Ref.TH.3281-CERN

CONSISTENCY REQUIREMENTS FOR $1/N$ EXPANDABLE FIELD THEORIES

Per Salomonson

CERN -- Geneva

ABSTRACT

It is demonstrated that $1/N$ expansion in a class of field theories with an internal $O(N)$ symmetry leads to pathologies. Therefore, these models are either not internally consistent, or their perturbation expansions (ordinary as well as $1/N$) are unreliable.

Ref.TH.3281-CERN
15 April 1982

INTRODUCTION

The $1/N$ expansion has proved to be a useful method for exploring a class of field theories beyond the reach of ordinary perturbation theory. For a review see Ref. 1). Sometimes, for example in sigma type models or in the $O(N)$ symmetric $\lambda(\phi^2)^2$ model, it is possible to analyze the first term, the limit $N \rightarrow \infty$, rather explicitly by introducing auxiliary fields. This has resulted in an improved understanding of phenomena such as symmetry breaking and the formation of bound states. The purpose of this paper is to point out that in some models one obtains a pathological result. This becomes most apparent in cases where it is possible to confront the result with supersymmetric properties of the models.

The paper is organized as follows. A supersymmetric model is analyzed in Section 2. An investigation of some qualitative properties of its effective potential is given in an Appendix. Section 3 contains a comparative treatment of the $O(N)$ symmetric $\lambda(\phi^2)^2$ model. Conclusions are drawn in Section 4.

2. - $1/N$ EXPANSION IN A SUPERSYMMETRIC MODEL

A simple example of $1/N$ expansion in a four-dimensional supersymmetric model is constructed as follows. The action is chosen invariant under an internal $O(N)$ symmetry. It is built from two kinds of chiral superfields, an $O(N)$ vector $\underline{\phi}$ and an $O(N)$ singlet χ ²⁾. The most general renormalizable action is

$$S = \int d^4x \left\{ \int d^4\theta (\bar{\underline{\phi}} \cdot \underline{\phi} + \bar{\chi} \chi) + \left[\int d^2\theta \left(\frac{g}{2} \underline{\phi}^2 \chi + \frac{1}{2} m \chi^2 + \xi \chi \right) + \text{h.c.} \right] \right\}. \quad (2.1)$$

Essentially the same model was first analyzed in Ref. 2); I carry the investigation a little bit further. Since rescaling the fields does not change the content of the model, there is a redundancy in the set of parameters. I use this redundancy to express the action in terms of three positive parameters as

$$S = N \int d^4x \left\{ \int d^4\theta (\bar{\underline{\phi}} \cdot \underline{\phi} + \bar{\chi} \chi) + \left[\int d^2\theta \left(\frac{1}{2} \underline{\phi}^2 \chi + \frac{1}{2} b \chi^2 + a \chi \right) + \text{h.c.} \right] \right\} \quad (2.2)$$

Comparing (2.1) and (2.2) reveals that the set of parameters are related, up to phases, as

$$(g, m, \xi) = \left(\frac{1}{\sqrt{N} Z_s}, \frac{b}{Z_s}, \sqrt{N} \frac{a}{\sqrt{Z_s}} \right) \quad (2.3)$$

I proceed to solve this model, i.e., compute the effective potential, find its minimum and determine the particle spectrum, in the limit $N \rightarrow \infty$. The factors of N in (2.2) have been chosen such that Z_s , a and b are kept fixed when this limit is taken. In terms of component fields $\phi = (\underline{A}, \underline{\zeta}, \underline{F})$, $\chi = (\underline{B}, \underline{\eta}, \underline{G})$ the action reads

$$\begin{aligned} S = N \int d^4x \{ & -\underline{A}^\dagger \partial^2 \underline{A} + i \underline{\zeta}^\dagger \tilde{\sigma}^\mu \partial_\mu \underline{\zeta} + \underline{F}^\dagger \underline{F} \\ & + \underline{Z}_s [-\underline{B}^\dagger \partial^2 \underline{B} + i \underline{\eta}^\dagger \tilde{\sigma}^\mu \partial_\mu \underline{\eta} + \underline{G}^\dagger \underline{G}] \\ & + [\frac{1}{2} \underline{G} \underline{A}^2 + \underline{F} \cdot \underline{A} \underline{B} - \frac{1}{2} \underline{B} \underline{\zeta}^2 - \underline{A} \cdot \underline{\zeta} \underline{\eta} + h.c.] \\ & + [b \underline{G} \underline{B} - \frac{1}{2} \underline{B} \underline{\eta}^2 + a \underline{G} + h.c.] \}. \end{aligned} \quad (2.4)$$

[I use a timelike metric, a complex scalar and auxiliary fields and a two-component spinor notation as in Ref. 3).]

In the limit $N \rightarrow \infty$ the complete effective action is given by the following formal expression

$$\frac{1}{N} S_{\text{eff}} = \frac{1}{N} S_0 + \frac{i}{2} \text{Tr} \ln M \quad (2.5)$$

where

$$M = \begin{pmatrix} \partial^2 + \underline{B}^\dagger \underline{B} & -\underline{G}^\dagger & 0 & \underline{\eta}^\dagger \underline{\zeta} \\ -\underline{G} & \partial^2 + \underline{B} \underline{B}^\dagger & \underline{\eta} \underline{\zeta}^\dagger & 0 \\ 0 & \underline{\eta} \underline{\zeta}^\dagger & -i \tilde{\sigma}^\mu_a \underline{\zeta} \partial_\mu & 0 \\ \underline{\eta} \underline{\zeta}^\dagger & 0 & 0 & -i \sigma^\mu_a \underline{\zeta} \partial_\mu \end{pmatrix} \quad (2.5')$$

This expression is still in Minkowski space. The trace is graded and is over the matrix components displayed in (2.5) as well as space-time indices. The next step is to divide the fields into vacuum expectation values plus remaining quantum fields

$$\begin{aligned}\underline{\phi} &= \underline{\phi}_0 + \underline{\phi}_q, \\ \chi &= \chi_0 + \chi_q.\end{aligned}\tag{2.6}$$

We then expand S_{eff} as a power series in the quantum fields.

Our immediate interest is focused on the first term, the effective potential, which determines the vacuum amplitudes, and in particular decides whether supersymmetry is broken ($\underline{F}_0 \neq 0$ or $B_0 \neq 0$) or not ($\underline{F}_0 = 0, B_0 = 0$). The effective potential is $\tilde{V}_{\text{eff}} = \tilde{V}^{(0)} + \tilde{V}_{\text{eff}}^{(1)}$,

$$\begin{aligned}\frac{1}{N} \tilde{V}^{(0)} &= -\underline{F}_0^+ \cdot \underline{F}_0 - \frac{1}{2} (\underline{A}_0^2 G_0 + 2 B_0 \underline{A}_0 \cdot \underline{F}_0 + \text{h.c.}) \\ &\quad - Z_s G_0^+ G_0 - ((b B_0 + \alpha) G_0 + \text{h.c.}),\end{aligned}\tag{2.7}$$

$$\frac{1}{N} \tilde{V}_{\text{eff}}^{(1)} = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \left\{ \text{Tr} \ln \begin{pmatrix} p^2 + |B_0|^2 & -G_0^+ \\ -G_0 & p^2 + |B_0|^2 \end{pmatrix} - (G_0 \rightarrow 0) \right\}.$$

The integral is now over Euclidean momentum space. The matrix has been reduced to 2×2 size. The $\sim$ sign on $\tilde{V}_{\text{eff}}$ is used to denote that the auxiliary fields have not been eliminated.

We are interested in stable extrema of $\tilde{V}_{\text{eff}}$. Since an imaginary part of $\tilde{V}_{\text{eff}}$ signals an instability⁴⁾, we are not interested in field configurations for which $\tilde{V}_{\text{eff}}$ is not real. Therefore, we restrict ourselves to auxiliary field amplitudes such that

$$|G_0| \leq |B_0|^2.\tag{2.8}$$

The equation for the auxiliary vector field is

$$\underline{F}_0 + \underline{A}_0 B_0 = 0.\tag{2.9}$$

After elimination of $\underline{F}_0$, the terms in V_{eff}/N involving $\underline{A}$ are

$$|B_0|^2 |\underline{A}_0|^2 - \frac{1}{2} G_0 \underline{A}_0^2 - \frac{1}{2} G_0^+ \underline{A}_0^{+2}.\tag{2.10}$$

Due to cancellation between boson and fermion loops, the integral in (2.7) is only logarithmically divergent. We regularize it by a cut-off at $p^2 = \Lambda^2$. The integral is easily evaluated with the result

$$\frac{1}{N} V_{\text{eff}}^{(1)} = \frac{1}{64\pi^2} \left\{ \left[\Lambda^4 \ln \left(\frac{\Lambda^2 + |B_0|^2 + |G_0|}{\Lambda^2} \right) + (|B_0|^2 + |G_0|)^2 \ln \left(\frac{|B_0|^2 + |G_0|}{\Lambda^2 + |B_0|^2 + |G_0|} \right) \right] + [|G_0| \rightarrow -|G_0|] - 2 [G_0 \rightarrow 0] \right\}. \quad (2.11)$$

The cut-off dependence simplifies for large Λ^2 . By expanding the logarithms one obtains

$$\frac{1}{64\pi^2} \left\{ \left[(|B_0|^2 + |G_0|)^2 \ln \left(\frac{|B_0|^2 + |G_0|}{\mu^2} \right) \right] + (|G_0| \rightarrow -|G_0|) - 2 [|G_0| \rightarrow 0] - 2 |G_0|^2 \ln \frac{\Lambda^2 \sqrt{e}}{\mu^2} \right\} \left\{ 1 + O \left(\frac{|B_0|^2}{\Lambda^2} \right) \right\} \quad (2.12)$$

μ is a mass parameter introduced to make the arguments of the logarithm dimensionless. We get rid of the cut-off dependence by neglecting $O(|B_0|^2/\Lambda^2)$ and absorbing the $\ln \Lambda^2/\mu^2$ terms in a renormalization of Z_s

$$Z_s^r(\mu) = Z_s + \frac{1}{32\pi^2} \ln \left(\frac{\Lambda^2 \sqrt{e}}{\mu^2} \right). \quad (2.13)$$

The renormalized effective potential with the auxiliary vector field eliminated by Eq. (2.9) is

$$\begin{aligned} \frac{1}{N} \tilde{V}_{\text{eff}}(B_0, G_0, A_0) = & |B_0|^2 |A_0|^2 - \frac{1}{2} A_0^2 B_0 - \frac{1}{2} A_0^+ G_0^+ - \\ & - Z_s(\mu) |G_0|^2 - [(b B_0 + a) G_0 + \text{h.c.}] \\ & + \frac{1}{64\pi^2} \left[(|B_0|^2 + |G_0|)^2 \ln \left(\frac{1}{\mu^2} (|B_0|^2 + |G_0|) \right) \right. \\ & \left. + [|G_0| \rightarrow -|G_0|] - 2 [G_0 \rightarrow 0] \right]. \end{aligned} \quad (2.14)$$

Since the parameters are real, $\tilde{V}_{\text{eff}}$ assumes its extrema for real values of the fields. Qualitative properties of the effective potential (2.14) are investigated in Appendix A. It is found there that $V_{\text{eff}}(B_0, \underline{A}_0)$, which is obtained from $\tilde{V}_{\text{eff}}(B_0, G_0, \underline{A}_0)$ by eliminating the auxiliary field G_0 , has a branch on which it is negative.

The result that $V_{\text{eff}}(B_0, A_0)$ can become negative in supersymmetric models contradicts common belief. The argument goes as follows. The supersymmetry algebra says that the Hamiltonian is a sum of squares of Hermitian operators, the supercharges,

$$H = \sum_{\alpha} Q_{\alpha} Q_{\alpha} \quad (2.15)$$

Hence the energy of the ground state is positive

$$E_0 = \langle 0 | H | 0 \rangle = \sum_{\alpha} \sum_n |\langle 0 | Q_{\alpha} | n \rangle|^2 \geq 0 \quad (2.16)$$

This argument is too naïve, infinite renormalizations demanded by divergences may spoil it. To make the argument more precise we have to regularize the model. This can be done without explicit supersymmetry breaking assuming that the infinite volume limit exists for the fixed ultra-violet cut-off⁵⁾. Ultra-violet divergences can be regularized by a point splitting of the interaction terms, infra-red divergences, by putting the system in a box with periodic boundary conditions. For such a regularized model nothing can spoil the positivity argument.

It is actually easy to show explicitly that the one-loop potential, calculated as was done here, with auxiliary fields eliminated at the last stage, is positive definite for regularized supersymmetric models⁶⁾. The proof rests on the observations that $\tilde{V}_{\text{eff}}(B_0, G_0, A_0)$ is zero for $G_0 = 0$ and has negative second derivatives with respect to G_0 . As a function of G_0 it can therefore have at most one extremum, a maximum at which it is ≥ 0 . The reason why this proof does not apply to the expression (2.14) is that we have done a renormalization (2.13) of the coefficient of the singlet kinetic energy. The proof of positivity in the regularized model requires that the bare kinetic energy be positive.

In the process of isolating a simple cut-off dependence we also took a limit $\Lambda^2 \rightarrow \infty$ in order to get rid of terms which were down by $O(|B_0|^2/\Lambda^2)$. These two requirements, $Z_S > 0$ and $\Lambda^2 \rightarrow \infty$, require $Z_S^r(\mu) = +\infty$. This means that the model is a free field theory, as is seen by rescaling the fields to conventional normalization, (2.1). So this proof that the energy is positive applies only when there is no interaction.

A positive alternative treatment of the model, which avoids getting $Z_S^r(\mu) = \infty$, is to accept a large but finite Λ . Then $O(|B_0|^2/\Lambda^2)$ terms can be neglected only if $|B_0|^2/\Lambda^2$ is small enough. It turns out however, that even if they are neglected the condition [from (2.13)]

$$Z_s^r(M) \geq \frac{1}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2} \quad (2.17)$$

will have the effect that the local minimum of the effective potential (see Appendix A) occurs for $|B_0|$ so large that this approximation would not be justified.

There remains to settle the question of whether something is seriously wrong in the model when Z_s is negative. Depending on the values of the parameters in the model, the effective potential $V_{\text{eff}}(B_0, \underline{A}_0)$ may or may not be bounded from below. If it is not bounded from below the vacuum state, built on a local minimum of $V_{\text{eff}}(B_0, \underline{A}_0)$ may conceivably be able to decay by tunnelling⁴⁾, although it is not clear to me how to calculate the tunnelling amplitude. Since the effective action, in terms of suitably normalized fields, is of order N the tunnelling amplitude should decrease with N as $e^{-\text{const.}N}$ for large N . So, for large N this problem, if it exists, may not necessarily be very serious. A metastable vacuum state with an enormous lifetime is conceivable.

The model has worse defects however; the $O(N)$ singlet propagators have pathological properties. Firstly, they have poles at negative P^2 ($= E^2 - \underline{P}^2$). Secondly, the residues of these poles have the wrong sign. Thirdly, the kinetic energy term in the effective action is unbounded from below.

An explicit calculation of these terms in the quantum corrections to the effective action (2.4) and (2.5) which are bilinear in the quantum fields (2.6) gives, if, for simplicity, G_0 is neglected

$$\begin{aligned} & \int \frac{d^4 p}{(2\pi)^4} \left[\tilde{B}_q^+(-p) p^2 \tilde{B}_q(p) + \tilde{\eta}_q^+(-p) \tilde{\sigma}^{\mu} p_{\mu} \tilde{\eta}_q(p) + \tilde{F}_q^+(-p) \tilde{F}_q(p) \right] \\ & \times \frac{N}{32\pi^2} \int_0^1 d\tau \ln \left(\frac{\Lambda^2/e}{-\tau(1-\tau) p^2 + B_0^2 - i\epsilon} \right). \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} B_q(x) &= \int \frac{d^4 x}{(2\pi)^4} \tilde{B}_q(p) e^{-i p x}, \\ B_q^+(x) &= \int \frac{d^4 x}{(2\pi)^4} \tilde{B}_q^+(p) e^{-i p x}, \\ & \text{etc.} \end{aligned} \quad (2.19)$$

Neglecting mass terms, for example, the scalar propagator is, according to (2.19),

$$\frac{1}{p^2 + i\epsilon} \left(Z_s(\mu) - \frac{1}{32\pi^2} \ln \left(\frac{-p^2 - i\epsilon}{\mu^2} \right) \right)^{-1} \quad (2.20)$$

Divided into real and imaginary parts this is

$$\begin{aligned} & \frac{1}{p^2} \frac{Z_s(\mu) - \frac{1}{32\pi^2} \ln \left| \frac{p^2}{\mu^2} \right| - \frac{i}{32\pi} \theta(p^2)}{\left(Z_s(\mu) - \frac{1}{32\pi^2} \ln \left| \frac{p^2}{\mu^2} \right| \right)^2 + \left(\frac{1}{32\pi} \theta(p^2) \right)^2} + \\ & + i 32\pi^3 \delta(p^2 + \mu^2 e^{32\pi^2 Z_s(\mu)}) \end{aligned} \quad (2.21)$$

A comparison with the spectral representation of the propagator

$$\begin{aligned} \int ds \rho(s) \frac{1}{p^2 - s + i\epsilon} &= \int ds \rho(s) \frac{1}{p^2 - s} - i\pi \rho(p^2), \\ \rho(s) &= (2\pi)^3 \sum_n \delta^4(p_n^2 - s) | \langle 0 | B_{\frac{1}{2}}(0) | n \rangle |^2. \end{aligned} \quad (2.22)$$

gives the spectral density

$$\begin{aligned} \theta(p^2) \frac{1}{p^2} \frac{1}{32\pi^2} \left[\left(Z_s(\mu) - \frac{1}{32\pi^2} \ln \left| \frac{p^2}{\mu^2} \right| \right)^2 + \left(\frac{1}{32\pi} \right)^2 \right]^{-1} \\ - 32\pi^2 \delta(p^2 + \mu^2 e^{32\pi^2 Z_s(\mu)}) \end{aligned} \quad (2.23)$$

The second term in (2.23) is a contribution from negative p^2 , i.e., from a tachyon. Furthermore, it has the wrong sign. Comparison with (2.21) shows that the tachyon state has negative norm, i.e., it is a ghost. The presence of ghosts in the model explains how (2.16) can be negative.

These pathologies are reminiscent of the Landau ghost in quantum electrodynamics⁷⁾ perturbation theory. There is an important difference however. The Landau pole appears at an enormously large momentum where perturbation theory breaks down, and also pure quantum electrodynamics is not expected to be realistic. In the present model on the other hand, higher order corrections are suppressed

by powers of $1/N$ and hence dominated by the leading effects calculated here for sufficiently large N . Therefore, this pathology will appear in calculations based on the $1/N$ expansion to any finite order in $1/N$ when N is chosen large enough. And since the coupling constant squared, $1/Z_S$ is not required to be small, the pole need not be at an enormous momentum.

Unfortunately, these pathologies appear to rule out a nice mechanism for spontaneous supersymmetry breaking by radiative corrections²⁾. For if supersymmetry is manifest at the tree level, then the tree potential $V^{(0)}$ has a zero. By the non-renormalization theorem this remains a zero of V_{eff} after addition of radiative corrections. The $1/N$ expansion is qualitatively similar to ordinary perturbation theory in this case since it amounts to a summation of an infinite set of ordinary perturbative corrections. So the effective potential in the $1/N$ expansion will also have a supersymmetric point. The only situation in which supersymmetry could be broken by ordinary perturbative corrections is when the potential becomes negative. But as we have seen this violates general principles in a model based on supersymmetry, and when it happens it is a sign of the appearance of pathologies such as ghosts.

3. - THE $(\phi^2)^2$ MODEL

The purpose of this section is to compare the $O(N)$ symmetric $(\phi^2)^2$ model and the supersymmetric model in Section 2 with respect to the properties investigated in there. Since the $(\phi^2)^2$ model has been thoroughly investigated⁸⁾ I will be brief.

The Lagrangian density reads, in terms of suitably normalized fields,

$$\begin{aligned} \frac{1}{N} \mathcal{L}_0 &= -\frac{1}{2} \underline{\varphi} \cdot \partial^2 \underline{\varphi} - \frac{1}{2} m^2 \underline{\varphi}^2 - \frac{\lambda}{8} (\underline{\varphi}^2)^2 + \frac{1}{2\lambda} \left(\frac{\lambda}{2} \underline{\varphi}^2 + m^2 - \chi \right)^2 \\ &= -\frac{1}{2} \underline{\varphi} \cdot \partial^2 \underline{\varphi} - \frac{1}{2} \underline{\varphi}^2 \chi - \frac{1}{2\lambda} \chi^2 - \frac{1}{\lambda} m^2 \chi + \text{const.} \end{aligned} \quad (3.1)$$

Each field is written as vacuum average plus remaining quantum field

$$\begin{aligned} \chi &= \chi_0 + \chi_q, \\ \underline{\varphi} &= \underline{\varphi}_0 + \underline{\varphi}_q. \end{aligned} \quad (3.2)$$

The leading quantum corrections to the effective potential $\tilde{V}_{\text{eff}}$, for large N , regularized with a momentum cut-off, is

$$\begin{aligned} \frac{N}{2} \int \frac{d^4 p}{(2\pi)^4} \ln(p^2 + \chi_0) = \\ = \frac{N}{64\pi^2} \chi_0^2 \left(\ln \frac{\chi_0}{\Lambda^2} - \frac{1}{2} \right) + \frac{N}{32\pi^2} \Lambda^2 \chi_0 + \text{const.} + o\left(\frac{\chi_0}{\Lambda^2}\right) \end{aligned} \quad (3.3)$$

An effective potential $\tilde{V}_{\text{eff}}$, which is finite up to an irrelevant constant in the limit $\Lambda \rightarrow \infty$, is obtained by renormalizing the coupling constant and mass.

$$\begin{aligned} \frac{1}{N} \tilde{V}_{\text{eff}}(\phi_0, \chi_0) = \frac{1}{2} \phi_0^2 \chi_0 + \frac{m^2(\mu)}{\lambda(\mu)} \chi_0 - \frac{1}{2} \frac{1}{\lambda(\mu)} \chi_0^2 \\ + \frac{1}{64\pi^2} \chi_0^2 \left(\ln \frac{\chi_0}{\mu^2} - \frac{1}{2} \right), \end{aligned} \quad (3.4)$$

$$\frac{1}{\lambda(\mu)} = \frac{1}{\lambda} + \frac{1}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2},$$

$$\frac{m^2(\mu)}{\lambda(\mu)} = \frac{m^2}{\lambda} + \frac{\Lambda^2}{32\pi^2}. \quad (3.5)$$

The essential difference between the model in Section 2 and the present model is that the divergent quantum correction to the effective action is a kinetic term in the supersymmetric model and a potential term in the $(\phi^2)^2$ model. To cancel the divergence in the supersymmetric model requires a bare kinetic term with the wrong sign, and this leads to pathological properties of the propagators at large (P^2) . In the $(\phi^2)^2$ model one may expect instead a pathological behaviour of the potential at large ϕ_0 .

The qualitative properties of the effective potential (3.4) was investigated in Ref. 8). The result is as follows. After elimination of the auxiliary field χ_0 , the effective potential $V_{\text{eff}}(\phi_0) \equiv \tilde{V}_{\text{eff}}(\phi_0, \chi_0(\phi_0))$ has one, two and zero real branches respectively in the intervals

$$\begin{aligned} \frac{1}{2} \phi_0^2 + \frac{m^2(\mu)}{\lambda(\mu)} < 0, \\ 0 < \frac{1}{2} \phi_0^2 + \frac{m^2(\mu)}{\lambda(\mu)} < \frac{\mu^2}{32\pi^2} \frac{1}{e} e^{\frac{32\pi^2}{\lambda(\mu)}}, \\ \frac{\mu^2}{32\pi^2} \frac{1}{e} e^{\frac{32\pi^2}{\lambda(\mu)}} < \frac{1}{2} \phi_0^2 + \frac{m^2(\mu)}{\lambda(\mu)}. \end{aligned} \quad (3.6)$$

The dependence on μ in (3.6) is of course only apparent. End-points of real branches are branch-points of the function $V_{\text{eff}}(\phi_0)$. Beyond them it becomes complex. The qualitative behaviour of the effective potential when $m^2(\mu)/\lambda(\mu) < 0$ is shown in Fig. 1. If $m^2(\mu)/\lambda(\mu) > 0$ the correct qualitative picture is obtained by moving the vertical axis the appropriate amount to the right and deleting the part of the figure which falls to the left of it. To the right of the branch-point

$$\frac{1}{2} \phi_0^2 + \frac{m^2(\mu)}{\lambda(\mu)} = \frac{1}{e} \frac{\mu^2}{32\pi^2} e^{\frac{32\pi^2}{\lambda(\mu)}}$$

there are two complex conjugate complex branches. As explained in Ref. 4), complex branches of the effective potential correspond to unstable states of the system. The branch with negative imaginary parts thus describes decaying states. When $\phi_0^2 \rightarrow \infty$ the real part of this branch of V_{eff} tends to $-\infty$. Higher order terms in the $1/N$ expansion can alter this behaviour, as was actually found in Ref. 8). However, since these corrections are suppressed by powers of $1/N$ the magnitude of ϕ_0 at which they become important tends to infinity when N tends to infinity. So the fact that the minimum of the real part of $V_{\text{eff}}(\phi_0)$ at $\phi_0 = 0$ is not global, will remain true to any finite order in the $1/N$ expansion, if N is chosen large enough. I find it reasonable therefore to believe that the vacuum state at $\phi_0^2 = 0$ is unstable against tunnelling⁹⁾ to states with large ϕ_0^2 . Since I do not know how to compute the tunnelling amplitude, this is only a conjecture. The tunnelling amplitude would contain the classical action for a bounce solution to the equations of motion⁴⁾, which depends on N as $e^{-N \times \text{const.}}$. Therefore, this would mean that the $\phi_0^2 = 0$ vacuum is unstable in each order of $1/N$ perturbation theory, for sufficiently large but finite N ; there would be no acceptable stable vacuum in the $1/N$ approximation.

4. - CONCLUSIONS

The model in Section 2 has a pathological behaviour in ordinary perturbation theory as well as in the $1/N$ expansion in the following sense.

The leading terms in the $1/N$ expansion, expressed in terms of bare parameters and a momentum cut-off parameter Λ , are such that the kinetic term in the effective action tends to infinity when $\Lambda \rightarrow \infty$ while the other terms have a finite limit. If one requires the bare kinetic term to be positive, this has the consequence that the leading terms in the $1/N$ expansion give a free field theory in the limit $\Lambda \rightarrow \infty$. If, on the other hand, one requires the theory to be non-trivial in the limit $\Lambda \rightarrow \infty$, then one must use a bare kinetic term with a Λ dependent coefficient which tends to $-\infty$ when $\Lambda \rightarrow \infty$. This results in a theory with pathological properties such as tachyons and ghosts.

If one calculates to any finite order in $1/N$, using the same prescriptions as for the leading order calculation, one will obtain an effective action which is a finite power series in $1/N$, the leading term in the effective action being the same as in the leading order calculation. Therefore, at least for N sufficiently large, the pathological properties will remain.

Ordinary perturbation expansion can be done in at least two different ways, namely one can eliminate the auxiliary fields either before or after adding the loop corrections to the effective action. I have nothing to say about the first alternative, which appears to be the least attractive one¹⁰⁾. The second alternative is very similar to the $1/N$ expansion, because the auxiliary fields of the supersymmetry play the rôle of the auxiliary fields used in the $1/N$ expansion to sum the leading order explicitly. In the second alternative the one-loop terms already contain all the leading terms in the $1/N$ expansion. Hence, for large enough N , the leading terms in the $1/N$ expansion will dominate, and one obtains the above-mentioned pathologies.

Hence loop expansion or $1/N$ expansion to any finite order for the model in Section 2 will lead to pathological results at least for large enough N .

Tracing the origin of these pathologies, one finds that they originate in kinetic energy corrections from loops of fermions in $O(N)$ vector representations. Scalar and auxiliary fields do not contribute since the propagator corrections they give are momentum independent. The fermion loop corrections to the momentum-dependent self-energy always come with the same sign. Therefore, any model of spin 0 and spin 1/2 particles will have these pathologies for large N if the leading terms in the $1/N$ expansion can be obtained by a one-loop calculation after adding auxiliary fields, and if these one-loop diagrams contain fermion loops. In particular, all four-dimensional renormalizable supersymmetric models based on chiral superfields, which consist of scalars and vectors under an internal $O(N)$ symmetry, will fall into this class.

I do not conclude that these models are necessarily inconsistent by themselves. It is conceivable that consistent results are obtained by solving the models exactly before taking the limit $\Lambda \rightarrow \infty$. In fact there exist results which indicate that the Wess-Zumino model may be consistent, while the ϕ^4 model may not be so¹¹⁾. However, the model in Section 2 differs from the Wess-Zumino model in that it has two independent kinetic term parameters. Therefore, the result in Ref. 11) does not apply directly to this model.

The aforementioned pathologies are absent if there are no Yukawa couplings, so that fermion loops do not contribute to the leading quantum corrections. The $O(N)$ symmetric $(\phi^2)^2$ model is an example of this. This model has another

peculiar feature. The effective potential decreases without bound when the field amplitude increases, at least as long as it is within the range of applicability of the lowest order in the $1/N$ approximation. It is not clear whether this is a pathology or not. Tunnelling may render the vacuum, based on a local minimum of the effective potential at $\phi = 0$, unstable for $N < \infty$, but it is not clear how to compute the tunnelling amplitude.

The presence of vector fields can alter the situation. In asymptotically free theories the sign of the quantum corrections is reversed. Therefore, they do not suffer from any of the pathologies discussed in this paper.

ACKNOWLEDGEMENTS

I thank several colleagues and especially G. Shore and H. Nicolai for useful discussions.

NOTE ADDED

After completion of this work I have obtained two other papers^{13),14)} reaching similar and different conclusions, respectively.

APPENDIX - SOME QUALITATIVE PROPERTIES OF THE EFFECTIVE
POTENTIAL FOR THE MODEL IN SECTION 2

In the classical model the auxiliary fields are determined from $\tilde{V}_{\text{eff}}^{(0)}$ in Eq. (2.7) as

$$\begin{aligned} -F_0^+ &= B_0 A_0, \\ -Z, G_0^+ &= \frac{1}{2} A_0^2 + b B_0 + a. \end{aligned} \quad (\text{A.1})$$

Generically there are two types of supersymmetric solutions to these equations corresponding to two types at absolute minima of $V_{\text{eff}}^{(0)}$:

$$\begin{aligned} \text{i) } \underline{A}_0 &= 0, \quad b B_0 + a = 0 \\ \text{ii) } B_0 &= 0, \quad \frac{1}{2} A_0^2 + a = 0 \end{aligned} \quad (\text{A.2})$$

These field configurations have to remain supersymmetric after the addition of quantum corrections¹²⁾. Due to a singularity of the parametrization (A.3), we will however not see ii) in the investigation below.

If the fields in the expression (2.14) for the effective potential are parametrized as absolute value times phase factor, then the extremization equations for the phase factors require either that G_0, B_0 and $\underline{A}_0^2$ are all real, or that they are real up to phase factors which are not determined. In either case it is sufficient for determining the extrema of V_{eff} to restrict oneself to real G_0, B_0 and $\underline{A}_0^2$. Doing this, and introducing the notation,

$$\begin{aligned} \frac{G_0}{B_0^2} &= g, & 64 \pi^2 \mu^2 a &= \alpha, \\ \frac{\underline{A}_0^2}{B_0^2} &= \pm a^2, & 64 \pi^2 \mu^3 b &= \beta, \\ \frac{B_0}{\mu} &= b, & 64 \pi^2 Z_5^r(\mu) &= \gamma. \end{aligned} \quad (\text{A.3})$$

$$\frac{1}{N} \tilde{V}_{\text{eff}} = \frac{\mu^4}{32 \pi^2} \tilde{u},$$

we can write (2.14) as

$$\tilde{u}(a, b, g) = \frac{1}{2} b^4 \left[a^2 (1+g) + (1+g)^2 \ln(1+g) + (1-g)^2 \ln(1-g) - 2 \frac{\alpha + \beta b}{b^2} g - (\gamma - 2 \ln b^2) g^2 \right]. \quad (A.4)$$

The equation for the auxiliary field, g , is

$$0 = \frac{\partial \tilde{u}}{\partial g} = b^4 \left[(1+g) \ln(1+g) - (1-g) \ln(1-g) + \frac{1}{2} a^2 - 2 \frac{\alpha + \beta b}{b^2} g - (\gamma - 2 \ln b^2) g^2 \right]. \quad (A.5)$$

Minimizing with respect to a yields two possibilities, either $a = 0$ or $g = \pm 1$. In the latter case (A.5) can be viewed as an equation for a :

$$\frac{1}{2} a^2 - 2 \ln 2 + \gamma - 1 - 2 \ln b^2 \pm \frac{\alpha + \beta b}{b^2} = 0. \quad (A.6)$$

The possible branches of u as a function of b , after a and g have been eliminated by the equations $\partial \tilde{u} / \partial a = 0$, $\partial \tilde{u} / \partial g = 0$, can be obtained as follows.

First one searches for extrema with respect to g of

$$\tilde{u}(0, b, g) = \frac{1}{2} b^4 \left[(1+g)^2 \ln(1+g) + (1-g)^2 \ln(1-g) - 2 \frac{\alpha + \beta b}{b^2} g - (\gamma - 2 \ln b^2) g^2 \right]. \quad (A.7)$$

The equation for g is

$$\frac{\partial \tilde{u}(0, b, g)}{\partial g} = b^4 \left[(1+g) \ln(1+g) - (1-g) \ln(1-g) - \frac{\alpha + \beta b}{b^2} - (\gamma - 1 - 2 \ln b^2) g \right] = 0 \quad (A.8)$$

The qualitative behaviour of the bracket in (A.8) as a function of g is illustrated in Fig. 2. For a given b there are at most three zeros $g_i(b)$, $i = 2, 3, 4$ giving at most three branches $u_i(b)$ of the effective potential. They can be numbered such that:

$$g_2 < g_3 < g_4,$$

$$g_2 < 0 < g_4,$$

u_2, u_4 are maxima of $\tilde{u}$, u_3 a minimum,

u_3 is negative, u_2 or u_4 is positive.

These yield the branches with $a = 0$. There remains the branches with $g = \pm 1$. I denote them as g_5 and g_1 . They exist when (A.6) can be satisfied. All branches are obtained from (A.7) as

$$u_i(b) = \tilde{u}(0, b, g_i) = \int_0^{g_i} dg \frac{\partial \tilde{u}(0, b, g)}{\partial g}. \quad (A.9)$$

Conclusions can now be drawn about the qualitative properties of u . For example, if $(\alpha + \beta b) > 0$, then for small b there is a single branch u_1 , which starts out at the origin and is positive for $|b| > 0$. As $|b|$ is increased it may or may not turn into the branch u_2 and later back to u_1 again. At some $|b|$ the branches u_3 and u_4 will appear. u_4 will later turn into u_5 . The branches u_1 and u_5 will tend to $+\infty$ when $|b| \rightarrow \infty$ while u_3 will remain negative and be approximated by

$$u_3 = - \frac{(\alpha + \beta b)^2}{4 \ln b^2} \left(1 + o\left(\frac{1}{\ln b^2}\right) \right). \quad (A.10)$$

The following inequalities will be valid for all $|b|$, $u_{1,2} > 0 > u_3$, $u_{1,2} > u_{4,5} > u_3$. If $(\alpha + \beta b) < 0$, $u_{1,2}$ and $u_{5,4}$ are exchanged in this discussion. If $(\alpha + \beta b)$ changes sign there are even more qualitative possibilities.

According to (A.10), u is bounded from below only if $\beta = 0$. Therefore, this case is of special interest. Unfortunately, u has no stationary point in this case. This can be seen if one magnifies Fig. 2 by a factor b^2 . Then the curve intersects the vertical axes at a fixed point with a slope which increases with b^2 . The potential (A.9) equals the area under this curve and from this interpretation it is easy to see that all branches are increasing functions of b^2 . The potential will look as in Fig. 3a.

Among other possibilities I choose to investigate the case when $\alpha = 0$. Then $u(a) = u(-a)$ and it suffices to investigate for example $\beta > 0$ and $b > 0$. The differences between this case and the previous one are due to the fact that the intersection between the curve and the vertical axis in the magnified Fig. 2 moves away from the origin as b is increased. For large b , u_3 decreases according to Eq. (A.10) while u_1 and u_5 increase. There must exist at least

one stationary point. Depending on the values of the parameters it can be a maximum on u_3 , or a minimum on u_4 or u_5 . These qualitatively different behaviours are illustrated in Fig. 3b, 3c and 3d, respectively

It should be noted that the rather complicated branch structure investigated in this Appendix is a pathology which is due to the neglect of terms of order B_0^2/Λ^2 in (2.12). Namely, for small $|b|$, when $\gamma - 3 - 2\ln b^2 > 0$, the function in Fig. 2 and (A.8) is monotonically decreasing. Then there can be at most one branch, and according to (A.9) it is positive. Since from (2.13) and (A.3)

$$\gamma - 3 - 2 \ln b^2 = 64\pi^2 \epsilon_s + 2 \ln \frac{\Lambda^2}{|B_0|^2} - 2$$

situations with more than one branch or with negative u can appear only if Z_s is negative or $|B_0|^2$ is of the same order as Λ^2 .

REFERENCES

- 1) S. Coleman, "1/N" in "Ettore Majorana", Int. School of Subnuclear Physics, Erice (1979).
- 2) D. Zanon, Phys. Lett. 104B (1981) 127.
- 3) M. Rocek, "An introduction to superspace and supergravity", in "Superspace and Supergravity", proc. of the Nuffield Workshop, Cambridge, (1980).
- 4) S. Coleman, "Secret Symmetries", Section 3.7 in "Ettore Majorana", Int. School of Subnuclear Physics, Erice (1973).
- 5) H. Nicolai, Nucl. Phys. B140 (1978) 294; B156 (1979) 157.
- 6) A.C. Davis, P. Salomonson and J.W. van Holten, CERN preprint TH.3268 (1982).
- 7) L.D. Landau, A.A. Abrikosov and I.M. Khalatnikov, Dokl. Akad. Nauk. SSSR 95 (1954) 497; 1177; 96 (1954) 261.
- 8) L. Abbott, J. Kang and H. Schnitzer, Phys. Rev. D13 (1976) 2212.
- 9) M.B. Voloshin, I.Yu. Kobzarev and L.B. Okun, Yad. Fiz. 20 (1974) 1229; (Sovj. Nucl. Phys. 20 (1975) 644);
S. Coleman, Phys. Rev. D15 (1977) 2929;
C. Callan and S. Coleman, Phys. Rev. D16 (1977) 1762.
- 10) B. de Wit and M.T. Grisaru, Phys. Rev. D20 (1979) 2082;
L. Alvarez-Gaumé, D.Z. Freedman and M.T. Grisaru, "Spontaneous breakdown of supersymmetry in two dimensions", Harvard preprint HUTMP 81/B111 (1982).
- 11) K. Wilson and J. Kogut, Phys. Rep. 12C (1974) 75;
H. Nicolai, Nucl. Phys. B156 (1979) 177.
- 12) M.T. Grisaru, W. Siegel and M. Roček, Nucl. Phys. B159 (1979) 429.
- 13) A. Higushi, Y. Kazama, Kyoto University preprint KHNS 627, HE(TH) 82/01 (1982).
- 14) D. Amati and K.-C. Chou, CERN preprint TH.3273 (1982).

FIGURE CAPTIONS

- Fig. 1 : The effective potential $(V_{\text{eff}}(\phi_0^2))$ for $(\phi^2)^2$ model.
- Fig. 2 : $(1/b^4)(\partial\tilde{u}(0,b,g)/\partial g)$ and $(2\ln b^2 - \gamma + 3)g - (\alpha + \beta b)/b^2$ as functions of g .
- Fig. 3 : Some possible qualitatively different behaviours of the effective potential $V_{\text{eff}}(B_0)$ for the supersymmetric model.

Fig. 1

Fig. 2

Fig. 3