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On Higher Order Mode Loss in a Stationary State

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Abstract

Higher order mode loss in a stationary state in an electron storage ring is analyzed both in the time domain and in the frequency domain. The effect of multiple passage is studied carefully. It is shown that the coupling impedance of frequencies harmonic to the revolution frequency contributes to the higher order mode loss. In a high-Q RF cavity, the loss is large when the mode frequency is close to the harmonic of the revolution frequency and the loss is small otherwise. Statistical argument shows that the expectation value of the loss in a high-Q RF cavity is equal to the energy loss in a single passage case. This is a good estimate for a room temperature RF cavity, but for a superconducting cavity, the rms error of the estimated energy loss is quite large.

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51. Introduction

The energy loss due to the excitation of higher order modes in RF cavities and in vacuum chamber components in an electron storage ring has been studied by many authors. 1-6) Morton and Neil¹⁾ have treated the effect of an RF cavity in a single passage case and have shown that only the solenoidal electric fields play a role in the energy loss. They have given some explicit expressions for a pill-box (closed cylindrical) cavity. Wilson³⁾ has treated the case of multiple passage appropriate for an electron storage ring. He has first calculated a single-passage energy loss and then obtained a correction factor showing the effect of multiple passage on the basis of superposition. This is a time domain analysis. Sands⁴⁾ then has shown that since the exact frequencies of the higher order modes are not known, the estimate of the loss should be made statistically and in this case, the most probable loss is equal to the single-passage energy loss.

Keil et al.⁵⁾ have also treated the effect of an RF cavity in a somewhat different approach and obtained expressions for the higher order mode loss in a stationary multiple passage case. They have shown that the higher order mode loss does not occur if the cavity is made of perfectly conducting walls and if the resonance condition between the higher order mode frequencies and the revolution frequency is not satisfied. This suggests that a dissipative mechanism in RF cavities plays a role in the higher order mode loss in a stationary state. Sacherer²⁾ has used a frequency domain analysis and has shown a general relation between the higher order mode loss and the real part of the coupling impedance which shows the dissipation of the beam energy.

Recently, a renewed interest has arisen in the theory of higher order mode loss in connection with superconducting RF cavities. Hutton⁶⁾ has argued that since the energy dissipation in a superconducting RF cavity is

small, the wake field in the cavity will accelerate and decelerate the beam alternately and the higher order mode loss is of little concern in a superconducting cavity if the resonance condition between the higher order mode frequencies and the revolution frequency is carefully avoided.

However, due to the construction errors of the RF cavity cells, it seems difficult to avoid the resonance condition in every RF cavity cell. Thus, the statistical treatment is reinvestigated. Further, to shed more light on the higher order mode loss in a stationary multiple passage case, the problem is reinvestigated in this paper both in the time domain and in the frequency domain. The relation between the two descriptions is clarified and the relations between the results obtained in the previous papers are discussed in a coherent manner. Especially, the results of the sophisticated theory of Keil et al.⁵⁾ are shown to be derivable from the present general theory.

In §2, we review the single passage higher order mode loss. The coupling impedance of a discontinuous structure such as RF cavities is introduced from the first principle. In §3, we describe the higher order mode loss in a stationary multiple passage case both in the time domain and in the frequency domain. In §4, we discuss the statistical estimate based on the randomness of the mode frequencies and show that the average energy loss is given by the single passage energy loss. It is also shown that the above estimate is a good one for a room temperature RF cavity, but that the rms error of the above estimate is large for a superconducting RF cavity. Section 5 gives conclusions and discussions. In the appendix, we show that the results of Keil et al.⁵⁾ are derivable from the present theory if the appropriate form of coupling impedance is taken.

We consider an RF cavity or other discontinuous structures of length g located at $z = z_c$ to $z = z_c + g$. Suppose that a delta-function charge enters the cavity at time $t = t_i$. Let the longitudinal electric field $W(z - z_c, t - t_i)$ be excited at time t at location z by the delta-function charge. When many particles enter the cavity each at time t_i , the total electric field $E_z(z - z_c, t)$ is given by

$$E_z(z - z_c, t) = \sum_i W(z - z_c, t - t_i), \quad (2.1)$$

because the Maxwell equations are linear and the principle of superposition applies. When we consider a distribution of current

$$I(t) = e \sum_i \delta(t - t_i), \quad (2.2)$$

where e is the elementary charge, eq. (2.1) can be written as

$$E_z(z - z_c, t) = \frac{1}{e} \int_{-\infty}^{\infty} W(z - z_c, t - t') I(t') dt'. \quad (2.3)$$

Since eq. (2.3) is an integral of convolution type, discussion with Fourier transforms is useful. Define

$$I(\omega) = \int_{-\infty}^{\infty} I(t) \exp(-i\omega t) d\omega, \quad (2.4)$$

$$E_z(z, t) = \int_{-\infty}^{\infty} E_z(z, \omega) \exp(-i\omega t) d\omega, \quad (2.5)$$

$$W(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(z, \omega) \exp(-i\omega t) d\omega. \quad (2.6)$$

Then, we obtain from eq. (2.3)

$$eE_z(z, \omega) = \tilde{W}(z, \omega) I(\omega). \quad (2.7)$$

Now, consider a test particle which enters the cavity at time $t = t_0$. The energy "loss" $eV(t_0)$ of this test particle is

$$\begin{aligned} eV(t_0) &= -e \int_{z_c}^{z_c+g} E_z(z - z_c, t_0 + \frac{z-z_c}{v}) dz \\ &= -e \int_{-\infty}^{\infty} d\omega \int_0^g dz \tilde{W}(z, \omega) I(\omega) \exp[-i\omega(t_0 + \frac{z}{v})]. \end{aligned} \quad (2.8)$$

Now, we introduce the coupling impedance $Z(\omega)$ by

$$Z(\omega) = -\frac{1}{e} \int_0^g \tilde{W}(z, \omega) \exp(-\frac{i\omega z}{v}) dz. \quad (2.9)$$

Then, we obtain from eq. (2.8)

$$eV(t_0) = e \int_{-\infty}^{\infty} Z(\omega) I(\omega) \exp(-i\omega t_0) d\omega, \quad (2.10)$$

$$\text{or} \quad \tilde{V}(\omega) = Z(\omega) I(\omega), \quad (2.11)$$

where $\tilde{V}(\omega)$ is the Fourier transform of $V(t)$. Equation (2.11) reveals that the coupling impedance introduced by eq. (2.9) coincides with the usual definition of coupling impedance for a smooth vacuum chamber.

The energy loss ΔU for a current distribution $I(t)$ in a single passage case is given by

$$\Delta U = \int_{-\infty}^{\infty} V(t) I(t) dt, \quad (2.12)$$

according to the principle of superposition as described in connection with eq. (2.3). Using eqs. (2.4) and (2.10), we obtain in the frequency domain

$$\Delta U = 4\pi \int_0^\infty \operatorname{Re} Z(\omega) |I(\omega)|^2 d\omega, \quad (2.13)$$

where the relations

$$I(-\omega) = I^*(\omega), \quad (2.14)$$

$$Z(-\omega) = Z^*(\omega), \quad (2.15)$$

have been used. Here, the asterisk denotes the complex conjugate.

For a Gaussian bunch relevant to an electron beam, eq. (2.13) is

$$\Delta U = \frac{q^2}{\pi} \int_0^\infty \operatorname{Re} Z(\omega) \exp(-\omega^2 \sigma^2) d\omega, \quad (2.16)$$

where q is the total charge in a bunch and σ is the rms bunch length in time. We see from eq. (2.16) that the real part of the coupling impedance and the energy loss as a function of bunch length σ is related by the Laplace transform for a Gaussian bunch. This is seen by the replacement $t = \omega^2$ and $s = \sigma^2$.

We give a closed expression of eq. (2.16) for an RF cavity. For an RF cavity, the coupling impedance is given approximately as

$$Z(\omega) = \frac{R_s}{s} \frac{1}{1 - 10 \frac{\omega}{\omega_s} \frac{\omega_s}{\omega}}, \quad (2.17)$$

where R_s is the shunt impedance, Q_s the quality factor and $\omega_s/2\pi$ is the eigen-frequency of the mode s . The integral (2.16) is then⁷⁾

$$\Delta U = q^2 \frac{R_s}{s} \frac{(\omega_s \sigma)^2}{2Q_s^2} \left[\operatorname{Re} \left\{ \frac{w(\omega_s \sigma \sqrt{1 - \frac{\omega_s \sigma}{4Q_s^2}} + i \frac{\omega_s \sigma}{2Q_s})}{\frac{\omega_s \sigma}{2Q_s} - i \omega_s \sigma \sqrt{1 - \frac{\omega_s \sigma}{4Q_s^2}}} \right\} \right. \\ \left. + \left(1 - \frac{1}{2Q_s^2} \right) \frac{Q_s}{\sqrt{1 - \frac{\omega_s \sigma}{4Q_s^2}}} \operatorname{Im} \left\{ \frac{w(\omega_s \sigma \sqrt{1 - \frac{\omega_s \sigma}{4Q_s^2}} + i \frac{\omega_s \sigma}{2Q_s})}{\frac{\omega_s \sigma}{2Q_s} - i \omega_s \sigma \sqrt{1 - \frac{\omega_s \sigma}{4Q_s^2}}} \right\} \right], \quad (2.18)$$

where $w(z)$ is the complex error function and Re and Im denote the real and imaginary parts.*)

*) The form of eq. (2.18) has been suggested to the author by K. Satoh

For $Q_s \gg 1$ and $\omega_s \sigma \ll 1$, eq. (2.18) reduces to the well known form³⁾

$$\Delta U \approx q^2 \frac{R_s \omega_s}{s} \exp(-\omega_s^2 \sigma^2). \quad (2.19)$$

For $Q = 6 \times 10^4$ and $\omega_s/2\pi = 2$ GHz, the approximation of eq. (2.19) is good within an error of 2 % up to $\sigma = 270$ ps (bunch length ~ 8 cm), but deviates drastically from the exact expression (2.18) beyond that. This shows that the approximation of (2.19) is a good one for large electron storage rings in operation and in design stage.

§3. Higher Order Mode Loss in a Stationary Multiple Passage Case

In the multiple passage case as in an electron storage ring, the residual fields due to the previous turns also play a role. With the energy loss $V(t_0)$ in the single passage case, the energy loss $V_m(t_0)$ in the multiple passage case can be expressed as

$$V_m(t_0) = \sum_{p=0}^{\infty} V(t_0 + pT), \quad (3.1)$$

where T is the revolution period because the principle of superposition applies to the electromagnetic field in the cavity. In a multibunch case of equal population, T can be considered to be the interbunch time distance, and with this interpretation, the following argument can be easily generalized to a multibunch case.

Since

$$V(t_0 + pT) = 0, \quad \text{for } p < 0 \quad (3.2)$$

from the principle of causality, the sum over p in eq. (3.1) can be extended to $p = -\infty \sim \infty$. Then from the Fourier transformation expression

$$V_m(t_0) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} Z(\omega) I(\omega) \exp(-i\omega t_0 - ip\omega T) d\omega. \quad (3.3)$$

From the Poisson sum formula

$$\sum_{p=-\infty}^{\infty} \exp(-ip\omega T) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \quad (3.4)$$

where $\omega_0 = 2\pi/T$, eq. (3.3) is transformed into

$$V_m(t_0) = \omega_0 \sum_{n=-\infty}^{\infty} Z(n\omega_0) I(n\omega_0) \exp(-in\omega_0 t_0). \quad (3.5)$$

The energy loss per turn ΔU_m for the multiple passage case can be written as

$$\Delta U_m = \int_{-T/2}^{T/2} V_m(t) I(t) dt. \quad (3.6)$$

Since the bunch length is much shorter than the interbunch distance, the limit of the integral (3.6) can be extended from $-\infty$ to $+\infty$. Then, we obtain the two alternative forms

$$\Delta U_m = 2\pi \sum_{p=0}^{\infty} \int_{-\infty}^{\infty} Z(\omega) |I(\omega)|^2 \exp(-ip\omega T) d\omega, \quad (3.7)$$

and

$$\Delta U_m = T \left\{ 2 \sum_{n=1}^{\infty} \operatorname{Re} Z(n\omega_0) |I_n|^2 + \operatorname{Re} Z(0) |I_0|^2 \right\}, \quad (3.8)$$

where

$$I_n = \frac{1}{T} \int_{-T/2}^{T/2} I(t) \exp(in\omega_0 t) dt = \frac{2\pi}{T} I(n\omega_0). \quad (3.9)$$

Both expressions (3.7) and (3.8) are useful. Equation (3.7) is a direct consequence of the intuitive time domain expression (3.1) and is useful for numerical evaluation of the cavity problem as shown later. Equation (3.8) shows that only the coupling impedance of the harmonics of revolution frequency contributes to the higher order mode loss.

It has been argued by Hutton⁶⁾ that in a superconducting RF cavity which has high Q , the stationary state higher order mode loss is not important if the resonance condition between the higher order mode frequencies and the revolution frequency is carefully avoided. This fact is also made clear from eq. (3.8).

For numerical evaluation for the RF cavity case, the form (3.7) is more convenient than the sum (3.8). For the coupling impedance of an RF cavity given by eq. (2.17) and for the Gaussian bunch, we obtain⁷⁾ for $p \neq 0$

$$\begin{aligned}
 I_p &= 2\pi \int_{-\infty}^{\infty} Z(\omega) |I(\omega)|^2 \exp(-ip\omega T) d\omega \\
 &= \sum_s \frac{q^2 \omega_s R_s}{\omega_r \omega_s} \left[\exp\{-\sigma^2 \omega_r^2 + \sigma^2 \omega_i^2 + \omega_i p T\} \times \right. \\
 &\quad \left\{ \omega_r \cos(\omega_r p T + 2\omega_r \omega_i \sigma^2) + \omega_i \sin(\omega_r p T + 2\omega_r \omega_i \sigma^2) \right\} \\
 &\quad + \frac{(\omega_r - i\omega_i)}{4} w(-\omega_r + i\omega_i) \sigma + \frac{ipT}{2\sigma} \Bigg] , \quad (3.10)
 \end{aligned}$$

where

$$\omega_r = \omega_s \sqrt{1 - \frac{1}{4Q_s^2}} , \quad (3.11)$$

$$\omega_i = -\frac{\omega_s}{2Q_s} , \quad (3.12)$$

and w is the complex error function.

For $T \gg 2\sigma$ and $Q_s \gg 1$, eq. (3.10) can be approximated as

$$I_p = q^2 \sum_s \frac{\omega_s R_s}{Q_s} \exp\left(-\sigma^2 \omega_s^2 - \frac{\omega_s p T}{2Q_s}\right) \cos(\omega_s p T) . \quad (3.13)$$

The sum (3.7) has been evaluated by Wilson³⁾ by using the approximate formulae (2.19) and (3.13) as

$$\Delta U_m = \sum_s \Delta U_s f(\delta_s, \theta_s) , \quad (3.14)$$

where

$$\Delta U_s = q^2 \frac{\omega_s R_s}{2Q_s} \exp[-\sigma^2 \omega_s^2] , \quad (3.15)$$

$$f(\delta_s, \theta_s) = \frac{1 - \exp(-2\delta_s)}{1 - 2 \exp(-\delta_s) \cos \theta_s + \exp(-2\delta_s)} , \quad (3.16)$$

$$\delta_s = \frac{\pi \omega_s}{\omega_0 Q_s} , \quad (3.17)$$

$$\theta_s = \frac{2\pi \omega_s}{\omega_0} . \quad (3.18)$$

Keil et al.⁵⁾ have treated the energy loss in a pill-box cavity with a more elaborate method and obtained the results similar to eqs. (3.8) and (3.14). It is shown in the Appendix that their results are the same as our present general results when an appropriate form of the coupling impedance of the pill-box cavity is taken into account.

§4. Statistical Estimate in a High-Q Resonator

The energy loss in a single passage is described by the integral (2.13) whereas that in a multiple passage case is described by the infinite series (3.8). The infinite series can be approximated by the integral in case of a broadband impedance such as a low-Q cavity. For a low-Q cavity, the condition is that the width of the peak $\omega_s/(2Q_s)$ is much larger than the revolution frequency ω_0 . This is also seen from eq. (3.16), since δ_s is large in this case and the correction factor f_s given by (3.16) is nearly unity. This result is plausible because the field excited by a bunch decays substantially

at the time of arrival of the succeeding bunch so that the situation is that of a single passage.

In a high-Q resonator, the situation is different. In a disk-and-washer RF cavity to be used for TRISTAN⁸⁾, Q is typically 6×10^4 . For the typical higher order mode frequency of 1 GHz, the width of the peak $\omega_s/(2Q_s)$ is 5×10^4 Hz while the revolution (angular) frequency is $2\pi \times 10^5$ Hz. Thus, the width of the impedance peak is narrower than the revolution frequency. If the peak lies near the harmonic of the revolution frequency, the loss is very large while if the peak lies apart from the harmonic of the revolution frequency, the loss is small. This fact is more prominent for a superconducting cavity for which Q is typically 10^9 .

However, since the higher order mode frequencies are expected to be randomly distributed due to the construction errors of the cells of the RF cavities, we must make a statistical estimate of the loss. The dependence of the frequency error on the construction error is estimated by using the formula of the pill-box cavity given by eq. (A. 37)

$$\left(\frac{\omega_s}{c}\right)^2 = \left(\frac{v_h}{b}\right)^2 + \left(\frac{p\pi}{g}\right)^2, \quad (4.1)$$

where b and g are the radius and length of the cavity, c the velocity of light, v_h the zero of the zeroth order Bessel function, p is an integer and s denotes the mode (n, p).

Then,

$$\frac{\Delta\omega_s}{\omega_s} = -\frac{(v_h/b)^2}{(\omega_s/c)^2} \frac{\Delta b}{b} - \frac{(p\pi/g)^2}{(\omega_s/c)^2} \frac{\Delta g}{g}, \quad (4.2)$$

and the relative frequency error is of the order of the construction errors $\Delta b/b$ and $\Delta g/g$. We assume that the absolute error Δb or Δg is ± 0.1 mm and

choose the typical length as 30 cm which is the half wavelength of the 500 MHz RF cavity. Then $\Delta\omega_s/\omega_s$ is typically $\pm 3 \times 10^{-4}$ and for the typical value of $\omega_s = 2\pi \times 1$ GHz, $\Delta\omega_s = \pm 1.8 \times 10^6$ Hz. Thus, the frequency error is larger than the revolution frequency.

Now, we introduce a distribution function $F(\theta_s, \bar{\theta}_s, \Delta_s)$ where θ_s is given by eq. (3.18), $\bar{\theta}_s$ is the average value of θ_s and Δ_s is its error. Then, the mean value $\langle \Delta U \rangle$ is given by

$$\langle \Delta U \rangle = N_c \sum_s \Delta U_s < f_s >, \quad (4.3)$$

where

$$< f_s > = \int f(\delta_s, \theta) F(\theta; \bar{\theta}_s, \Delta_s) d\theta, \quad (4.4)$$

and the rms error ΔS is given by

$$\Delta S^2 = N_c \sum_s (\Delta U_s)^2 (< f_s^2 > - < f_s >^2), \quad (4.5)$$

where

$$< f_s^2 > = \int f^2(\delta_s, \theta) F(\theta; \bar{\theta}_s, \Delta_s) d\theta. \quad (4.6)$$

In the above formula, N_c denotes the number of cavities.

First, we make a simplified assumption that the frequencies of each mode are distributed uniformly so that θ_s defined by eq. (3.18) lies between (integer)- π to (integer)+ π . For this uniform distribution, we obtain

$$< f_s > = 1, \quad (4.7)$$

and

$$< f_s^2 > = \frac{1 + \exp(-2\delta_s)}{1 - \exp(-2\delta_s)}. \quad (4.8)$$

The same results are obtained if we assume a uniform distribution between

§5. Conclusions and Discussions

The higher order mode loss in a stationary state in an electron storage ring is discussed by the time domain and frequency domain analyses. It is shown that, in a multiple passage case, the wake fields due to the previous turns play an important role and the loss shows a different behavior compared to the single passage case. The loss in a multiple passage case is described by the series (3.8) instead of the integral (2.13) or the correction factor (3.16) is added to the loss in a single passage. These considerations include the case of the pill-box cavity considered by Keil et al.⁵⁾ as a special case as shown in the appendix.

In a high-Q resonator, the loss is large when the higher order mode frequency lies close to the harmonics of the revolution frequency and the loss is small otherwise. However, the mode frequencies usually have a spread due to the construction errors of the cavity. The estimated spread is larger than the revolution frequency in TRISTAN electron storage ring.

The statistical estimate shows that the mean expected value of the loss is equal to the loss in a single passage case. This estimate is quite a good one for a room temperature RF cavity. In case of the superconducting cavity, the rms error of the estimated loss is quite large and the loss will be sometimes much larger and sometimes much smaller than the mean value given by the single passage energy loss. Since it seems difficult to control all the mode frequencies in all the RF cells, a doubt is cast on the conclusion of Hutton⁶⁾ that the higher order mode loss is not of concern in the superconducting cavity although the control of higher order mode frequencies will be one of the good countermeasures if successful.

α and $\alpha+2\pi$, where α is an arbitrary number. This means that the statistical estimates (4.7) and (4.8) are valid irrespective of the position of the mode frequency. The estimate (4.7) has been given by Sands⁴⁾, which shows that the average loss is given by the single passage energy loss irrespective of the value of δ or Q , i.e. even in a superconducting cavity.

We make a crude estimate for the TRISTAN⁸⁾ disk-and-washer cavity. There are about twenty peaks per cell up to the cut-off frequency of the vacuum pipe and the total number of cells is about 1250. We assume that the loss ΔU_s is the same for all the modes, take 1 GHz as a typical higher order mode frequency and 6×10^4 as a typical value of Q , and take a case of two bunches per beam. Thus $\Delta S / \langle \Delta U \rangle_m = 0.62\%$, which shows that the estimate (4.7) is a very good one. If we make the same estimate for the superconducting cavity of $Q = 10^9$, we obtain $\Delta S / \langle \Delta U \rangle_m = 160\%$ which is a very large number. This shows that for the superconducting cavity, the mean value estimate (4.7) is a poor one and that there may be cases where the loss is substantially larger or smaller than the average loss given by the single passage energy loss.

To further confirm the above argument, we have numerically calculated $\langle f_s \rangle$ and $\langle f_s^2 \rangle$ as a function of the rms error Δ for the Gaussian distribution of mode frequencies. The results are shown in Figs. 1(a) and (b) for the room temperature RF cavity of $Q_s = 6 \times 10^4$ and in Figs. 2(a) and (b) for the superconducting cavity of $Q_s = 10^9$. In the figures, $\bar{\theta} = 0$ means that the average peak coincides with the harmonic of the revolution frequencies and $\bar{\theta} = \pi$ means that the average peak lies in the middle of the harmonics of the revolution frequencies. We see from these figures that both $\langle f_s \rangle$ and $\langle f_s^2 \rangle$ approach the values given by eqs. (4.7) and (4.8) for $\Delta \gtrsim \pi$. Since Δ is about equal to or more than π as estimated before, the simple statistical argument using eqs. (4.7) and (4.8) is justified.

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In this appendix, we derive an expression for the coupling impedance of the pill-box cavity considered by Keil et al.⁵⁾ and show that their result is a special case applied to the pill-box cavity of the general treatment described in this paper.

The cavity and beam geometry is shown in Fig. 3. The cavity has radius b and length g . The end walls perpendicular to the beam are assumed to have infinite conductivity and the side wall is assumed to have finite conductivity σ . For the calculation of the coupling impedance of the cavity, we follow the method of Keil et al. closely. However, since we are also interested in the effect of the charge and current in the cavity, we consider a cylindrical beam of radius a in which the particles are uniformly distributed along the cross-section. This is because we wish to avoid a divergent quantity. Such a method has been used by Morton et al.⁹⁾ to calculate the coupling impedance of a resistive vacuum chamber. We use cylindrical coordinates (r, θ, z) and the charge and current densities $\rho(x, t)$ and $j_z(x, t)$ are given as

$$\rho(x, t) = n_0 f(z - vt) H(a - r), \quad (\text{A.1})$$

$$j_z(x, t) = n_0 v f(z - vt) H(a - r), \quad (\text{A.2})$$

where the function $f(z)$ is normalized as

$$\int_{-\infty}^{\infty} f(z) dz = 1. \quad (\text{A.3})$$

In the above formula, v is the velocity of the particle and n_0 is the charge density related to the total number N of particles in a bunch as

$$\pi a^2 n_0 = Ne \quad (A.4)$$

$H(x)$ is the Heaviside step function, which is unity for $x > 0$ and zero for $x < 0$.

We use the vector potential A and the scalar potential ϕ in the Lorentz gauge to calculate the electromagnetic fields in the cavity. Since we have currents only in the z -direction, $A_r = A_\theta = 0$ and only A_z is non-zero. Thus, the Maxwell equations are

$$\square A_z = -\frac{4\pi}{c} j_z, \quad (A.5)$$

$$\square \phi = -4\pi\rho, \quad (A.6)$$

and the Lorentz condition is

$$\frac{\partial A_z}{\partial z} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0. \quad (A.7)$$

The boundary conditions are

$$E_r = 0 \quad \text{for } z = 0 \text{ and } z = g. \quad (A.8)$$

$$\tilde{E}_z = -R \tilde{H}_\theta \quad \text{for } r = b. \quad (A.9)$$

Here, $\tilde{E}_z$ and $\tilde{H}_\theta$ are the Fourier transforms of E_z and H_θ , respectively; i.e.

$$\tilde{E}_z(r, z, t) = \int_{-\infty}^{\infty} E_z(r, z, \omega) \exp(-i\omega t) d\omega. \quad (A.10)$$

The wall impedance R is given by

$$R(\omega) = (1 - i) \sqrt{\frac{\omega}{8\pi\sigma}}. \quad (A.11)$$

We now write A_z and ϕ by virtue of the boundary condition (A.8) and the Lorentz condition (A.7) in the form

$$A_z = \sum_{p=0}^{\infty} \int_{-\infty}^{\infty} A_p(\omega, r) \cos\left(\frac{\pi p z}{g}\right) \exp(-i\omega t) d\omega, \quad (A.12)$$

$$\phi = \sum_{p=0}^{\infty} \int_{-\infty}^{\infty} B_p(\omega, r) \sin\left(\frac{\pi p z}{g}\right) \exp(-i\omega t) d\omega. \quad (A.13)$$

From the Lorentz condition (A.7), we obtain

$$B_p(\omega, r) = \left(\frac{1}{c}\right) \left(\frac{\pi p}{g}\right) A_p(\omega, r). \quad (A.14)$$

Thus, we have to solve only the equation for the vector potential.

The Maxwell equation (A.5) now reads

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial A_p}{\partial r} \right) + \lambda_p^2 A_p = -\frac{4n_0}{cg(1+\delta_{p0})} \frac{C_p(\omega)}{g} H(a-r), \quad (A.15)$$

where

$$\lambda_p^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi p}{g}\right)^2, \quad (A.16)$$

and

$$C_p(\omega) = v \int_0^g dz \int_{-\infty}^{\infty} \exp(i\omega t) \cos\left(\frac{\pi p z}{g}\right) f(z-vt) dt. \quad (A.17)$$

Since we are interested in the case

$$f(z-vt) = \delta(z-vt), \quad (A.18)$$

for the calculation of the coupling impedance, $C_p(\omega)$ reduces to

$$C_p(\omega) = \int_0^g \exp\left(\frac{i\omega z}{v}\right) \cos\left(\frac{\pi p z}{g}\right) dz, \quad (\text{A.19})$$

and

$$|C_p(\omega)|^2 = 2\left(\frac{\omega}{v}\right)^2 \frac{[1 - (-1)^p \cos(\frac{\omega g}{v})]}{[(\frac{\omega}{v})^2 - (\frac{\pi p}{g})^2]^2}. \quad (\text{A.20})$$

The boundary condition (A.9) is written as

$$A_p(\omega, b) = -\left(\frac{i\omega}{c}\right) \left(\frac{R}{\lambda_p}\right) \frac{\partial A_p}{\partial r} \Big|_{r=b}. \quad (\text{A.21})$$

The solution for eq. (A.15) is

$$A_p = C J_0(\lambda_p r) - \frac{4n_0 C_p}{\lambda_p^2 \operatorname{cg}(1+\delta_{p0})} \quad (\text{for } r < a), \quad (\text{A.22})$$

and

$$A_p = C J_0(\lambda_p r) + \frac{2\pi n_0 a C_p}{\lambda_p \operatorname{cg}(1+\delta_{p0})} \{ N_1(\lambda_p a) J_0(\lambda_p r) - J_1(\lambda_p a) N_0(\lambda_p r) \} \quad (\text{A.23})$$

(for $a < r < b$),

where J_n and N_n are Bessel and Neumann functions of order n and C is a constant. In deriving eq. (A.23), the continuity of A_p and $\frac{\partial A_p}{\partial r}$ at $r = a$ is assumed. From the boundary condition (A.21), we obtain

$$C = \frac{2\pi n_0 a C_p}{\lambda_p \operatorname{cg}(1+\delta_{p0})} \left[\frac{N_0(\lambda_p b) - \frac{i\omega R}{c\lambda_p} N_1(\lambda_p b)}{J_0(\lambda_p b) - \frac{i\omega R}{c\lambda_p} J_1(\lambda_p b)} J_1(\lambda_p a) - N_1(\lambda_p a) \right]. \quad (\text{A.24})$$

The longitudinal electric field E_z is given by

$$E_z = -\frac{\partial \phi}{\partial z} - \frac{1}{c} \frac{\partial A_z}{\partial t}, \quad (\text{A.25})$$

and thus it is given by

$$E_z(r=0, z, \omega) = \left(\frac{ic}{\omega}\right) \sum_{p=0}^{\infty} \lambda_p^2 A_p(\omega, 0) \cos\left(\frac{\pi p z}{g}\right). \quad (\text{A.26})$$

The energy loss $V(t_0)$ of the particle entering the cavity at time t_0 is

$$\begin{aligned} V(t_0) &= -\int_0^g E_z(z, t_0 + \frac{z}{v}) dz \\ &= -\sum_{p=0}^{\infty} \int_{-\infty}^{\infty} d\omega \exp(-i\omega t_0) \left(\frac{ic\lambda_p^2}{\omega}\right) A_p(\omega, 0) \int_0^g dz \cos\left(\frac{\pi p z}{g}\right) \exp\left(-i\frac{\omega z}{v}\right), \end{aligned} \quad (\text{A.27})$$

and its Fourier transform $V(\omega)$ is given by

$$V(\omega) = -\sum_{p=0}^{\infty} \left(\frac{ic\lambda_p^2}{\omega}\right) C_p^* A_p(\omega, 0). \quad (\text{A.28})$$

With

$$A_p(\omega, 0) = C - \frac{4n_0 C_p}{\lambda_p^2 \operatorname{cg}(1+\delta_{p0})}, \quad (\text{A.29})$$

and

$$I(\omega) = \frac{n_0(\pi a^2)}{2\pi}, \quad (\text{A.30})$$

the coupling impedance $Z(\omega)$ is given by

$$Z(\omega) = \frac{\tilde{V}(\omega)}{\tilde{I}(\omega)} = Z_{\text{res}} + Z_{\text{sc}}, \quad (\text{A.31})$$

where

$$Z_{\text{res}} = -\frac{4\pi i}{\omega g} \sum_{p=0}^{\infty} \frac{|C_p|^2 \lambda_p}{(1+\delta_{po})^2 a} \frac{N_0(\lambda_p b) - \frac{i\omega R}{c\lambda_p} N_1(\lambda_p b)}{J_0(\lambda_p b) - \frac{i\omega R}{c\lambda_p} J_1(\lambda_p b)} J_1(\lambda_p a), \quad (\text{A.32})$$

and

$$Z_{\text{sc}} = -\frac{4\pi i}{\omega g} \sum_{p=0}^{\infty} \frac{|C_p|^2}{1+\delta_{po}} \left\{ \frac{\lambda_p}{a} N_1(\lambda_p a) + \frac{2}{\pi a^2} \right\}. \quad (\text{A.33})$$

For large p , it is better to introduce $\lambda'_p{}^2 = -\lambda_p^2$. In this case

$$Z_{\text{res}} = -\frac{8i}{\omega g} \sum_{p=0}^{\infty} \frac{|C_p|^2 \lambda'_p}{1+\delta_{po}} \frac{a}{I_0(\lambda'_p b) - \frac{i\omega R}{c\lambda'_p} I_1(\lambda'_p b)} \frac{K_0(\lambda'_p b) + \frac{i\omega R}{c\lambda'_p} K_1(\lambda'_p b)}{I_1(\lambda'_p a)}, \quad (\text{A.34})$$

and

$$Z_{\text{sc}} = \frac{8i}{\omega g} \sum_{p=0}^{\infty} \frac{|C_p|^2}{1+\delta_{po}} \left\{ \frac{1}{a^2} - \frac{\lambda'_p}{a} K_1(\lambda'_p a) \right\}. \quad (\text{A.35})$$

We study the structure of the coupling impedance obtained above more closely. The term Z_{res} is not divergent for $a \rightarrow 0$ and has a resonant form. This is the term taken up by Keil et al. We assume high conductivity and R to be very small. Then this term has a pole for

$$J_0(\lambda_n b) = 0, \quad (\text{A.36})$$

λ_n being given by v_n/b , where v_n is the zero of the Bessel function $J_0(x)$ and

this determines a resonant frequency ω_s by

$$\left(\frac{\omega_s}{c}\right)^2 = \left(\frac{v_n}{b}\right)^2 + \left(\frac{\pi p}{g}\right)^2, \quad (\text{A.37})$$

where s denotes the mode (n, p) .

We expand $J_0(\lambda_p b)$ in terms of ω^2 around the pole ω_s^2 and express Z_{res} as a sum of the contributions of poles. Then, with some algebra, Z_{res} is given as

$$Z_{\text{res}} = -\frac{16i\omega c^2}{g v^2} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{[1 - (-1)^p \cos(\frac{\omega g}{v})]}{[(\frac{\omega}{v})^2 - (\frac{\pi p}{g})^2]^2} \times \frac{\lambda_p^2}{(1+\delta_{po}) b^2 J_1^2(v_n^2)} \frac{1}{\{\omega_s^2 - \omega^2 - \frac{(1+i)}{Q} \omega \omega_s\}}, \quad (\text{A.38})$$

where the relation

$$2i c R = \frac{(1+i)}{Q} \omega, \quad (\text{A.39})$$

is used. This expression coincides with the expression obtained by the normal mode analysis of the solenoidal electric field.¹⁰⁾

The term Z_{sc} is divergent for $a \rightarrow 0$ and is considered to be caused by the charge and current in the resonant cavity because the effect after the bunch has passed the cavity is described only by the solenoidal electric field.¹⁾ This is a pure imaginary coupling impedance and does not play a role in energy loss. This term has been neglected by Keil et al.

Now, Keil et al. have given the following two expressions for the energy loss.

$$\Delta U = T I_{av}^2 Z_{eff} \quad (A.40)$$

$$Z_{eff} = \sum_{n=0}^{\infty} Z_n \exp(-n^2 \omega_0^2 \sigma^2) \quad (A.41)$$

where

$$Z_n = \frac{16\omega_0^2}{bgv^2 c} \sum_{p=0}^{\infty} \frac{n^2}{1 + \delta_{po}} \frac{[1 - (-1)^p \cos(\frac{n\omega_0 g}{v})]}{[(\frac{\pi p}{g})^2 - (\frac{n\omega_0}{v})^2]^2} \frac{(n\omega_0 g)^{1/2}}{(8\pi\sigma)^{1/2}} \frac{1}{np} \quad (A.42)$$

and

$$D_{np} = [J_0(\lambda_p b) - \frac{n\omega_0}{c\lambda_p} (\frac{n\omega_0}{8\pi\sigma})^{1/2} J_1(\lambda_p b)]^2 + [\frac{n\omega_0}{c\lambda_p} (\frac{n\omega_0}{8\pi\sigma})^{1/2} J_1(\lambda_p b)]^2 \quad (A.43)$$

The alternative expression is

$$Z_{eff} = \frac{16\pi}{g\omega_0} \sum_{p=0}^{\infty} \sum_{n=1}^{\infty} \frac{R_s \exp(-\omega_s^2 \sigma^2)}{(1 + \delta_{po}) v_n J_1^2(v_n)} \times \frac{[1 - (-1)^p \cos(\frac{\omega_s g}{v})]}{\{1 + [(p\pi b)/(vgv_n)]^2\}^2} \quad (A.44)$$

where

$$R_s = \frac{\sinh(\pi\omega_s/\omega_0 Q_s)}{\cosh(\pi\omega_s/\omega_0 Q_s) - \cos(2\pi\omega_s/\omega_0)} \quad (A.45)$$

and

$$\gamma^2 = \frac{1}{1 - (v/c)^2} \quad (A.46)$$

Equation (A.41) is equal to our eq. (3.8) when $J_0(\lambda_p b)$ is expanded about its zeros as the argument following eq. (A.36). Equation (A.44) is equal to our eq. (3.14) when the shunt impedance R_s is taken from eq. (A.38). Thus, the results of Keil et al. are included into our present general theory.

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Figure Captions

- Fig. 1 (a): $\langle f \rangle$ as a function of Δ for a room temperature cavity. $Q = 6 \times 10^4$.
- Fig. 1 (b): $\langle f^2 \rangle$ as a function of Δ for a room temperature cavity. $Q = 6 \times 10^4$.
- Fig. 2 (a): $\langle f \rangle$ as a function of Δ for a superconducting cavity. $Q = 10^9$.
- Fig. 2 (b): $\langle f^2 \rangle$ as a function of Δ for a superconducting cavity. $Q = 10^9$.
- Fig. 3: Geometry of pill-box cavity.

Fig. 1 (b)

Fig. 1 (a)

Fig. 2 (b)

Fig. 2 (a)

Fig. 3