mathbb{C}(1,3)$, although the physics (computing a cross-section, for example) should be independent of the particular choice $+++$ or $---$. It should also be kept in mind that we have discussed so far real Clifford algebras, in particular the imaginary symbol i did not appear in the list of generators $\{1, \epsilon, e_i, \epsilon e_i, e_i e_j, \dots\}$. This, of course, should not be confused with the fact that $\mathbb{C}(p,q)$ can very often be realized as an algebra of matrices with complex entries; for example $\mathbb{C}(0,2)$ is defined as the real algebra of dimension 4 whose elements are of the form $a + be_1 + ce_2 + de_1e_2$ with a,b,c,d real and $e_1^2 = e_2^2 = -1$ and $e_1e_2 + e_2e_1 = 0$. It is obviously isomorphic to the algebra of quaternions $\mathbb{H}$ (for Hamilton) and as a consequence it is also isomorphic to the subalgebra of 2 by 2 complex matrices generated by elements of the form

$$\begin{pmatrix} a - id & -ib - c \\ -ib + c & a + id \end{pmatrix}$$

Upon complexification, we construct, starting with $\mathbb{C}(p,q)$, the complexified Clifford algebra $\mathbb{C}^{\mathbb{C}}(n) = \mathbb{C}(p,q) \otimes \mathbb{C}$. It is clear that several choices for the pair (p,q) , $p+q = n$ lead to the same complex extension $\mathbb{C}^{\mathbb{C}}(n)$, which is an algebra of dimension 2^n over the complex but $2 \cdot 2^n$ over the real. We noticed previously that when n is even it is not always possible to break the spin representation of $\mathbb{C}_+$ into two pieces (this depends upon the sign of ϵ^2). However, upon complexification of $\mathbb{C}(p,q)$ and even if $\epsilon^2 = -1$, it is possible using $(i\epsilon)^2 = +1$, to find an idempotent $(1+i\epsilon)/2$, allowing us to write the following decomposition of $\mathbb{C}_+^{\mathbb{C}}$

$$\mathbb{C}_+^{\mathbb{C}} = \mathbb{C}_+^{\mathbb{C}} \left(\frac{1-i\epsilon}{2} \right) \oplus \mathbb{C}_+^{\mathbb{C}} \left(\frac{1+i\epsilon}{2} \right)$$

At the complexified level it is thus always possible to break a "Dirac spinor" into two "Weyl spinors", provided n is even. If a matrix realization is used for $\mathbb{C}^{\mathbb{C}}(n)$, ϵ (or $i\epsilon$) becomes, under the spinorial representation, the usual γ_5 matrix with $\gamma_5^2 = 1$; this, of course, when n is even. When n is odd, there is only one subcase at the complexified level: $\mathbb{C}^{\mathbb{C}}(n)$ is not simple. Although the classification of the $\mathbb{C}(p,q)$ is not so obvious, as will be seen in the next part of this paper, the classification of the complexified $\mathbb{C}^{\mathbb{C}}(n)$ is

almost trivial; the following result is well known:

$n = 2r$ $\mathbb{C}(n)$ is isomorphic to the full algebra of matrices $2^r \times 2^r$ over the complex: $\mathbb{M}(2^r, \mathbb{C})$.

$n = 2r + 1$ $\mathbb{C}(n)$ is isomorphic to the algebra $\mathbb{M}(2^r, \mathbb{C}) \oplus \mathbb{M}(2^r, \mathbb{C})$.

results which exhibit clearly the fact that the spinorial representation of $\mathbb{C}(n)$ is of dimension $2^{\lfloor n/2 \rfloor}$. $\lfloor n/2 \rfloor$ denotes the integer part of $n/2$.

Another point of interest which is quite disconnected from the above but worth mentioning is that $\mathbb{C}(p, q)$ closes under commutation. Moreover, the $[,]$ operation defined by $[a, b] = ab - ba$ satisfies the Jacobi identity. This shows that $\mathbb{C}(p, q)$ also possesses a structure of Lie algebra. One can prove²⁾ that it is the Lie algebra of the group of its invertible elements. Here again one should make a clear distinction between $\mathbb{C}(p, q)$ and its complexified form. For example, it is not true that the Lie algebra of the conformal group $SO(4, 2)$ is included in $\mathbb{C}(1, 3)$, it is only included in the complex extension $\mathbb{C}(4)$.

In the next section we will give the classification of the real Clifford algebra $\mathbb{C}(p, q)$. Our aim in this paper is also to convince the reader that one should always distinguish between $\mathbb{C}(p, q)$ and $\mathbb{C}(n)$ when dealing with physical applications; more than exhibiting the concept involved, this can lead to physically relevant observations as we shall see in third section. Before concluding this section, and in order to eliminate possible ambiguities, we would like to mention the fact that we used the words "spin representation" to denote the unique (when $\mathbb{C}$ is simple) irreducible representation of the algebra $\mathbb{C}(p, q)$; the same words are also very commonly used in the literature to denote the irreducible representations of the associated Lorentz group $SO(p, q)$ whose double covering can be naturally embedded in $\mathbb{C}(p, q)$. Contrary to the Clifford algebra this generalized Lorentz group admits, as it is well known, an infinite number of irreducible inequivalent representations. For a detailed discussion of the Clifford group, the generalized Lorentz group, the covering group $\text{Spin}(n)$, $\text{Pin}(n)$ as well as representation independent definitions of transpose, bar operation, etc., one is referred to Refs. 1) and 2)

CLASSIFICATION OF REAL CLIFFORD ALGEBRAS

Let us start with the following simple observation: if $\{e_i\}$ is a set of generators for $\mathbb{C}(p, q)$ and $\{\omega_1, \omega_2\}$ a set of generators for $\mathbb{C}(0, 2)$, then $\{1 \otimes \omega_1, 1 \otimes \omega_2, e_i \otimes \omega_1 \omega_2\}_{i=1 \dots p+q}$ is a set of generators for $\mathbb{C}(q, p+2)$. It is then easy to prove that $\mathbb{C}(q, p+2)$ is indeed isomorphic to $\mathbb{C}(p, q) \otimes \mathbb{C}(0, 2)^{1), 5)}$. In the same way, one can prove the isomorphisms:

$$\mathbb{C}(p,q) \otimes (1,1) \simeq \mathbb{C}(p+1, q+1)$$

$$\mathbb{C}(p,q) \otimes (2,0) \simeq \mathbb{C}(q+2, p)$$

$$\mathbb{C}(p,q) \otimes (0,2) \simeq \mathbb{C}(q, p+2)$$

The structure of the lower dimensional $\mathbb{C}(p,q)$ is really obtained by inspection:

$$\mathbb{C}(1,0) \simeq \mathbb{R} \oplus \mathbb{R} \qquad \mathbb{C}(0,1) \simeq \mathbb{C}$$

$$\mathbb{C}(2,0) \simeq \mathbb{C}(1,1) \simeq \mathbb{M}(2,\mathbb{R}) \qquad \mathbb{C}(0,2) \simeq \mathbb{H}$$

We call $\mathbb{H}$ the quaternion field and $\mathbb{M}(r,\mathbb{F})$ the full $r \times r$ matrix algebra over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$). Therefore, a repeated use of the above recurrence relations and of the known identities

$$\mathbb{M}(n,\mathbb{R}) \otimes \mathbb{M}(m,\mathbb{R}) \simeq \mathbb{M}(nm,\mathbb{R}), \quad \mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$$

$$\mathbb{M}(n,\mathbb{R}) \otimes \mathbb{R} \simeq \mathbb{M}(n,\mathbb{R}), \quad \mathbb{H} \otimes \mathbb{C} \simeq \mathbb{M}(2,\mathbb{C})$$

$$\mathbb{M}(n,\mathbb{R}) \otimes \mathbb{C} \simeq \mathbb{M}(n,\mathbb{C}), \quad \mathbb{H} \otimes \mathbb{H} \simeq \mathbb{M}(4,\mathbb{R})$$

$$\mathbb{M}(n,\mathbb{R}) \otimes \mathbb{H} \simeq \mathbb{M}(n,\mathbb{H}),$$

leads to the following table of the $\mathbb{C}(n,0)$ and $\mathbb{C}(0,n)$ for $n \leq 8$:

n	$\mathbb{C}(n,0)$	$\mathbb{C}(0,n)$	$\mathbb{C}(n)^{\mathbb{C}}$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$	$\mathbb{C} + \mathbb{C}$
2	$\mathbb{M}(2,\mathbb{R})$	$\mathbb{H}$	$\mathbb{M}(2,\mathbb{C})$
3	$\mathbb{M}(2,\mathbb{C})$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{M}(2,\mathbb{C}) \oplus \mathbb{M}(2,\mathbb{C})$
4	$\mathbb{M}(2,\mathbb{H})$	$\mathbb{M}(2,\mathbb{H})$	$\mathbb{M}(4,\mathbb{C})$
5	$\mathbb{M}(2,\mathbb{H}) \oplus \mathbb{M}(2,\mathbb{H})$	$\mathbb{M}(4,\mathbb{C})$	$\mathbb{M}(4,\mathbb{C}) \oplus \mathbb{M}(4,\mathbb{C})$
6	$\mathbb{M}(4,\mathbb{H})$	$\mathbb{M}(8,\mathbb{R})$	$\mathbb{M}(8,\mathbb{C})$
7	$\mathbb{M}(8,\mathbb{C})$	$\mathbb{M}(8,\mathbb{R}) \oplus \mathbb{M}(8,\mathbb{R})$	$\mathbb{M}(8,\mathbb{C}) \oplus \mathbb{M}(8,\mathbb{C})$
8	$\mathbb{M}(16,\mathbb{R})$	$\mathbb{M}(16,\mathbb{R})$	$\mathbb{M}(16,\mathbb{C})$

This table is given in Ref. 1), along with the following observation: recurrence relations given at the beginning of this section imply

$$\mathbb{C}(0, n+8) \simeq \mathbb{C}(0, n) \otimes \mathbb{C}(0, 8)$$

Furthermore $\mathbb{C}(0, 8) \simeq \mathbb{M}(16, \mathbb{R})$, whence if $\mathbb{C}(0, n) = \mathbb{M}(m, F)$, F being $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, one obtains $\mathbb{C}(0, n+8) \simeq \mathbb{M}(16m, F)$. Thus, both columns are of period 8: if we move up eight steps, the field is left unaltered, while the dimension is multiplied by 16. For example, $\mathbb{C}(13, 0) \simeq \mathbb{M}(32, \mathbb{H}) \oplus \mathbb{M}(32, \mathbb{H})$ since $13 \equiv 5 \pmod{8}$. This modulo 8 periodicity theorem for the $\mathbb{C}(n, 0)$ and the $\mathbb{C}(0, n)$ is the corner stone of many recent developments and other periodicity theorems in algebraic topology [see Ref. 6) and references therein]. However, the authors of Ref. 1) do not discuss the case of an arbitrary signature, that is, the general classification of the $\mathbb{C}(p, q)$ to the study of which we devote the end of this section.

The isomorphism

$$\mathbb{C}(\alpha, \beta) \otimes \mathbb{C}(1, 1) \simeq \mathbb{C}(\alpha+1, \beta+1)$$

implies (suppose $p > q$)

$$\mathbb{C}(p-q, 0) \otimes \underbrace{\mathbb{C}(1, 1) \otimes \dots \otimes \mathbb{C}(1, 1)}_{q \text{ times}} \simeq \mathbb{C}(p, q)$$

but $\mathbb{C}(1, 1) \simeq \mathbb{M}(2, \mathbb{R})$. This shows that

$$\mathbb{C}(p, q) \simeq \mathbb{C}(p-q, 0) \otimes \mathbb{M}(2^q, \mathbb{R})$$

If $\mathbb{C}(p-q, 0)$ is isomorphic to $\mathbb{M}(m, F)$, one obtains

$$\begin{aligned} \mathbb{C}(p, q) &\simeq \mathbb{M}(m, F) \otimes \mathbb{M}(2^q, \mathbb{R}) \\ &\simeq \mathbb{M}(2^q m, F), \end{aligned}$$

and this brings us back to the previous discussion. Of course, if $p < q$, the same line of arguments brings the study of $\mathbb{C}(p, q)$ back to the study of $\mathbb{C}(0, q-p)$. It is then clear that the whole classification of real Clifford algebras depends upon the number $p-q \pmod{8}$, and that one can give the structure of all possible cases in the following table:

$p+q = n$								
$p-q \pmod{8}$	0	1	2	3	4	5	6	7
$\mathbb{C}(p,q)$	$\mathbb{M}(2^{n/2}, \mathbb{R})$	$\mathbb{M}(2^{\lfloor n/2 \rfloor}, \mathbb{R}) \oplus \mathbb{M}(2^{\lfloor n/2 \rfloor}, \mathbb{R})$	$\mathbb{M}(2^{n/2}, \mathbb{R})$	$\mathbb{M}(2^{\lfloor n/2 \rfloor}, \mathbb{C})$	$\mathbb{M}(2^{n/2-1}, \mathbb{H})$	$\mathbb{M}(2^{\lfloor n/2 \rfloor-1}, \mathbb{H}) \oplus \mathbb{M}(2^{\lfloor n/2 \rfloor-1}, \mathbb{H})$	$\mathbb{M}(2^{n/2-1}, \mathbb{H})$	$\mathbb{M}(2^{\lfloor n/2 \rfloor}, \mathbb{C})$

where $\lfloor n/2 \rfloor$ means the integer part of $n/2$. For example, the structure of $\mathbb{C}(4,7)$ is $\mathbb{M}(2^4, \mathbb{H}) \oplus \mathbb{M}(2^4, \mathbb{H})$

$$\text{since } \begin{cases} 4 - 7 = -3 \equiv 5 \pmod{8} \\ n = 4 + 7 = 11, \lfloor n/2 \rfloor = 5 \end{cases}$$

A direct consequence of these results is that $\mathbb{C}(p,q)$ is isomorphic to $\mathbb{M}(m, \mathbb{R})$ (one can find a so-called Majorana representation) if and only if $p-q \equiv 0$ or $2 \pmod{8}$, this remark using a direct proof has already been made in the literature⁷⁾⁻⁹⁾. For the reader unfamiliar with the use of quaternions $\mathbb{H}$, it may be useful to remind him/her that a full matrix algebra $\mathbb{M}(m, \mathbb{H})$ can be written as a subalgebra of $\mathbb{M}(2m, \mathbb{C})$ by using the 2×2 matrix representation of a quaternion given in the previous section. Notice that if $p-q \equiv 1 \pmod{8}$, although $\mathbb{C}(p,q)$ is not isomorphic to a full matrix algebra over the real numbers, it is nevertheless a matrix algebra over the real numbers and one can speak again of a real representation. Finally, let us note that "Weyl-Majorana" spinors exist, (i.e., there exists a matrix representation of $\mathbb{C}_+(p,q)$ which is real and reducible into two real components) if we have at the same time $p-q \equiv 0$ or $2 \pmod{8}$ and $p-q \equiv 0(4)$ which leaves us only with the possibility $p-q \equiv 0 \pmod{8}$. This comment extends the results given in Ref. 10).

This generalization of the periodicity theorem is probably floating around in mathematical folklore since it follows trivially from old and well-known identities, however the author did not find any article dealing with it among the physical or mathematical papers available to him and would acknowledge any information on this topic. In any case, and regardless of its originality from the mathematical point of view, it is probably worth bringing this (extended) periodicity theorem to the attention of physicists working with spinors in dimensions $n \neq 3 + 1$.

APPLICATIONS IN PARTICLE PHYSICS

Before dealing with the heart of the subject (maximal supergravity theories) let us briefly review some standard notations for metric-preserving Lie groups¹¹⁾.

Lie groups preserving

symmetric (p,q) metrics on real vector spaces are denoted $SO(p,q,\mathbb{R})$
 antisymmetric $(2n)$ metrics on complex vector spaces are denoted $Sp(2n,\mathbb{C})$
 sesquilinear (p,q) metrics on complex vector spaces are denoted $U(p,q;\mathbb{C})$
 sesquilinear (p,q) metrics on quaternionic vector spaces one denoted $U(p,q,\mathbb{H})$
 Moreover, one calls

$$USp(2p,2q) = U(2p,2q;\mathbb{C}) \cap Sp(2n,\mathbb{C}), \quad n = p + q$$

Finally, let us notice the isomorphisms:

$$USp(2p,2q) \simeq U(p,q;\mathbb{H})$$

$$USp(4) \simeq SO(5,\mathbb{R}) \simeq U(2,\mathbb{H})$$

The vector space where the spinorial representation of $\mathbb{C}(n,0)$ acts, i.e., the space of spinors, is endowed with a "natural" metric, in the following sense. Let us look for example at $\mathbb{C}(4,0) \simeq \mathbb{H}(2,\mathbb{H})$. Then, spinors are of the kind $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ where q_i are quaternions. A "natural" metric on this space can be represented (in a suitable basis) by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The group leaving this metric invariant is $U(2,\mathbb{H}) \simeq USp(4)$. More generally, we look at the Lie group G which leaves invariant a positive bilinear symmetric (sesquilinear) form in the space of spinors. Thus, we construct immediately, using the definitions given above, the following table

n	1	2	3	4	5	6	7	8
$\mathbb{C}(n,0)$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{H}(2,\mathbb{R})$	$\mathbb{H}(2,\mathbb{C})$	$\mathbb{H}(2,\mathbb{H})$	$\mathbb{H}(2,\mathbb{H}) \oplus \mathbb{H}(2,\mathbb{H})$	$\mathbb{H}(4,\mathbb{H})$	$\mathbb{H}(8,\mathbb{C})$	$\mathbb{H}(16,\mathbb{R})$
G	$Z_2 \times Z_2$	$SO(2)$	$U(2,\mathbb{C})$ $\simeq$ $SU(2) \times U(1)$	$U(2,\mathbb{H})$ $\simeq$ $SO(5)$ $\simeq$ $USp(4)$	$U(2,\mathbb{H}) \times U(2,\mathbb{H})$ $\simeq$ $SO(5) \times SO(5)$	$U(4,\mathbb{H})$ $\simeq$ $USp(8)$	$U(8,\mathbb{C})$	$SO(16,\mathbb{R})$

In the case $n = 1$, we obtain a discrete group ($Z_2 = \{-1, +1\}$). Now, let us look at the content and the symmetries of maximal supergravities in dimensions $d + n = 11$ where d is the dimension of space-time and n the number of "extra dimensions". These theories are obtained from $N = 1$ supergravity in 11 dimensions^{12),13)} after dimensional reduction. The following table is taken from Ref. 13):

Dimension (d,n)	Multiplet structure ($e_\mu^r, \psi_\mu, \chi, A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}, \phi$)	Hidden local invariance group	Hidden global invariance group
(10,1)	—	—	—
(9,2)	1,2,4,3,2,1,3	SO(2)	GL(2,R)
(8,3)	1,2,6,6,3,1,7	SU(2) $\times$ U(1)	SL(3,R) $\times$ SL(2,R)
(7,4)	1,4,16,10,5,0,14	SO(5)	SL(5,R)
(6,5)	1,4,20,16,5,0,25	SO(5) $\times$ SO(5)	SO(5,5)
(5,6)	1,8,48,27,0,0,42	USp(8)	$E_{6(6)}$
(4,7)	1,8,56,28,0,0,70	SU(8,C)	$E_{7(7)}$
(3,8)	1,16,128,0,0,0,128	SO(16)	$E_{8(8)}$

In the above table, $e_\mu^r, \psi_\mu, \chi, A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}, \phi$ denote gravitons, gravitini, spinors, 1, 2 and 3 completely antisymmetric fields and finally scalar fields. Thus, particles fall into representations of the local invariance group [the adjoint representation, (the 63 of SU(8), for example), is filled by composite gauge fields built from the scalars]. As a matter of fact, the emergence of what is called the "hidden local invariance group", historically came about after hard work and highly ingenious manipulations, but was not at all obvious from the starting Lagrangian¹²⁾.

A comparison between the two previous tables exhibits the principal result of this section: these hidden groups of (local) symmetries coincide with the isometry groups G of the natural metric defined on the space of the spinorial representation of the real Clifford algebra $\mathbb{C}(n,0)$ associated with the metric in the tangent space $\mathbb{C}(n,0)$ in the n dimensional (extra) world. The emergence of members of the E series (ending with E_8) as a group of global invariance and a geometrical proof of the invariance of these theories with respect to the groups previously defined, is left as a (probably non-trivial!) exercise for active workers in this field. The above remark about "hidden" symmetries is actually not a consequence of the modulo 8 periodicity theorem but follows merely

from the structure of the $\mathbb{C}(n,0)$ itself. However, we believe that this periodicity theorem rather than providing us only with a classifying device, will also cast new light on the structure of theories such as Kaluza-Klein theories, string models and, of course, supergravity theories.

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