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QUARTIC MASS MATRIX AND RENORMALIZATION CONSTANTS
IN SUPERSYMMETRIC YANG-MILLS THEORIES

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A B S T R A C T

We derive the general formula for the supertrace of the quartic mass matrix in a general supersymmetric gauge theory, with arbitrary representations for the chiral multiplets. This formula clarifies the non-renormalization theorems in presence of gauge interactions and gives "extended renormalization theorems" for $N = 2$ and $N = 4$ supersymmetric Yang-Mills theories. In particular we find the known result that $g_{\text{ren}} = g_{\text{bare}}$ for the $N = 4$ theory and the new result $m_{\text{ren}} = m_{\text{bare}}$ for the $N = 2$ gauge interactions of massive hypermultiplets. We give arguments to the extent that the latter non-renormalization theorem persists to all orders in perturbation theory.

One of the main properties of supersymmetric theories, relevant to solving the so-called hierarchy problem of gauge interactions, is the lowering of quantum divergences with respect to conventional renormalizable gauge theories. In particular the softening of quantum divergences manifests itself in a set of non-renormalization theorems first derived for a single self-interacting chiral multiplet ¹⁾ and later extended to gauge interactions using superfield Feynman rules ²⁾. The non-renormalization theorems assure that suitable ratios between coupling constants appearing in the superpotential are non-renormalized and hence that it is technically natural to fix their ratios. Moreover the absence of quadratic mass renormalization for scalar fields avoids large radiative mass corrections and therefore solves, at least in principle, the hierarchy problem of G.U.T.'s ³⁾.

In the present paper we would like to discuss the one-loop evaluation of the divergent renormalization constants (f1) of a general supersymmetric Yang-Mills theory and in particular to show how the non-renormalization theorems come about in a component formulation which uses a non-supersymmetric Wess-Zumino-Landau gauge. In this gauge the wave function renormalization of matter fields is gauge dependent and is not supersymmetric. In spite of this fact we will show that the supersymmetric effective potential has the right structure to give rise to new terms which cancel the gauge dependent part of the normal wave function renormalization and generate a universal, gauge independent, multiplicative renormalization of coupling constants and masses. In general, a quadratic renormalization of the Fayet-Iliopoulos term ⁴⁾ is also needed if the corresponding $U(1)$ gauge group has an over-all tracefull charge ⁵⁾.

The multiplicative renormalization is not a simple wave function renormalization and it leads to a component formulation of the non-renormalization theorems in the presence of gauge interactions, extending at the one-loop level previous analysis done for pure chiral interactions and Abelian QED ⁶⁾. Our analysis can be applied to the case of $N = 2$ and $N = 4$ Yang-Mills theories. For the $N = 4$ case one obtains the known fact that in the Wess-Zumino gauge there is only an infinite wave function renormalization but not coupling constant renormalization. For the $N = 2$ case, with arbitrary massive hypermultiplets in general representations of the gauge group G , one gets a new non-renormalization theorem which tells us that the mass is unrenormalized.

We take, for simplicity, the case in which the gauge group is simple so there is only one gauge coupling constant g . The generalization to semi-simple groups

and groups including U(1)-factors is straightforward.

The supersymmetric potential has the form

$$V(z^a, z_a) = f_a f^a + \frac{1}{2} D^\alpha D^\alpha \quad (1)$$

where $f_a = \partial f(z^a) / \partial z^a$, $f^a = f_a^*$, $z_a = (z^a)^*$
and $f(z^a)$ is the superpotential function

$$f(z^a) = \eta_a z^a + m_{ab} z^a z^b + g_{abc} z^a z^b z^c \quad (2)$$

as appropriate to a renormalizable theory. D^α is the gauge potential term given by

$$D^\alpha = g z_a T^{\alpha a}_b z^b + \xi^\alpha \quad (3)$$

where ξ^α is non vanishing only in correspondence to U(1) Abelian factors.

If we start with the square mass matrix for spin 0, $\frac{1}{2}$ and 1 fields of a general supersymmetric theory given respectively by ⁷⁾:

$$M_0^2 = \begin{pmatrix} f^{ab} f_{bc} + D^{\alpha a} D_c^\alpha + D_c^{\alpha a} D^\alpha & f^{abc} f_b + D^{\alpha a} D_c^\alpha \\ f_{abc} f^b + D_a^\alpha D_c^\alpha & f_{ab} f^{bc} + D_a^\alpha D_c^\alpha + D_a^{\alpha c} D^\alpha \end{pmatrix} \quad (4)$$

$$M_{1/2}^2 = \begin{pmatrix} f^{ab} f_{bc} + 2 D^{\alpha a} D_c^\alpha & -\sqrt{2} f^{ab} D_b^\beta \\ -\sqrt{2} D^{\alpha b} f_{bc} & 2 D^{\alpha c} D_c^\beta \end{pmatrix} \quad (5)$$

$$M_1^2 = D_a^\alpha D^\beta a + D^{\alpha a} D_a^\beta \quad (6)$$

we get the following general results for the supertraces of the quadratic and quartic mass matrix :

$$STr M^2 = \sum_J (-)^{2J} (2J+1) m_J^2(z^a, z_a) = 2 g_\alpha D^\alpha Tr T^\alpha \quad (7)$$

$$\begin{aligned}
 \text{Str } M^4 = & 2g^2 \sum_A C_2(A) [-4f^{Ai} f_{Ai} - f_{Ai} c_j f^{Ai} z^{cj} + \\
 & - z_{cj} f_{Ai} f^{Ai} c_j] + 2 f_{ck} X_{de}^{ck} f^{de} + \\
 & + \sum_A \left\{ 4g^3 C_2(A) z_{Ai} T^{\alpha Ai}_{Aj} z^{Aj} - g z_{Ai} \{X, T^{\alpha}\}_{Bj}^{Ai} z^{Bj} \right\} D^{\alpha} \\
 & + 2g^2 \left[\sum_A T(A) - 3C_2(G) \right] D^{\alpha} D^{\alpha}
 \end{aligned} \tag{8}$$

Eq. (7) controls the quadratic divergences of the potential while Eq. (8) controls the logarithmic divergences^{8),9)}.

In Eq. (18) the following conventions have been used

$$z^a = z^{Ai} \quad (A: \text{irrep. of } G, i: \text{element of } A) \tag{9}$$

$$g^2 T(A) \delta^{\mu\nu} = \sum_{ij} D^{\mu Ai}_{Aj} D^{\nu Aj}_{Ai}; \quad g^2 C_2(A) \delta^i_k = \sum_{\mu j} D^{\mu Ai}_{Aj} D^{\mu Aj}_{Ak} \tag{10}$$

$$X^c_d = X^{ck}_{de} = f^{Ai} B_j^{ck} f_{Ai} B_j^{de} \tag{11}$$

The supertrace $\text{Str} M^4$ takes the final form (8) by using a set of identities which follow from the G-invariance of the superpotential, i.e.,

$$f_a(z) T^{\alpha a}_b z^b = 0 \tag{12}$$

In particular, the X^c_d in Eq. (11) are non-zero only if z^c and z^d transform in the same way under the group G.

In the Wess-Zumino-Laudau gauge the one-loop renormalized effective potential is

$$\begin{aligned}
 \hat{V}(\hat{z}) = & \hat{f}^a \hat{f}_a + \frac{1}{2} \hat{D}^{\alpha} \hat{D}^{\alpha} = \\
 = & V(z) + h \text{Str } M^2(z) - k \text{Str } M^4(z)
 \end{aligned} \tag{13}$$

with $h = \Lambda^2/32\pi^2$, $k = \ln\Lambda^2/64\pi^2$ and the finite renormalization term $S\text{Tr}M^4(z)\log M^2(z)$ has been omitted.

If we now define

$$\begin{aligned}\hat{z}^a &= \left(\delta_{a'}^a - \frac{1}{2} \epsilon_{a'}^a \right) z^{a'} \\ \hat{g} &= (1+\rho) g \\ \hat{\xi}^\alpha &= \xi^\alpha + \delta \xi^\alpha \\ \hat{\eta}_a &= \left(\delta_a^{a'} + \alpha_a^{a'} + \frac{1}{2} \epsilon_a^{a'} \right) \eta_{a'} \\ \hat{m}_{ab} &= \left(\delta_a^{a'} \delta_b^{b'} + \beta_{ab}^{a'b'} + \frac{1}{2} \delta_a^{a'} \epsilon_b^{b'} + \frac{1}{2} \epsilon_a^{a'} \delta_b^{b'} \right) m_{a'b'} \\ \hat{g}_{abc} &= \left(\delta_a^{a'} \delta_b^{b'} \delta_c^{c'} + \gamma_{abc}^{a'b'c'} + \frac{1}{2} \delta_a^{a'} \delta_b^{b'} \epsilon_c^{c'} + \frac{1}{2} \delta_a^{a'} \epsilon_b^{b'} \delta_c^{c'} + \right. \\ &\quad \left. + \frac{1}{2} \epsilon_a^{a'} \delta_b^{b'} \delta_c^{c'} \right) g_{a'b'c'}\end{aligned}\tag{14}$$

Eq. (13) defines a set of equations for the constants $\epsilon, \rho, \delta\xi, \alpha, \beta$ and γ which have the solution

$$\rho = 2k g^2 \left[3C_2(G) - \sum_A T(A) \right]\tag{15}$$

$$\delta \xi^\alpha = 2h g_\alpha \text{Tr} T^\alpha\tag{16}$$

$$\epsilon_{a'}^a = -2k \left(X_{a'}^a - 2g^2 C_2(A) \delta_{a'}^a \right)\tag{17}$$

$$\alpha_a^{a'} = 0\tag{18}$$

$$\beta_{ab}^{a'b'} = 2k g^2 \delta_a^{a'} \delta_b^{b'} [C_2(A) + C_2(B)] = 4k g^2 \delta_a^{a'} \delta_b^{b'} C_2(A)\tag{19}$$

$$\gamma_{abc}^{a'b'c'} = 2k g^2 \delta_a^{a'} \delta_b^{b'} \delta_c^{c'} [C_2(A) + C_2(B) + C_2(C)]\tag{20}$$

From the above equations we see that one gets the multiplicative renormalization properties

$$\begin{aligned}\hat{\eta}_a &= (Z^{1/2})_a^{a'} \eta_{a'} & \hat{m}_{ab} &= (Z^{1/2})_a^{a'} (Z^{1/2})_b^{b'} m_{a'b'} \\ \hat{g}_{abc} &= (Z^{1/2})_a^{a'} (Z^{1/2})_b^{b'} (Z^{1/2})_c^{c'} g_{a'b'c'}\end{aligned}\tag{21}$$

where $Z_{aa'}^{1/2} = \delta_{aa'} + k(4g^2 C_2(A) \delta_{aa'} - X_{aa'})$. The complicated structure of $STr M^4$ for $g \neq 0$ is just appropriate to make this multiplicative and universal renormalization possible.

Eq. (21) extends the non-renormalization theorems of pure chiral theories¹⁾ to gauge interactions. We note that in the presence of gauge interactions the renormalization of masses and chiral couplings, $Z^{1/2}$, differs from the wave function renormalization of scalar fields, given by Eq. (17). This is not in contrast with the superspace analysis. The wave function renormalization is gauge-dependent, not supersymmetric and therefore cannot be compared with $Z^{1/2}$. The difference between the two constants is an artifact of the Wess-Zumino-Landau gauge: indeed, Eq. (21) gives by itself all the renormalization theorems that can be derived from superspace Feynman-rules considerations¹⁰⁾.

The previous formulae can be easily adapted to extended supersymmetric Yang-Mills theories. For $N = 4$, where the over-all tree potential V_0 is $SU(4)$ invariant¹¹⁾, one finds

$$STr M^4 = -8C_2(G)V(z) \quad (22)$$

therefore

$$\begin{aligned} \hat{V}(\hat{z}) &= V(z) - kSTr M^4 = \\ &= V(z(1+2kg^2C_2(G))) = V(\hat{z}) \end{aligned} \quad (23)$$

and there is no coupling constant renormalization, $g_{ren} = g_{bare}$, as it can also be checked by using the property

$$3g^2C_2(G) - \sum_A g^2T(A) = 0 \quad (24)$$

for three chiral multiplets in the adjoint representation.

A more interesting situation occurs for $N = 2$ where the vector multiplet is obtained by adding to the $N = 1$ vector multiplet V^α an $N = 1$ chiral multiplet N^α in the adjoint representation. The $N = 2$ matter multiplet is now given by a pair of $N = 1$ chiral multiplets S^a, T_a in complex conjugate representations R and $\bar{R}$ ¹²⁾.

The $N = 2$ gauge minimal coupling is obtained by adding to the $N = 1$ minimal coupling a superpotential term with coupling constant related to g as follows

$$\sqrt{2} g t^{\alpha a} S^b T_a N^\alpha \quad (25)$$

where t^α are the group-generators in the representation R . The other two possible $N = 2$ invariants are

$$m_{ST} S^a T_a + \eta_\alpha N^\alpha \quad (26)$$

By using our previous formulae it is straightforward to obtain in this case

$$\begin{aligned} \hat{g} &= g [1 + 2k g^2 (2C_2(G) - 2T(S))] \\ \hat{f}_{ST\alpha}^S &= (\hat{g}/g) f_{ST\alpha} \\ \hat{m}_{ST} &= m_{ST} \\ \hat{S} &= [1 + 2k g^2 C_2(S)] S \\ \hat{N} &= [1 - 2k g^2 C_2(G) + 4k g^2 T(S)] N \\ \hat{\eta}_\alpha &= [1 - 4k g^2 T(S)] \eta_\alpha \end{aligned} \quad (27)$$

where we have used the fact that

$$\begin{aligned} \sum_{ST} f^{ST\alpha} f_{ST\beta} &= 4g^2 T(S) \delta_\beta^\alpha \\ \sum_{\alpha T} f^{ST\alpha} f_{S'T\alpha} &= 4g^2 C_2(S) \delta_{S'}^S \end{aligned}$$

From (27) we see that there is no mass renormalization for the hypermultiplet (S, T) (f2). This miraculous cancellation, which is an $N = 2$ non-renormalization theorem, is likely to be true to all orders in perturbation theory if $N = 2$ supersymmetry is preserved.

This can be seen in different ways. In a $N = 1$ superspace formulation, using a background field gauge, one can repeat for the $N = 2$ Yang-Mills theory

the argument given in Ref. 13) for the $N = 4$ case. One gets the following three relations between the gauge coupling renormalization Z_g , the vector hypermultiplet wave function renormalization $Z_{V,N}$ and the hypermultiplet wave function renormalization $Z_{S,T}$

$$Z_g Z_V^{1/2} = 1 \quad , \quad Z_g Z_S^{1/2} Z_T^{1/2} Z_N^{1/2} = 1 \quad (28)$$

$$m_{\text{ren}} = Z_S^{1/2} Z_T^{1/2} m_{\text{bare}}$$

Because of $N = 2$ supersymmetry $Z_V = Z_N, Z_S = Z_T$ so we get $Z_S = 1$ and

$$m_{\text{ren}} = m_{\text{bare}} \quad (29)$$

A second argument which also leads to Eq. (29), beyond perturbation theory, is based on the fact that the hypermultiplet (S,T) yields a massive representation of $N = 2$ supersymmetry, the mass playing the role of a central charge. The mass cannot be renormalized because it appears in the right-hand side of the anti-commutation relation of the conserved supersymmetry charges ¹⁴⁾.

Finally we observe that Eq. (29) agrees with the folklore that in a superspace formulation only local counterterms may arise that are integrals over all superspace volume ¹⁵⁾. The kinetic term of the $N = 2$ hypermultiplet cannot be written as an integral over the full superspace volume ¹⁶⁾ so that no wave function renormalization is expected, consistently with Eq. (29).

FOOTNOTES

(f1) A full one-loop analysis, including finite renormalization will be given elsewhere.

(f2) The $N = 2$ hypermultiplet thus escapes the so-called survival hypothesis, namely that vector-like fermions must necessarily have masses of the order of the ultra-violet cut-off.

REFERENCES

- 1) J. Wess and B. Zumino, Phys. Lett. 49B, 52 (1974).
J. Iliopoulos and B. Zumino, Nucl. Phys. B76, 310 (1974).
S. Ferrara, J. Iliopoulos and B. Zumino, Nucl. Phys. B77, 41 (1974).
- 2) S. Ferrara and O. Piguet, Nucl. Phys. B93, 261 (1975).
M.T. Grisaru, W. Siegel and M. Rocek, Nucl. Phys. B159, 42 (1979).
- 3) E. Gildener and S. Weinberg, Phys. Rev. D13, 3333 (1976).
S. Weinberg, Phys. Lett. 83B, 387 (1979).
- 4) P. Fayet and J. Iliopoulos, Phys. Lett. 51B, 461 (1974).
- 5) W. Fischler, H. Nilles, J. Polchinski, S. Raby and L. Susskind, Phys. Rev. Lett. 47, 757 (1981).
- 6) J. Wess and B. Zumino, Nucl. Phys. B78, 1 (1974).
- 7) See, e.g., S. Ferrara, L. Girardello and F. Palumbo, Phys. Rev. D20, 403 (1979).
- 8) S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).
- 9) L. Maiani, in Proceedings of Ecole d'Eté de Physique des Particules, Gif-sur-Yvette (1979), p. 3.
- 10) M.T. Grisaru et al., quoted in Ref. 2). See also the discussion by E. Witten, Nucl. Phys. B188, 513 (1981).
- 11) F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122, 253 (1977).
L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. B121, 77 (1977).
- 12) A. Salam and J. Strathdee, Nucl. Phys. B97, 293 (1975).
P. Fayet, Nucl. Phys. B113, 535 (1976).
M. Sohnius, Nucl. Phys. B138, 109 (1978).
- 13) K.S. Stelle, LPTENS preprint 81/24, to appear in Proceedings of 1981 Nuffield Quantum Gravity Workshop, Cambridge University Press, M. Duff and C.J. Isham eds.
- 14) Similar arguments have been used in :
A. D'Adda, R. Horsley and P. Di Vecchia, Phys. Lett. 76B, 298 (1978).
E. Witten and D. Olive, Phys. Lett. 78B, 97 (1978).
- 15) M.T. Grisaru and W. Siegel, Caltech preprint CALT-68-892 (1982).
- 16) B. Zumino, in Proceedings of the Europhysics Study Conference on "Unification of Fundamental Particle Interactions", Plenum Press (1981), S. Ferrara, J. Ellis and P. Van Nieuwenhuizen eds., p.101.
M. Sohnius, K.S. Stelle and P.C. West in Superspace and Supergravity (S.W. Hawking and M. Rocek Eds., University Press (1980), p. 283.
P.S. Howe, K.S. Stelle and P.K. Townsend, CERN preprint TH.3271 (1982).