ACADEMY OF SCIENCES OF THE USSR
P. N. LEBEDEV PHYSICAL INSTITUTE

High energy and cosmic rays physics

Preprint No 71

THE NON-PERTURBATIVE APPROACH TO THE INFRARED
PROBLEM IN QCD AT FINITE TEMPERATURE

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CM-P00066758

MOSCOW, 1982

A non-perturbative approach is developed for investigation of the infrared problem in QCD at $T \neq 0$ in the ghost-free axial gauge. The problem is solved by using a 3-dimensional theory within the exact Slavnov-Taylor identities and Schwinger-Dyson equations. The system of two non-linear integral equations for the structural functions of the gluon polarization tensor is obtained whose solution determines the infrared behavior of the temperature Green functions in the 4-dimensional QCD. The simplest solution of these equations which is the same as the first term of the perturbation expansion shows the presence of singularities in the gluon propagator at momenta $p \sim g^2 T$, that cannot be eliminated by any choice of the gauge. The infrared instability of QCD at $T \neq 0$ caused by these singularities is discussed.

INTRODUCTION

Investigations of non-Abelian gauge theories at high temperatures and densities are now of great practical interest. These investigations may throw light upon the confinement problem in QCD and are useful in solving a number of tasks in astrophysics and cosmology. The possibility to obtain reliable results is due to the asymptotic freedom of the non-Abelian gauge fields [1], which allows to study the QCD (and other gauge models) at the asymptotically large values of certain parameters (such as temperature, and density [2], also momentum [3]) within perturbative methods. In this sense the QCD is considered as the quantitative theory of strong interaction and its study becomes a subject of the first-rate importance. Unfortunately, the investigation of long-range behavior of QCD is much more difficult and the results obtained are not yet reliable because of the infrared singularities.

The infrared problem in QCD at finite temperatures is qualitatively different from that at $T=0$ [4,5]. It manifests itself in the presence of the infrared power divergencies (the logarithmic ones are not leading) in all the higher orders of the perturbative expansion and in the appearance of the "fictitious" singularities of the Green functions. The problem is not always solved by introducing the infrared cut-off. In particular, the difficulties retain even if the infrared cut-off of the transversal gluons takes place

at $p \sim g^2 T$ because such scale of cut-off is not large enough.

The essential difficulties already arise in the first order of the perturbation theory, where infrared singularities of the transversal gluon propagator and other Green functions were found [6]. These singularities (poles) are gauge dependent but they are present at any choice of the gauge fixing parameter. Independently the same results were obtained in [7]. The singularities found in [6,7] are of great importance and likely indicate the instability of QCD at $T \neq 0$ with the respect to generation of non-zero vacuum field [8]. The perturbative results are not reliable due to such singularities and it is needed to find more powerful methods of investigations within which the sense of the difficulties indicated may be understood.

The present paper is devoted to development of non-perturbative approach to QCD in small momentum domain and, in particular, to determination of the true vacuum state of QCD at $T \neq 0$. The main attention is paid to the infrared problem.

We use some circumstances allowing to simplify the solution of our problem. Since the infrared problem is studied within the temperature Green functions technique [9] the momentum space is discrete and only the static infrared limit is important for the infrared behavior of QCD at $T \neq 0$. In this limit the quark propagator is not essential and the studying of the QCD is reduced to the investigation of the Yang-Mills theory only. More important is however the following. At $T \neq 0$ the leading (power) infrared divergences

arise from the terms with zero frequencies. Therefore as far as the static infrared limit is concerned the four-dimensional Euclidean space effectively reduces to three dimensional one. Moreover the one-loop calculations [10] show that the $\Pi_{44}(p_4=0; |\vec{p}| \rightarrow 0)$ component of the polarization operator is nonzero and the screening of longitudinal gluons takes place. As a result, the infrared problem in QCD is related only to transversal gluon propagator. We conclude that the infrared behavior of the four-dimensional QCD at $T \neq 0$ is exactly determined by that of the three-dimensional Yang-Mills theory with the dimensional coupling constant $\tilde{g}^2 = g^2 T$.

The further progress can be achieved if the investigation is carried on within the axial gauges

$$n_\mu A_\mu = 0$$

which was examined in [11]. The most remarkable feature of such gauges consists in the absence of the ghost particles from the theory. This allows one to simplify the Slavnov-Taylor identities. The latter play an essential role in the non-perturbative approach proposed here.

Our approach to the infrared problem in QCD at $T \neq 0$ reduces to the investigation of the polarization tensor in the infrared momentum domain in the three-dimensional Yang-Mills theory with the dimensional coupling constant $\tilde{g}^2 = g^2 T$. We work in the axial gauge and use the Slavnov-Taylor identities by means of which the vertices are expressed through

the polarization tensor. In the four-dimensional QCD at $T = 0$ an analogous approach was discussed by many authors [12, 13].

The contents of the paper is as follows. In Section II we develop the temperature Green function technique for the Yang-Mills theory in axial gauge. The Schwinger-Dyson equations and the Slavnov-Taylor identities are given for the case when arbitrary vacuum expectation value of the gluonic field is present. With the simplest vacuum expectation

$\langle A_\mu(x) \rangle = 0$ these equations are used in Section III for the three dimensional theory with $\tilde{g}^2 = g^2 T$. We find the system of two self-consistent equations for the structural functions of the polarization tensor which describe the infrared behavior of the theory under consideration. The first iteration of the system obtained is given and discussed in Section IV. In Section V we summarize and discuss the perspectives. In Appendix one can find a table of integrals needed for calculations of Section IV.

II. TEMPERATURE GREEN FUNCTION TECHNIQUE FOR YANG-MILLS THEORY IN AXIAL GAUGE

Axial gauge for the Yang-Mills theory is qualitatively singled out due to its physical essence. The quantization in such gauge does not use the "ghost" particles and all exact identities are quite simple. In building the non-perturbative methods these simplifications are of great importance and the axial gauge is broadly used at present by many authors. The first and systematical study of the Yang-Mills

theory in the axial gauge at $T = 0$ was given in ref. [11]. However, the developing of the temperature Green function technique is not an evident expansion of the one at $T = 0$. Moreover, it is interesting to build the Green function technique in which $\langle A_\mu(x) \rangle \neq 0$ since the importance of such solutions for the Yang-Mills theory has been proved today. Bearing in mind these facts the general formalism of the temperature Green function technique is given here. We are doing this in the same way as in ref. [11].

Our task is to consider the Yang-Mills theory with $SU(N)$ internal symmetry group. It is proposed to use the Euclidean metric and the axial gauge. The Lagrangian of the theory has the following form

$$\mathcal{L} = -\frac{1}{4}(G_{\mu\nu}^a)^2,$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \quad (2.1)$$

where g stands for the gauge coupling constant and f^{abc} are the structural constants of $SU(N)$ group; the Latin letters $a, b, c = 1, 2, \dots, N^2 - 1$ stand for the colour indices and the Greek ones $\mu, \nu = 1, 2, 3, 4$ are the Lorentz indices. The summation over repeated indices is implied. External sources are introduced as usually [9] after which the generating functional can be written as follows

$$W[J, K] = \int [dA] [dC] \exp(S[...]),$$

$$S[...] = \int_0^{1/T} dx_4 \int d^3x \left[-\frac{1}{4}(G_{\mu\nu}^a)^2 - (n_\mu A_\mu^a(x)) C^a(x) + \right]$$

$$+ K^a(x)C^a(x) + \mathcal{I}_M^a(x)A_M^a(x)], \quad (2.2)$$

where the auxiliary G-fields with their own K-sources are embedded for the preparation of the gauge constraints

$\eta_\mu A_\mu = 0$. Eq.(2.2) is the starting point for what follows.

The action (2.2) needs to be displayed in a more symmetrical form

$$\begin{aligned} S[\dots] = & -\frac{1}{2}A(1)[\tilde{\mathcal{D}}_{A^2}]_0^{-1}(1;2)A(2) + \\ & + \frac{1}{3!}\Gamma_{A^3}^{(0)}(1;2;3)A(1)A(2)A(3) + \\ & + \frac{1}{4!}\Gamma_{A^4}^{(0)}(1;2;3;4)A(1)A(2)A(3)A(4) - \quad (2.3) \\ & - A(1)[\tilde{\mathcal{D}}_{AC}]_0^{-1}(1;2)C(2) + K(1)C(1) + \mathcal{I}(1)A(1), \end{aligned}$$

where "bare" Green function are

$$[\tilde{\mathcal{D}}_{A^2}]_0^{-1}(p)/_{\mu\nu}^{ab} = \delta^{ab}(\delta_{\mu\nu}p^2 - p_\mu p_\nu) \quad (2.4)$$

$$[\tilde{\mathcal{D}}_{AC}]_0^{-1}(p)/_{\mu}^{ab} = \delta^{ab}\eta_\mu$$

and the "bare" vertices have the form

$$\Gamma_{A^3}^{(0)}(p, q, z)/_{\mu\nu\gamma}^{abc} = -igf^{abc}[\delta_{\mu\gamma}(z-p)_\nu +$$

$$\begin{aligned}
 & + \delta_{\mu\nu}(p-q)_\gamma + \delta_{\gamma\nu}(q-z)_\mu], \\
 & \Gamma_{A4}^{(0)}(p; q; z; s) /_{\mu\nu\gamma\rho} = -g^2 \left\{ f^{abc} f^{adf} [\delta_{\mu\gamma} \delta_{\nu\rho} - \right. \\
 & - \delta_{\mu\rho} \delta_{\nu\gamma}] + f^{abd} f^{acf} [\delta_{\mu\nu} \delta_{\gamma\rho} - \delta_{\mu\rho} \delta_{\gamma\nu}] + \\
 & \left. + f^{abf} f^{acd} [\delta_{\mu\nu} \delta_{\rho\gamma} - \delta_{\mu\gamma} \delta_{\rho\nu}] \right\}.
 \end{aligned} \tag{2.5}$$

The exact Green functions are defined as usual

$$\begin{aligned}
 [\mathcal{D}_{A2}]_{\mu\nu}^{ab}(x; y) &= \frac{\delta^2 \ell_n W[\mathcal{J}, K]}{\delta \mathcal{J}_\mu^a(x) \delta \mathcal{J}_\nu^b(y)}, \\
 [\mathcal{D}_{AC}]_{\mu}^{ab}(x; y) &= \frac{\delta^2 \ell_n W[\mathcal{J}, K]}{\delta \mathcal{J}_\mu^a(x) \delta K^b(y)}
 \end{aligned} \tag{2.6}$$

and according to this definition the exact Schwinger-Dyson equations are obtained. Here it is useful to understand that the bare value of (2.6) is not an usual inversion of (2.4). This fact is due to the mixing which occurs between the A and G-fields. This is a specific point of our task and it shows itself in the "unusual" form of the Schwinger-Dyson equations.

The set of functional equations is derived within the stationary action principle. According to (2.3) we obtain the two simple equations

$$[\tilde{\mathcal{D}}_{CA}]_0^{-1}(1; 2) \langle A(2) \rangle = K(1),$$

$$\begin{aligned}
 & [\tilde{\mathcal{D}}_{A^2}]_0^{-1}(1;2) \langle A(2) \rangle - \frac{1}{2} \Gamma_{A^3}^{(0)}(1;2;3) [\tilde{\mathcal{D}}_{A^2}(2;3) + \langle A(2) \rangle \langle A(3) \rangle] - \\
 & - \frac{1}{6} \Gamma_{A^4}^{(0)}(1;2;3;4) \left[\frac{\delta \mathcal{D}_{A^2}(2;3)}{\delta \mathcal{J}(4)} + 3 \mathcal{D}_{A^2}(2;3) \langle A(4) \rangle \right]_{(2.7)} + \\
 & + \langle A(2) \rangle \langle A(3) \rangle \langle A(4) \rangle + [\tilde{\mathcal{D}}_{AC}]_0^{-1}(1;2) \langle C(2) \rangle = \mathcal{J}(1);
 \end{aligned}$$

which define the average fields in the presence of the sources. These equations are basic for what follows but they may be also interesting in themselves when the non-trivial vacuum is to be found. In the absence of sources no assumption is made about the value of $\langle A_m^q(x) \rangle$ and it is not excepted that $\langle A_m^q(x) \rangle \neq 0$. However we believe that $\langle C^q(x) \rangle$ is zero always if the K-source is switched off.

The differentiation of eqs. (2.7) with respect to the external sources gives the unrenormalized equations for the Green functions. Each of eqs. (2.7) is twice differentiated and the expressions obtained are written in the matrix form:

$$\begin{bmatrix} [\tilde{\mathcal{D}}_{A^2}]^{-1} & [\tilde{\mathcal{D}}_{AC}]_0^{-1} \\ [\mathcal{D}_{CA}]_0^{-1} & 0 \end{bmatrix} \begin{bmatrix} [\mathcal{D}_{A^2}] & [\mathcal{D}_{AC}] \\ [\mathcal{D}_{CA}] & 0 \end{bmatrix} = \mathbb{1} \quad (2.8)$$

where the $[\tilde{\mathcal{D}}_{A^2}]^{-1}$ -function is defined as follows

$$\begin{aligned}
 [\tilde{\mathcal{D}}_{A^2}]^{-1}(1;2) &= [\tilde{\mathcal{D}}_{A^2}]_0^{-1}(1;2) - \Gamma_{A^3}^{(0)}(1;2;3) \langle A(3) \rangle_{(2.9)} - \\
 &- \frac{1}{2} \Gamma_{A^4}^{(0)}(1;2;3;4) \langle A(3) \rangle \langle A(4) \rangle + \Gamma_{A^2}(1;2).
 \end{aligned}$$

When obtaining eqs (2.9) we used the natural assumption about the n-leg function

$$\frac{\delta^2 \mathcal{D}_{A^2}(i;j)}{\delta \langle C(1) \rangle \dots \delta \langle C(n) \rangle} \equiv 0 \quad (2.10)$$

which can be verified within the perturbative expansion. By the same reason the Green function $\mathcal{D}_{C^2}(i;j)$ is equal to zero, too.

The self-energy tensor Π_{A^2} must be defined by its integral form. This representation requires that the vertex functions be introduced by inverting the derivative $\delta \mathcal{D}_{A^2}(i;j) / \delta \langle A(l) \rangle$. However we encounter difficulties since the Green function $\mathcal{D}_{A^2}(i;j)$ has no inverse. The inversion is possible only for the matrix Green function (2.8) being performed in a usual way. Nevertheless the simple dependence of the function $\tilde{\mathcal{D}}_{ij}(K, \ell)$ on $\langle A(q) \rangle$ reduces this task to the following prescription

$$-\frac{\delta \mathcal{D}_{A^2}(1;\bar{1})}{\delta \langle A(2) \rangle} = \mathcal{D}_{A^2}(1;3) \frac{\delta \tilde{\mathcal{D}}_{A^2}^{-1}(3;\bar{3})}{\delta \langle A(2) \rangle} \mathcal{D}_{A^2}(\bar{3};\bar{1}) \quad (2.11)$$

and thus all problems are solved. Now the needed expression for the tensor $\Pi_{A^2}(i;j)$ is obtained within simple algebra

$$\begin{aligned} \Pi_{A^2}(1;\bar{1}) &= -\frac{1}{2} \Gamma_{A^4}^{(6)}(1, 2; 3, \bar{1}) \mathcal{D}_{A^2}(2; 3) - \\ &- \frac{1}{2} \Gamma_{A^3}^{(2)}(1, 2; 3) \mathcal{D}_{A^2}(2; \bar{3}) \Gamma_{A^3}(\bar{3}; \bar{2}; \bar{1}) \mathcal{D}_{A^2}(\bar{2}; 3) - \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \Gamma_{A^4}^{(0)}(1;2;3;4) \langle A(2) \rangle \mathcal{D}_{A^2}(4; \bar{3}) \Gamma_{A^3}(3; \bar{2}; \bar{1}) \mathcal{D}_{A^2}(\bar{2}; 3) + \\
 & + \mathcal{D}_{A^2}(2; \bar{4}) \Gamma_{A^3}(4; 5; 6) \mathcal{D}_{A^2}(5; \bar{2}) \Gamma_{A^3}(3; \bar{2}; \bar{1}) \mathcal{D}_{A^2}(5; 4) * \\
 & * \mathcal{D}_{A^2}(6; 3)] - \frac{1}{6} \Gamma_{A^4}^{(0)}(1;2;3;4) \mathcal{D}_{A^2}(2; \bar{2}) * \quad (2.12)
 \end{aligned}$$

$$* \Gamma_{A^4}(4; \bar{3}; \bar{2}; \bar{1}) \mathcal{D}_{A^2}(3; \bar{3}) \mathcal{D}_{A^2}(4; 4) ;$$

where the vertices have the form:

$$\Gamma_{A^3}(1;2;3) = - \frac{\delta \tilde{\mathcal{D}}_{A^2}^{-1}(1;2)}{\delta \langle A(3) \rangle} \Gamma_{A^4}(1;2;3;4) = - \frac{\delta^2 \tilde{\mathcal{D}}_{A^2}^{-1}(1;2)}{\delta \langle A(3) \rangle \delta \langle A(4) \rangle} \quad (2.13)$$

Expression (2.12) thus obtained determines the function $\mathcal{D}_{A^2}$ completely, but one must know else the mixed Green function $\mathcal{D}_{AC}$ which is connected with the definition of $\mathcal{D}_{A^2}$ as it follows from (2.8)

$$[\tilde{\mathcal{D}}_{A^2}]^{-1}(1;2) \mathcal{D}_{A^2}(2; \bar{1}) = I - [\tilde{\mathcal{D}}_{AC}]^{-1}(1;2) \mathcal{D}_{CA}(2; \bar{1}) \quad (2.14)$$

However, the function $\mathcal{D}_{AC}$ is not given by the system (2.8) which results in only one constraint as far as the function $\mathcal{D}_{AC}$ concerned:

$$\eta_{\mu} \mathcal{D}_{AC}(p)_{\mu}^{ab} = \delta^{ab} \quad (2.15)$$

It is more reliable to find the function $\mathcal{D}_{AC}$ within the Slavnov-Taylor identities. Such identities are obtained using the stationary action principle with respect to infinitesimal gauge transformations of the field A and G. The result of

these transformation is written as follows

$$\left\{ \mathcal{D}_\mu^{ab} \left(\frac{\delta}{\delta y} \right) \mathcal{J}_\mu^b + n^\mu \mathcal{D}_\mu^{ab} \left(\frac{\delta}{\delta y} \right) \frac{\delta}{\delta K^b(x)} \right\} W = 0 \quad (2.16)$$

where $\mathcal{D}(\dots)$ is the usual definition of the covariant derivative

$$\mathcal{D}_\mu^{ab}(A) = \partial_\mu \delta^{ab} - g f^{acb} A_\mu^c \quad (2.17)$$

Performing once more differentiation (2.15) with respect to $\mathcal{J}_\nu^d(y)$ before the external source $K=0$ we obtain the definition of the function $\mathcal{D}_{CA}$ to be

$$\mathcal{D}_{CA}(x;y)_\mu^{ab} = (n_\nu \delta_\nu) \mathcal{D}_\mu^{ab}(\langle A \rangle) \delta(x-y) \quad (2.18)$$

Here we assumed in accordance with (2.10) that

$$\frac{\delta^3 W[\mathcal{J}; K]}{\delta \mathcal{J}(z) \delta K(j) \delta \mathcal{J}(k)} = 0 \quad (2.19)$$

since this function is absent from the perturbative expansion.

So, the temperature Green functions technique is presented now completely with only the regularization left to be worked out in a usual way. The above equations are correct for any value of $\langle A(z) \rangle$ and not only for $\langle A(z) \rangle = 0$ this point being our main achievement. The needed Slavnov-Taylor identities for the case $\langle A(z) \rangle \neq 0$ have the following form

$$\begin{aligned} \mathcal{D}_\mu^{ab}(\langle A \rangle) [\tilde{\mathcal{D}}_{A^2}^{-1}]_{\mu\nu}^{bc}(x,y) &= 0 \quad ; \\ \mathcal{D}_\mu^{ab}(\langle A \rangle) \Gamma_{A^3}^{bdf}(x,y,z)_{\mu p \theta} &= \end{aligned}$$

$$\begin{aligned}
 &= g f^{a b c} \left\{ \delta(x-z) \delta^{f s} [\tilde{D}_{A2}^{-1}]^{b d} \rho_{\rho}(x; y) + \delta(x-y) \delta^{s d} [\tilde{D}_{A2}^{-1}]^{b f} \rho_{\rho}(x; z) \right\}, \\
 &\quad \mathcal{D}_M^{a b} \langle A \rangle \Gamma_A(x; y; u; v)_{\rho \theta \tau}^{b d c f} = \quad (2.20) \\
 &= -g f^{a b c} \left\{ \delta^{c s} \delta(x-z) \Gamma_{A3}(x; y; v)_{\rho \theta \tau}^{b d f} + \right. \\
 &\quad \left. + \delta^{s d} \delta(x-y) \Gamma_{A3}(x; u; v)_{\rho \theta \tau}^{b c f} + \delta^{s f} \delta(u-x) \Gamma_{A3}(x; y; u)_{\tau \rho \theta}^{b d c} \right\},
 \end{aligned}$$

which is obtained within a simple algebra. Thus for developing the nonperturbative approach all expressions are ready and we can now explain our main ideas about solving the infrared problem of QCD.

III. SYSTEM OF SELF-CONSISTENT EQUATIONS FOR THE GLUON PROPAGATOR

In studying the infrared behavior in the four-dimensional Yang-Mills theory at $T = 0$ several authors [12] are developing a non-perturbative procedure based on the use of the Schwinger-Dyson equations for the gluon propagator and the Slavnov-Taylor identities for the proper vertices Γ_{A3} and Γ_{A4} . These identities allow to express the vertices Γ_{A3} and Γ_{A4} in terms of the "inverse" gluon propagator and in this way the self-consistent equations for the polarization tensor have been obtained. In realizing this program the progress was achieved [12] after the tensor structure of the "inverse" gluon propagator had been chosen the same as that of the bare one. This approach wherein the problem is reduced to determination of one scalar function solves the problem and

thus the $1/q^4$ - low-momentum behavior of the gluon propagator at $T = 0$ has been obtained. However, the reliability of such approach is not understandable since the gluon propagator structure in the one-loop approximation is quite different from the considered one.

In the present paper we propose a new non-perturbative scheme of studying the QCD infrared problem at $T \neq 0$. Our starting point is the three-dimensional Yang-Mills theory with $\tilde{g}^2 = g^2 T$ and the Slavnov-Taylor identities for axial gauge within the temperature technique. Only the case with $\langle A(i) \rangle = 0$ is considered. All the progress achieved in the previous investigations [12, 13] has been used together with the more accurate approximation of the gluon propagator structure. Our final result is two self-consistent equations for two scalar functions Π_1 and Π_2 which determine the gluon polarization tensor at $T \neq 0$ for three-dimensional QCD in axial gauge. We used the polarization tensor structure as follows

$$\begin{aligned} \Pi_{ij}(p) = & (p^2 \delta_{ij} - p_i p_j) \Pi_1(p) + \\ & + (p_n)^2 \left(\delta_{ij} - \frac{p_i n_j + p_j n_i}{(p_n)} + \frac{n_i n_j}{(p_n)^2} p^2 \right) \Pi_2(p) \end{aligned} \quad (3.1)$$

which is the exact tensor structure of $\Pi_{ij}(p)$ since the transversality of $\Pi_{ij}(p)$ with respect to momentum takes place due to the Slavnov-Taylor identities. According to (3.1) the expression for the gluon propagator can be found also within two structures by simple algebra:

$$D_{ij}(p) = \frac{1}{p^2 + [p^2 \Gamma_1(p) + p n^2 \Gamma_2(p)]} \left[\left(\delta_{ij} - \frac{n_i n_j}{n^2} \right) + \right. \\ \left. + \frac{1 + \Gamma_1(p)}{1 + [\Gamma_1(p) + n^2 \Gamma_2(p)]} \left(\frac{n_i n_j}{n^2} - \frac{n_i p_j + n_j p_i}{(np)} + \frac{p_i p_j}{(pn)^2 n^2} \right) \right], \quad (3.2)$$

where the facts that $n_i D_{ij} = 0$ and $\langle A(p) \rangle = 0$ were taken into account.

Let us now consider the Slavnov-Taylor identities in order to find the vertex function Γ_{A3} in term of Γ . Such identities when a vacuum with $\langle A(p) \rangle = 0$ is treated have an usual form

$$p_i [\tilde{D}^{-1}_{A2}]_{ij}(p) = 0, \\ z_i \Gamma_{A3}(z; p; q)_{ijk} = i \tilde{g} f^{abf} \{ [\tilde{D}^{-1}_{A2}]_{jk}(p) - \\ - [\tilde{D}^{-1}_{A2}]_{jk}(q) \}; \quad z_i \Gamma_{A4}(z; p; q; t)_{ijkl} = i \tilde{g} f^{abf} * \\ * \Gamma_{A3}(t; p; -t-p)_{ljk}^{fdb} + f^{adb} \Gamma_{A3}(t; q; -t-q)_{ljk}^{fcb} + f^{afh} \Gamma_{A3}(p; q; -p-q)_{ljk}^{deb},$$

expressing the longitudinal parts of higher Green functions (in our case Γ_{A3} and Γ_{A4}) over lower ones. The transversal parts of the functions Γ_{A3} and Γ_{A4} are completely free from constraints that is a serious difficulty of this approach. However some reasons exist that transversal parts of the higher Green function being less singular do not give the essential contribution to the problem under consideration. The longitudinal parts of the function Γ_{A3} is built with aid of (3.3) as this made by many authors [12, 13]. Nevertheless we found here a more exact form of Γ_{A3} where addi-

tional tensor structures of the function Γ_{A^3} are taken into account. This tensor structure of Γ_{A^3} is more correct and gives possibility to find an agreement with the tensor structure of Π_{ij} adopted in form (3.1). Our function Γ_{A^3} is built as follows

$$\begin{aligned} \Gamma_{A^3}(p, q, z)_{ijk}^{abc} = & -i\tilde{g}^2 f^{abc} \{ \delta_{ij} [\Pi_1(p) p_k - \Pi_1(q) q_k] - \\ & - \frac{\Pi_1(p) - \Pi_1(q)}{p^2 - q^2} [(pq) \delta_{ij} - q_i p_j] (p - q)_k - \frac{1}{3} n_i n_j n_k * \\ & * (\frac{(p^2)}{(qn)} - \frac{(q^2)}{(zn)}) \Pi_2(p) - \delta_{ij} n_k (\frac{(qn)^2}{(pn)} \Pi_2(q) - \frac{(zn)^2}{(pn)} \Pi_2(p)) + \\ & + n_i n_j (\frac{(pn)}{(zn)} \Pi_2(p) + \frac{(qn)}{(zn)} \Pi_2(q)) (q - p)_k + \delta_{ij} (p - q)_k + \\ & + \text{cycl. perm} \} \end{aligned} \quad (3.4)$$

and satisfies the Slavnov-Taylor identities (3.3). The part of this function which is not depend on the vector n_i is the same as in Ref. [12, 13]. The expression (3.4) can be obtained if one proposes for Γ_{A^3} the tensor structure with five auxiliary function

$$\begin{aligned} \Gamma_{A^3}(p, q, z)_{ijk}^{abc} = & -i\tilde{g}^2 f^{abc} \{ [A(p, q) p_k - \\ & - A(q, p) q_k] \delta_{ij} + [B(p, q) p_k - B(q, p) q_k] q_i p_j + \end{aligned} \quad (3.5)$$

$$+ \frac{1}{3} N(\dots/n) n_i n_j n_k + R(\dots/n) \delta_{ij} n_k + \\ + [T(p \dots/n) p_k - T(q \dots/n) q_k] n_i n_j + \text{cycl. perm.} \}$$

and solves the Slavnov-Taylor identities (3.2) by simple algebra. The tensor structure of Γ_{A^3} -function as (3.5) has many reasons to be exact in the infrared limit since only five auxiliary functions it is possible to define from (3.3) in a unique way under condition that the low momentum asymptotic of the transversal part of the vertex Γ_{A^3} is not essential. The last conclusion about the transversal part of the vertex- Γ_{A^3} is the main assumption of our approach.

Equations for the functions Π_1 and Π_2 are obtained after the Schwinger-Dyson equation for the "inverse" gluon propagator has been solved. According to (2.18) the self-energy tensor $\Pi(q)$ is represented by the four diagrams of the usual structure

$$\Pi = \frac{1}{2} \text{Diagram 1} + \frac{1}{2} \text{Diagram 2} + \frac{1}{6} \text{Diagram 3} + \frac{1}{2} \text{Diagram 4} \quad (3.6)$$

where the exact gluon propagator (3.2) and the vertex function Γ_{A^3} (3.4) should be used in calculations. After all diagrams are expressed in terms of Π_1 and Π_2 the self-consistent equations for these functions arise as a result. The multiplication of (3.6) by $n_i n_j$ leads to the first equations

$$\begin{aligned}
 & [n^2 p^2 - (np)^2] [\Gamma_1(p) + n^2 \Gamma_2(p)] = \\
 & = -\tilde{g}^2 N \int \frac{d^3 q}{(2\pi)^3} \left\{ (n/q - z) n q \mathcal{D}_{ij}(q) \mathcal{D}_{kl}(z) [\delta_{ij}(1 + \right. \\
 & + \Gamma_1(q) - n^2 \frac{(qn)}{(pn)} \Gamma_2(q)] + \frac{\Gamma_1(z) - \Gamma_1(q)}{z^2 - q^2} (z_i q_j - \\
 & - \delta_{ij}(zq)) + p_i (p - q)_j \frac{\Gamma_1(q) - \Gamma_1(p)}{q^2 - p^2}] - n^2 \mathcal{D}_{kk}(q) \left. \right\};
 \end{aligned}
 \tag{3.7}$$

which is obtained without any additional assumptions. The last two diagrams in (3.6) give no contribution to equation (3.7) due to their transversality with the respect to Π_i .

Contracting the Lorentz indices in (3.6) we obtain the second equation

$$\begin{aligned}
 & 2p^2 (\Gamma_1(p) + \Gamma_2(p)) (p^2 n^2 + (pn)^2) = -\tilde{g}^2 N \int \frac{d^3 q}{(2\pi)^3} \left\{ [(z-p)_i + \right. \\
 & + \mathcal{D}_{ij}(q) \mathcal{D}_{kl}(z) + (p-q)_i \mathcal{D}_{ik}(z) \mathcal{D}_{je}(q) + \\
 & + \mathcal{D}_{je}(q) \mathcal{D}_{ik}(z) (q-z)_e] \frac{\Gamma_1(q) - \Gamma_1(z)}{q^2 - z^2} (z_j q_k - \\
 & - (zq) \delta_{jk}) q_e + \frac{\Gamma_1(p) - \Gamma_1(q)}{p^2 - q^2} (q_e p_j - \delta_{ej}(pq)) (p-q)_k + \\
 & + (1 + \Gamma_1(q)) (\delta_{jk} q_e - \delta_{ej} q_k) + (1 + \Gamma_1(p)) \delta_{ej} p_k \left. \right\} =
 \end{aligned}
 \tag{3.8}$$

$$- \mathcal{D}_{ji}(q) \mathcal{D}_{ij}(z) \frac{(n/2-2)(n/2)^2}{(p_n)} \Pi_2(q) - 2 \mathcal{D}_{ii}(q) \} ;$$

where the contributions of the last two diagrams in (3.6) are omitted.

It is no problem to take the omitted diagrams into account, but it leads to nonprincipal complications.

The equations (3.7) and (3.8) form a self-consistent system for the functions Π_1 and Π_2 in the infrared momentum domain and give a possibility of nonperturbative treating of QCD at finite temperature. On this way we should like to avoid a difficulties associated with the usual perturbation expansion. The non-perturbative solution of these equations in the infrared momentum domain is a great importance but a separate paper is needed for this aim. Here we confine ourselves of solving these equations in the lowest approximation.

IV. ONE-LOOP APPROXIMATION FOR THE POLARIZATION TENSOR IN AXIAL GAUGE

Let us consider the first iteration for the Π_1 and Π_2 within the equations (3.7) and (3.8). This iteration as we will see coincides with the one-loop approximation for Π in the axial gauge.

In the zero-order approximation the expression for $\tilde{\mathcal{D}}_{ij}^{-1}(p)$ is the following:

$$\tilde{\mathcal{D}}_{ij}^{-1}(p) = \delta_{ij} p^2 - p_i p_j \quad (4.1)$$

for which we have

$$\Pi_1^{(0)} = \Pi_2^{(0)} = 0 \quad (4.2)$$

With this values for Π_1 and Π_2 the gluon propagator coincides with the bare one

$$\Phi_{ij}^{(0)}(p) = \frac{1}{p^2} \left[\delta_{ij} - \frac{n_i p_j + n_j p_i}{(np)} + \frac{p_i p_j}{(np)^2} n^2 \right] \quad (4.3)$$

and the vertex Γ_A given by (3.4) reduces to $\Gamma_A^{(0)}$. It means that the substitution of (4.2) in the right-hand side of (3.7) and (3.8) leads to expression which is equivalent to the one-loop approximation for the polarization tensor

$$\Pi^{(2)} = \frac{1}{2} \text{ (loop diagram) } + \frac{1}{2} \text{ (self-energy diagram) } \quad (4.4)$$

Equations (3.7) and (3.8) with the help of (4.2) are simplified. For $\Pi_1^{(2)}$ and $\Pi_2^{(2)}$ we obtain the equations with known right-hand sides

$$\begin{aligned} [n^2 p^2 - (np)^2] [\Pi_1^{(2)}(p) + n^2 \Pi_2^{(2)}(p)] = \\ = -\tilde{g}^2 N \int \frac{d^3 q}{(2\pi)^3} \frac{1}{q^2 (q+p)^2} \{ 2(nq)^2 + (nq)(np) - \\ - 4n^2(pq) + \frac{n^2}{(nq)^2} [q^2(np)^2 + n^2(q/p+q)^2] + \\ + \frac{2n^2(np)}{(nq)} [q^2 - (qp)] + \frac{n^4 (q/p+q)^2}{(nq)(np+q)} \} ; \end{aligned} \quad (4.5)$$

$$\begin{aligned} \text{and } 2p^2 \Pi_1^{(2)}(p) + \Pi_2^{(2)}(p) [n^2 p^2 + (np)^2] = -\tilde{g}^2 N \int \frac{d^3 q}{(2\pi)^3} * \\ * \frac{1}{q^2 (p+q)^2} \{ 4p^2 + q^2 + (qp) - \frac{4(np)}{(nq)} [q^2 + 2(qp)] + \\ + \frac{n^2}{(qn)^2} [4(qq)^2 + q^4 + 9(pq)q^2 + 4p^2 q^2] - \frac{n^2}{(nq)(n/p+q)} * \end{aligned}$$

$$\begin{aligned}
 & * [3q^2 p^2 - 2q^4 + 3q^2(pq) + 2(pq)^2] - \frac{n^2(np)}{(p+q/n)^2(nq)} * \\
 & * [q^4 + q^2 p^2 + 7q^2(pq) + 10(pq)^2 + 5(pq)p^2] - \\
 & - \frac{n^4}{(p+q/n)^2(qn)} \left[(p+q/q)^2 (2p^2 + 3(pq)) + \right. \\
 & \left. + (p+q)^2 q^2 (pq) \right] \} ;
 \end{aligned}
 \tag{4.6}$$

which can be solved by simple algebra. In (4.5) and (4.6) we omitted the contribution of the last diagrams in (4.4) because it vanishes identically when the angular integration is performed. All poles of the form $(qn)^{-m}$ which appear after substituting (4.2) must be treated in the principal value sense [11]

$$\frac{1}{(nq)^m} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \left[\frac{1}{(nq + i\varepsilon)^m} + \frac{1}{(nq - i\varepsilon)^m} \right] \tag{4.7}$$

Under this condition all the integrals in (4.5) and (4.6) are finite in the infrared region. The only arising divergencies are the ultraviolet ones, and they are regulated by the introducing an ultraviolet cut-off. After the renormalization procedure is performed all the ultraviolet divergencies disappear as usually. On the other hand it is possible to use the dimensional regularization where such power divergencies is absent at all.

Performing the three-dimensional integration we obtain the following expressions for (4.5)

$$\begin{aligned}
 & [n^2 p^2 - (np)^2] [\Pi_1^{(2)}(p) + n^2 \Pi_2^{(2)}(p)] = \\
 & = \frac{\tilde{g}^2 N}{4} \left[\frac{1}{p} \frac{np}{|n|} - \frac{1}{8} \frac{(np)^2}{|p|} - \frac{7}{8} \frac{1}{|p|} n^2 \right]; \quad (4.8)
 \end{aligned}$$

and (4.6)

$$\begin{aligned}
 & 2p^2 \Pi_1^{(2)}(p) + \Pi_2^{(2)}(p) [p^2 n^2 + (pn)^2] = \\
 & = \frac{\tilde{g}^2 N}{4} \left[\frac{1}{|n|} - \frac{7}{4} \frac{1}{|p|} \right]. \quad (4.9)
 \end{aligned}$$

Solving (4.8) and (4.9) the simple expressions for $\Pi_1^{(2)}$ and $\Pi_2^{(2)}$ are found

$$\Pi_1^{(2)}(p) = - \frac{\tilde{g}^2 N}{32|p|} \frac{7 + 14d - d^2}{(1+d)^2}, \quad (4.10)$$

$$\Pi_2^{(2)}(p) = \frac{\tilde{g}^2 N}{4|p|n^2} \frac{d}{(1+d)^2}, \quad (4.11)$$

where $d = |pn|/|p||n|$ is the gauge fixing parameter in the gauges under consideration ($0 \leq d \leq 1$).

The substitution of $\Pi_1^{(2)}(p)$, $\Pi_2^{(2)}(p)$ into (3.2) gives the gluon propagator in the one-loop approximation

$$\begin{aligned}
 D_{ij}(p) = & \frac{1}{p^2 - \frac{\tilde{g}^2 N |p| T}{32} \frac{(7+7d-8d^2)}{1+d}} \left\{ (\delta_{ij} - \frac{n_i n_j}{n^2}) + \right. \\
 & + \frac{p^2 + \frac{\tilde{g}^2 N |p| T}{32} \frac{(d^2 - 14d - 7)}{(1+d)^2}}{p^2 - \frac{\tilde{g}^2 N |p| T}{32} \frac{(7-d)}{1+d}} \left. \right\} \quad (4.12)
 \end{aligned}$$

$$* \left(\frac{n_i n_j}{n^2} - \frac{n_i p_j + n_j p_i}{(p n)^2} + \frac{p_i p_j n^2}{(p n)^2} \right) \Bigg\} ,$$

which has two tensor structures rather than one structure as the bare propagator. In (4.12) we used for $\tilde{g}^2$ its value $g^2 T$ to demonstrate the dependence of the gluon propagator on temperature.

The main feature of the propagator (4.12) consists in the presence of two poles for momenta $p \sim g^2 T$. These singularities cannot be eliminated by a suitable choice of the gauge, i.e. for all values of α ($0 \leq \alpha \leq 1$) the propagator $D_{ij}(p)$ has a singular behavior for $p \sim g^2 T$. For the particular choice of the gauge $p \perp n$ ($\alpha \rightarrow 0$) the expression (4.12) reduces to a more simple one

$$D_{ij}(\alpha \rightarrow 0) = \frac{\delta_{ij} - \frac{n_i p_j + n_j p_i}{(p n)^2} + \frac{p_i p_j n^2}{(p n)^2}}{p^2 - \frac{7g^2 N/4 T}{32}} ; \quad (4.13)$$

where the singularity coincides exactly with the one found in the transversal gluon propagator in the Feynman gauge [6]. Similarly, at $p \parallel n$ ($\alpha = 1$) the expression (4.12) has the only pole at $p = \frac{3g^2 N T}{32}$.

The singularities found here as well as in Refs. [6, 7] indicate the possible instability of the vacuum state with $\langle A \rangle = 0$ in the Yang-Mills theory at $T \neq 0$. This fact follows from the analysis of the effective action Γ believing that the minimum of Γ defines the ground state of the theory. The state with $\langle A \rangle = 0$ is a solution of

the equation $\delta\Gamma/\delta A = 0$, so the expansion of Γ in the vicinity of $\langle A \rangle = 0$ is a form

$$\Gamma \approx \frac{1}{2} \langle A_i(p) \rangle [\tilde{D}_0^{-1}]_{ij}(p) \langle A_j(-p) \rangle \quad (4.14)$$

In order to the vacuum state with $\langle A \rangle = 0$ was stable the expression (4.14) should be positive-definite. But the expressions (4.10) and (4.11) indicate, that (4.14) takes negative values for gauge fields which are nonzero only at $p \lesssim g^2 T$. That is why the vacuum state with $\langle A \rangle = 0$ is not even a local minimum of Γ thus being unstable. Such a situation is similar to the instability of QCD at nonzero magnetic field and at $T = 0$ [14].

V. SUMMARY

The complexity of the infrared problem in QCD at $T \neq 0$ requires the use of nonperturbative methods for its investigation since the perturbative expansion loses its validity at the momentum $p \sim g^2 T$. To develop non-perturbative methods the infrared problem of the four-dimensional QCD at $T \neq 0$ was reduced to studying the three-dimensional Yang-Mills theory and advantage of the axial gauge was exploited. In our formalism the two scalar functions $\Pi_1(p)$ and $\Pi_2(p)$ which define the gluon polarization tensor describe the infrared behavior of the theory. For the infrared momentum region both the gluon propagator and the three-

gluon vertex Γ_A^3 were expressed in terms of $\Pi_1(p)$ and $\Pi_2(p)$. For these functions self-consistent system of equations was obtained with the use the Schwinger-Dyson equations and Slavnov-Taylor identities. This system of equations determines the infrared behavior of Π_1 and Π_2 and thus of the QCD at $T \neq 0$. In setting up these equations we have used only exact relations, so that they constitute a solid basis to obtain the interesting non-perturbative information.

The first order perturbative solution of these equations was found which is the same as the first term of the ordinary perturbative expansion. The solution obtained exhibits poles in the gluon propagator (4.12) at $p = g^2 T$. These singularities depend on the gauge-fixing parameters but it is impossible to remove them by any choice of the gauge. Taking into account the presence of such singularities in the covariant gauge [5] we conclude that the singular behavior of gluon propagator is an essential feature of the QCD within the perturbative expansion. Such "fictitious" poles of the gluon propagator indicate that the infrared instability of QCD at $T \neq 0$ is likely to take place. The effective action Γ have been investigated in order to determine the stability of the vacuum state of the theory. It is found that a vacuum state with $\langle A \rangle \neq 0$ may supply us with additional parameters for eliminating the infrared singularities. With this possibility in mind, the Schwinger-Dyson equation and Slavnov-Taylor identities were obtained for the vacuum with arbitrary $\langle A \rangle$. These

equations are relevant for examination of the consequences which a nonzero vacuum may have for the infrared problem.

To summarize, we conclude that the non-perturbative effects can change drastically the singular behavior of the gluon propagator found in the first orders of the perturbation expansion. A nonperturbative solution of the equations for the gluon polarization tensor must be the next important step in order to clarify the infrared structure of the QCD.

The authors wish to express their gratitude to Prof. E.S.Fradkin for valuable discussions during the course of this work.

APPENDIX

All the n -dependent integrals may be expressed as combinations of the following integrals and of their derivatives with respect to p_i and n_i :

$$I_1 = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{q^2(qn)(n/p+q)} = -\frac{1}{8pn_1/n_1}$$

$$I_2 = \int \frac{d^3 q}{(2\pi)^3} \frac{q_i}{q^2(qn)(n/p+q)} = \frac{n_i}{8/n_1^3} \mathcal{E}\left(\frac{p_i}{p_1/n_1}\right)$$

$$I_3 = \int \frac{d^3 q}{(2\pi)^3} \frac{(pq)}{q^2(p+q)^2(qn)} = \frac{1}{8/n_1} \mathcal{E}\left(\frac{p/n}{p_1/n_1}\right)$$

$$I_4 = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(qn)(n/p+q)} = -\frac{1}{2\pi^2 n^2}$$

$$I_5 = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{q^2(p+q)^2(qn)(n/p+q)} = \frac{-1}{4p^2/pn_1/n_1}$$

$$I_6 = \int \frac{d^3 q}{(2\pi)^3} \frac{q_i}{(p+q/n)^2(qn)^2} = 0$$

$$I_7 = \int \frac{d^3 q}{(2\pi)^3} \frac{(q+p/q)^2}{q^2(p+q)^2(qn)^2(q+p/n)^2} = 0$$

where Λ is an ultraviolet cut-off and $\mathcal{E}(x)$ is the sign-function.

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Т - 31053

Подписано в печать 29 декабря 1981 года

Заказ № 176. Тираж 100 экз. п/л 1,4

Отпечатано в Редакционно-издательском
отделе ФИАН СССР

Москва, В-312, Ленинский проспект, 53