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BEHAVIOUR OF GLUONS AT HIGH TEMPERATURE

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ABSTRACT

The screening of colour electric and magnetic fields and plasma oscillations in a high temperature gluon plasma are investigated using linear response theory and self-consistent non-perturbative solutions to the Schwinger-Dyson equation. Both static electric and static magnetic fields are screened, with an electric mass-squared equal to $g^2 T^2$ minus a correction of order $g^3 T^2$, and with a magnetic mass-squared of order $g^3 T^2$. A temperature- and process-dependent running coupling constant is defined. The gauge choice is discussed in detail and it is shown that a convenient gauge for the problem is the physical ghostfree temporal axial gauge $A_0^a = 0$.

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1. INTRODUCTION

In the last few years a lot of attention has been paid to the theory of quantum chromodynamics at finite temperature. (For reviews see refs. [1, 2]; see also ref. [3].) The reasons for this are numerous. On the theoretical side one would like to understand the properties of confinement better by raising the temperature of the system. This sort of study makes a direct contact between elementary particle physics and statistical and conventional many-body physics. At low temperature one does not see the elementary particles of the real world, the quarks and gluons; rather one sees only their collective excitations manifested as hadrons. At high temperature one ought to be able to see the quarks and gluons directly since they ought to be deconfined due to asymptotic freedom. On the observational side we are quite sure that temperatures in the early universe were sufficiently high that such quark-gluon matter was formed. It is also quite probable that such matter will be formed in the laboratory during high energy collisions between heavy nuclei. In these latter cases the goal is to find some measurable consequences.

The picture which is emerging from a number of studies [4-7] is that a phase transition occurs at a temperature on the order of 200 MeV. The low temperature phase is characterized by a gas of hadronic resonances extending very high in mass. These may be thought of as islands of confined quarks and gluons in a confining sea of non-perturbative vacuum. The high temperature phase is characterized by a gas of asymptotically free quarks and gluons moving in the perturbative vacuum.

In the high temperature phase one would naively expect that the thermodynamic potential had a series expansion in powers of g^2 , the square of the coupling constant. This is not true. The first instance where weak coupling but non-perturbative effects arise is the well-known plasmon effect. If one looks at the set of fourth order diagrams one finds that a subset of them are infrared divergent. These diagrams come from the perturbative expansion of a particular term in the general formal expression for the thermodynamic potential. It has the form

$$\frac{1}{2} \text{Tr} \left\{ \ln [1 + \mathcal{D}_0(k) \Pi(k)] - \mathcal{D}_0(k) \Pi(k) \right\}, \quad (1.1)$$

where $\mathcal{D}_0(k)$ is the bare gluon propagator and $\Pi(k)$ is the gluon self-energy to lowest order. This expression can be expanded diagrammatically as indicated in fig. 1. Each term in the series is more infrared divergent than the preceding one. Therefore one must sum them all to recover eq. (1.1), and then evaluate the expression directly. It leads to an infrared finite contribution of order g^3 . The non-perturbative result comes about because in the plasma the gluons acquire an electric screening mass of order gT . Since this screening eliminates the infrared divergence in the above diagrams one is bound to obtain a non-analytic (in g^2) result for Ω . Overall then the thermodynamic potential takes the form [5]

$$\begin{aligned} \Omega = & -\frac{\pi^2}{45} \left(N_g + \frac{7}{4} N_c N_f \right) T^4 + \frac{g^2}{144} N_g \left(N_c + \frac{5}{4} N_f \right) T^4 \\ & - \frac{g^3}{12\pi} N_g \left(\frac{1}{3} N_c + \frac{1}{6} N_f \right)^{3/2} T^4 + \dots \end{aligned} \quad (1.2)$$

Here the gauge group is $SU(N_c)$, $N_g = N_c^2 - 1$, and N_f is the number of flavors of massless quarks. The renormalization group can be used to improve eq. (1.2) by implementing the running coupling

$$g^2 \rightarrow \bar{g}^2 = \frac{24\pi^2}{(11N_c - 2N_f) \ln(T/\Lambda)} \quad (1.3)$$

The scale Λ depends on the renormalization scheme, etc., and cannot be precisely specified without the order g^4 corrections to eq. (1.2).

However it was later shown that new infrared effects may appear at the order g^6 in the thermodynamic potential [8]. If one calculates the n -loop contribution to the thermodynamic potential then there ought to be a contribution of the form

$$g^6 T^4 \left[\frac{g^2 T}{m(T)} \right]^{n-3}, \quad (1.4)$$

where $m(T)$ is some infrared cut-off. The electric, or longitudinal, cut-off is of order gT so that eq. (1.4) would give contributions of order g^6, g^7, g^8 etc. as n increases. To lowest order there is no magnetic, or transverse, cut-off on the gluon propagator. Since the gluon self-energy possesses a naive expansion in powers of g^2 we would expect that $m^2(T)$ in that case would be of order $g^4 T^2$. That means that one would get a contribution to Ω of order $g^6 T^4$ for every n greater than 2. In effect we would lose perturbative computability. If $m(T)$ was strictly zero then, unless unexpected cancellations arise, Ω would be ill-defined.

It was shown that one cannot compute the static infrared limit of the gluon propagator using perturbation theory [8]. It was shown that the inverse propagator goes like

$$\mathcal{D}^{-1}(0, k) \sim k^2 + a_1 g^2 T k + a_2 g^4 T^2 + a_3 \frac{g^6 T^3}{k} + \dots \quad (1.5)$$

This means that the perturbative expansion breaks down when $k_0 = 0$ and $k \leq g^2 T$. Therefore, not only is the magnetic screening mass indeterminate in perturbation theory, but probably also the electric screening mass. We note that these problems would have some serious consequences for the early universe studies based on $SU(N)$ grand unified theories, and possibly also other groups.

In a recent letter [9] we pointed out that these weak coupling difficulties are cured by non-perturbative effects similar to the plasmon effect. The magnetic mass-squared is of order $g^3 T^2$, and the electric mass-squared gets corrections of the same order. This is why the perturbative expansion breaks down at momenta $\leq g^2 T$. Furthermore the problem posed by eq. (1.4) is solved. It is the purpose

of this paper to provide a detailed analysis of the resolution of the problems and the implications.

We have found it to be most convenient to work in the temporal axial gauge where $A_0^a = 0$. There are several reasons for this. One reason is that the tensorial structure of the exact propagator and vertices is precisely the same as at zero temperature. This is not the case with other axial gauges or with the covariant gauges, for example. Another reason is that this gauge is physical in the sense that it has the correct number of polarization degrees of freedom and is ghost free. It also admits a Hamiltonian formulation which we found to be of great utility at finite temperature. The partition function is

$$Z = \text{Tr}' \exp(-H/T), \quad (1.6)$$

which is gauge invariant [10]. The prime on the trace indicates that it is to be carried out only over those physical states satisfying Gauss' law. Any observable in the theory may be calculated from the set of correlation functions

$$Z^{-1} \text{Tr} \left\{ A_i^a(\vec{x}_1, t_1) A_j^b(\vec{x}_2, t_2) \cdots \exp(-H/T) \right\}, \quad (1.7)$$

even though these correlation functions in themselves may be gauge dependent. We will only study a pure SU(N) gauge theory since even massless quarks should not influence the infrared behaviour of the gluon propagator.

It is sometimes stated that the existence of the gauge invariant operator $\text{Tr} U(1/T, \vec{x})$ where, in matrix notation,

$$U(\tau, \vec{x}) = P \exp \left\{ ig \int_0^\tau d\tau' A_0(\tau', \vec{x}) \right\}, \quad (1.8)$$

forbids the use of the temporal axial gauge at non-zero temperature. This is because in that particular gauge apparently $\text{Tr } U(1/T, \vec{x}) = 1$ whereas in any other gauge it need not equal one but should result in a gauge independent answer to a dynamical calculation. This statement is rather surprising in view of our previous comments about the temporal axial gauge, and especially since $U(\tau, \vec{x})$ is just the gauge transformation which takes one from a gauge in which $A_0 \neq 0$ to one with $A_0 = 0$. The resolution of this paradox is not so trivial so we feel obliged to comment on it here.

One cannot naively write a path-integral formula for eq. (1.6) because the sum over states is restricted. To remove this restriction one inserts a projection operator which involves the integration over a Lagrange multiplier field $\Lambda(\vec{x})$ [2]. One then deduces a path-integral formula for Z which has $A_0 = 0$ but which is over A_i fields that are periodic in τ of period $1/T$ only up to a twist, i.e. a time independent gauge transformation involving $\Lambda(\vec{x})$.

$$A_i(1/T, \vec{x}) = A_i^{\wedge}(0, \vec{x}). \quad (1.9)$$

The operator $\text{Tr } U(1/T, \vec{x})$ is only gauge invariant if the fields are strictly periodic. In the $A_0 = 0$ gauge the fields are not strictly periodic, hence the above operator is only gauge invariant in a restricted class of gauges. We note that the residual time independent gauge freedom in the $A_0 = 0$ gauge manifests itself in frequency space as the well-known principal value prescription, to be used later.

Our principal tool for evaluating the gluon propagator in the infrared limit at high temperature will be a self-consistent one-loop approximation to the full Schwinger-Dyson equation. We would like to point out to the reader already at this time that such an approximation does not violate any of the symmetries of the Lagrangian. Indeed such an approach is consistent with the Goldstone Theorem in theories which exhibit spontaneous symmetry breaking at finite temperature whereas other approximations such as the mean field are not [11].

The plan of the paper is as follows. In section 2 we discuss at some length the choice of gauge at finite temperature. In section 3 we calculate the gluon propagator to lowest order in perturbation theory in the temporal axial gauge. This includes both the temperature part and the vacuum part as a check with previous authors. In section 4 we perform a non-perturbative weak coupling analysis of the Schwinger-Dyson equation to extract the leading infrared behaviour of the gluon propagator at high temperature. In section 5 we employ linear response theory to study rigorously the screening of static color electric and magnetic fields using the infrared and ultraviolet limits calculated in earlier sections. We also study the plasma oscillations in this physical gauge. In section 6 we give a brief summary and mention possibilities for future work.

2. GAUGE CHOICE

The basic quantity we shall discuss in this paper is the gluon propagator

$$\begin{aligned} i \mathcal{D}_{\mu\nu}^{ab}(x) &= \langle T(A_\mu^a(x) A_\nu^b(0)) \rangle \\ &= Z^{-1} \text{Tr} \left[\exp(-\beta H) T(A_\mu^a(x) A_\nu^b(0)) \right] \quad (2.1) \end{aligned}$$

in a gluon plasma in thermal equilibrium at $T \gg T_c \approx \Lambda_{\text{QCD}}$. Although physical results should be independent of gauge, $\mathcal{D}^{ab}$ is not, and calculations which are simple in one gauge may be complicated in another. Thus it is important to choose the gauge properly. For the purely perturbative calculation of the thermodynamic potential of gluon plasma [5] the Feynman gauge is simplest. For the partly non-perturbative self-consistent calculation of the electric and magnetic screening lengths carried out below it seems that the temporal axial gauge [12-17]

$$A_0^a(x) = 0 ; \quad a = 1, \dots, N \quad (2.2)$$

is the most convenient. We shall now analyse the gluon propagator in this gauge and point out its advantages.

Let us first recall the basic equations of the Hamiltonian formalism in the temporal axial gauge. The independent fields are A_i^a and their conjugate momenta are $\pi_i^a = \partial A_i^a / \partial t$. The electric and magnetic fields and the Hamiltonian density are

$$E_i^a = -\pi_i^a \quad (2.3)$$

$$B_i^a = \nabla_j A_k^a - \nabla_k A_j^a + g f^{abc} A_j^b A_k^c; \quad ijk \text{ cyclic}$$

$$\mathcal{H} = \frac{1}{2} E_i^a E_i^a + \frac{1}{2} B_i^a B_i^a. \quad (2.4)$$

The colour charge density and current density are

$$\rho^a = f^{abc} E_i^b A_i^c \quad (2.5)$$

$$J_i^a = f^{abc} \varepsilon_{ijk} B_j^b A_k^c$$

which together form the conserved four-current

$$J_\mu^a = f^{abc} F_{\mu\nu}^b A_\nu^c. \quad (2.6)$$

In this gauge the state vector $|\psi\rangle$ must satisfy Gauss' law

$$(\nabla_i E_i^a - g \rho^a) |\psi\rangle = 0. \quad (2.7)$$

The first advantage of the temporal axial gauge is that it is physical and ghostfree. This is important for the formulation of self-consistent equations, since the necessity of introducing ghost terms would lead to additional coupled integral equations. All gauges corresponding to the gauge fixing condition

$$n \cdot A^a = 0$$

$$n = \text{fixed four-vector} \quad (2.8)$$

are ghostfree. The temporal axial gauge corresponding to $n = (1, \vec{0})$ is singled out since anyway this fixed four-vector enters the discussion of gluon plasma: it defines the rest frame of the plasma.

The second advantage of the temporal axial gauge is that the most general form of the propagator (2.1) compatible with the gauge condition and symmetry of $\mathcal{D}_{\mu\nu}$ contains only two unknown functions. Since $A_0^a = 0$,

$$\mathcal{D}_{00} = \mathcal{D}_{0i} = \mathcal{D}_{i0} = 0. \quad (2.9)$$

Here and in the following we suppress the ab dependence which is δ_{ab} . In the ij subspace we have in momentum space two tensors available, δ_{ij} and $k_i k_j$. The coefficients of the two tensors are the two unknown functions. We shall write these in the form

$$\mathcal{D}_{ij}(k_0, \vec{k}) = \frac{1}{k^2 + G(k)} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) + \frac{1}{k^2 + F(k)} \frac{k^2}{k_0^2} \frac{k_i k_j}{\vec{k}^2} \quad (2.10)$$

so that $F = G = 0$ gives the free propagator

$$\mathcal{D}_{ij}^{(0)}(k_0, \vec{k}) = \frac{1}{k^2} \left(\delta_{ij} + \frac{k_i k_j}{k_0^2} \right). \quad (2.11)$$

In the perturbative treatment of high temperature gluon matter one is naturally led to treat k_0 as a pure imaginary number, $k_0 = 2\pi nT i$. The Minkowski invariants $k^2 = k_0^2 - \vec{k}^2 = -(2\pi nT)^2 - \vec{k}^2$ are thus negative definite and it is convenient to use the ++++ Euclidean metric with $k_0 = 2\pi nT$. We shall mainly use this convention,

apart from section 3, where k_0 is in places even continued to arbitrary complex values. The rules of transforming between Euclidean and Minkowski metric are given in eq. (2.38).

Also in the Landau ($\partial_\mu A_\mu^a = 0$) and Coulomb ($\partial_i A_i^a = 0$) gauges the general form of the propagator contains two unknown functions: in the former $k_\mu \mathcal{D}_{\mu\nu} = 0$ gives

$$\begin{aligned} \mathcal{D}_{0i} &= \mathcal{D}_{i0} = -\mathcal{D}_{00} \frac{k_0 k_i}{\vec{k}^2} \\ \mathcal{D}_{ij} &= \mathcal{D}_T \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) + \mathcal{D}_{00} \frac{k_0^2}{k^2} \frac{k_i k_j}{\vec{k}^2}, \end{aligned} \quad (2.12)$$

and in the latter $k_i \mathcal{D}_{i\nu} = 0$ gives

$$\begin{aligned} \mathcal{D}_{0i} &= \mathcal{D}_{i0} = 0 \\ \mathcal{D}_{ij} &= \mathcal{D}_T \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right). \end{aligned} \quad (2.13)$$

Of these the Landau gauge has ghosts and the Coulomb gauge even additional problems with perturbation theory [17]. In a general covariant gauge $\mathcal{D}^{\mu\nu}$ only satisfies a Ward identity implying $k_\mu k_\nu \mathcal{D}_{\mu\nu} = \xi = \text{gauge parameter}$ ($= 1$ for Feynman gauge). This only eliminates one of the four unknown functions (the four correspond to $\mathcal{D}_{00}$, one for $\mathcal{D}_{0i}$, and two for $\mathcal{D}_{ij}$) and leaves three [2].

It is useful to rewrite the above analysis in a formally covariant form. Corresponding to the four unknown functions one introduces the following four tensors (recall eq. (2.8):

$$\begin{aligned} A_{\mu\nu} &= \delta_{\mu\nu} + \frac{1}{k^2} \left[k_0 (n_\mu k_\nu + n_\nu k_\mu) - k_\mu k_\nu - k^2 n_\mu n_\nu \right] \\ &= \delta_{\mu i} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right) \delta_{j\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \end{pmatrix}, \end{aligned} \quad (2.14)$$

$$\begin{aligned}
 B_{\mu\nu} &= \frac{k^2}{\vec{k}^2} \left(n_\mu - \frac{k_0 k_\mu}{k^2} \right) \left(n_\nu - \frac{k_0 k_\nu}{k^2} \right) \\
 &= \frac{1}{k^2} \begin{pmatrix} \vec{k}^2 & k_0 k_i \\ k_0 k_i & \frac{k_0^2 k_i k_j}{\vec{k}^2} \end{pmatrix},
 \end{aligned} \tag{2.15}$$

$$\begin{aligned}
 C_{\mu\nu} &= \frac{1}{\sqrt{2} |\vec{k}|} \left[\left(n_\mu - \frac{k_0 k_\mu}{k^2} \right) k_\nu + \left(n_\nu - \frac{k_0 k_\nu}{k^2} \right) k_\mu \right] \\
 &= \frac{1}{\sqrt{2} |\vec{k}| k^2} \begin{pmatrix} 2 \vec{k}^2 k_0 & (\vec{k}^2 - k_0^2) k_i \\ (\vec{k}^2 - k_0^2) k_i & -2 k_0 k_i k_j \end{pmatrix},
 \end{aligned} \tag{2.16}$$

$$D_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}, \tag{2.17}$$

satisfying the multiplication table (in tensorial form apart from the C·B and D·C entries):

$$\begin{aligned}
 A \cdot A &= A, \\
 B \cdot A &= 0, \quad B \cdot B = B, \\
 C \cdot A &= 0, \quad (C \cdot B)_{\mu\nu} = \frac{1}{\sqrt{2} |\vec{k}|} k_\mu \left(n_\nu - \frac{k_0 k_\nu}{k^2} \right), \\
 C \cdot C &= \frac{1}{2} (B + D), \quad D \cdot A = 0, \quad D \cdot B = 0, \\
 (D \cdot C)_{\mu\nu} &= \frac{1}{\sqrt{2} |\vec{k}|} k_\mu \left(n_\nu - \frac{k_0 k_\nu}{k^2} \right), \quad D \cdot D = 0, \\
 C \cdot B + B \cdot C &= C, \quad D \cdot C + C \cdot D = C.
 \end{aligned} \tag{2.18}$$

Note that $A_{\mu\nu} = P_{\mu\nu}^T$, $B_{\mu\nu} = P_{\mu\nu}^L$ in the notation of ref. [5]. Some further practical relations are

$$\begin{aligned}\delta_{\mu\nu} &= A_{\mu\nu} + B_{\mu\nu} + D_{\mu\nu}, \\ n_\mu k_\nu + n_\nu k_\mu &= \sqrt{2} |\vec{k}| C_{\mu\nu} + 2k_0 D_{\mu\nu}, \\ k^2 n_\mu n_\nu &= \vec{k}^2 B_{\mu\nu} + \sqrt{2} |\vec{k}| k_0 C_{\mu\nu} + k_0^2 D_{\mu\nu}.\end{aligned}\quad (2.19)$$

Using these tensorial relations it is simple to verify that the most general form of the propagator compatible with $n_\mu \partial_{\mu\nu} = n_\nu \partial_{\mu\nu} = 0$ is

$$\mathcal{D}_{\mu\nu} = \alpha A_{\mu\nu} + \beta \left(B_{\mu\nu} - \frac{\sqrt{2} |\vec{k}|}{k_0} C_{\mu\nu} + \frac{\vec{k}^2}{k_0^2} D_{\mu\nu} \right), \quad (2.20)$$

where α and β are two independent functions of k_0 and $|\vec{k}|$. Identifying

$$\alpha = \frac{1}{k^2 + G} \quad \beta = \frac{1}{k^2 + F} \quad (2.21)$$

this is the same as eqs. (2.11)-(2.12) and leads to the covariant form

$$\mathcal{D}_{\mu\nu}^{(0)} = \frac{1}{k^2} \left\{ \delta_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{n \cdot k} + \frac{k_\mu k_\nu}{(n \cdot k)^2} \right\} \quad (2.22)$$

of the free propagator (2.11). Further, the inverse propagator, which satisfies

$$\mathcal{D}_{\mu\sigma}^{-1} \mathcal{D}_{\sigma\nu} = \delta_{\mu\nu} - \frac{n_\mu k_\nu}{k_0} \quad (2.23)$$

has the form

$$\mathcal{D}_{\mu\nu}^{-1} = \frac{1}{\alpha} A_{\mu\nu} + \frac{1}{\beta} B_{\mu\nu}. \quad (2.24)$$

Finally, the gluon self-energy, which enters the Schwinger-Dyson equation, is given by

$$\begin{aligned}\pi_{\mu\nu} &= \mathcal{D}_{\mu\nu}^{-1} - \mathcal{D}_0^{-1}{}_{\mu\nu} \\ &= G A_{\mu\nu} + F B_{\mu\nu}.\end{aligned}\quad (2.25)$$

Using eqs. (2.14)-(2.15) the independent components of π are related to F and G by

$$\begin{aligned}F &= \frac{k^2}{k^2} \pi_{00} \\ G &= \frac{1}{2} \left(\pi_{ii} - \frac{k_0^2}{k^2} \pi_{00} \right) = \frac{1}{2} \left(\pi_{\mu\mu} - \frac{k^2}{k^2} \pi_{00} \right).\end{aligned}\quad (2.26)$$

The complexities associated with general covariant gauges can also be discussed in terms of the set of four covariant tensors introduced above. Let us write

$$\mathcal{D}_{\mu\nu}^{-1} = a A_{\mu\nu} + b B_{\mu\nu} + c C_{\mu\nu} + d D_{\mu\nu}, \quad (2.27)$$

where a, b, c and d are four independent functions of k_0 and $|\vec{k}|$. Requiring $\mathcal{D}_{\mu\sigma} \mathcal{D}_{\sigma\nu}^{-1} = \mathcal{D}_{\mu\sigma}^{-1} \mathcal{D}_{\sigma\nu} = \delta_{\mu\nu}$ [now there cannot be any non-covariant terms of the type in eq. (2.23)] the propagator itself is seen to be

$$\mathcal{D}_{\mu\nu} = \frac{1}{a} A_{\mu\nu} + \frac{2}{c^2 - 2bd} \left(-d B_{\mu\nu} + c C_{\mu\nu} - b D_{\mu\nu} \right). \quad (2.28)$$

One relation between the four functions is now obtained from the Ward identity:

$$k_\mu k_\nu \mathcal{D}_{\mu\nu} = \mathcal{F} = \frac{-2b}{c^2 - 2bd} k^2. \quad (2.29)$$

This leads to

$$\mathcal{D}_{\mu\nu} = \frac{1}{a} A_{\mu\nu} + \left(\frac{1}{b} + \frac{c^2 \mathcal{F}}{2b^2 k^2} \right) B_{\mu\nu} - \frac{c \mathcal{F}}{b k^2} C_{\mu\nu} + \frac{\mathcal{F}}{k^2} D_{\mu\nu}. \quad (2.30)$$

Only in the Landau gauge $\xi = 0$ does this simplify to a form containing only two independent functions [cf. eq. (2.12) above]. However, the third function resides in the inverse propagator or the gluon self-energy and needs the ghost for its determination.

In covariant gauges the free propagators are given by

$$\begin{aligned} \mathcal{D}_{0\mu\nu} &= \frac{1}{k^2} (A_{\mu\nu} + B_{\mu\nu} + \xi D_{\mu\nu}), \\ \mathcal{D}_0^{-1}{}_{\mu\nu} &= k^2 (A_{\mu\nu} + B_{\mu\nu} + \frac{1}{\xi} D_{\mu\nu}). \end{aligned} \quad (2.31)$$

The gluon self-energy is then

$$\begin{aligned} \Pi_{\mu\nu} &= \mathcal{D}_{\mu\nu}^{-1} - \mathcal{D}_0^{-1}{}_{\mu\nu} \\ &= (a - k^2)A_{\mu\nu} + (b - k^2)B_{\mu\nu} + cC_{\mu\nu} + \frac{c^2}{2b}D_{\mu\nu}, \end{aligned} \quad (2.32)$$

which even in the Landau gauge contains three independent functions and which in the vacuum ($c = 0$, $a = b$) reduces to the standard form

$$\begin{aligned} \Pi_{\mu\nu} &= (a - k^2)(A_{\mu\nu} + B_{\mu\nu}) \\ &= (a - k^2)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \end{aligned} \quad (2.33)$$

We now come to the disadvantage of the temporal axial gauge: the tensorial structure of the propagator (2.20) is rather complicated and, in particular, it contains poles in $k_0 = k \cdot n$. The correct way to treat these poles [13] is to use the principal value prescription when k_0 is integrated over:

$$\begin{aligned}\frac{1}{k_0} &\rightarrow \frac{1}{2} \left[\frac{1}{k_0 + i\epsilon} - \frac{1}{-k_0 + i\epsilon} \right], \\ \frac{1}{k_0^2} &\rightarrow \frac{1}{2} \left[\frac{1}{(k_0 + i\epsilon)^2} + \frac{1}{(-k_0 + i\epsilon)^2} \right].\end{aligned}\tag{2.34}$$

Later on we shall see many examples of how the rule (2.34) operates. As indicated in the introduction, this prescription is related to the residual freedom of performing time-independent gauge transformations in this gauge.

The form (2.10) of the exact propagator clearly suggests that G , which multiplies a tensor transverse to $\vec{k}$, is associated with magnetic and F , which multiplies a tensor parallel to $\vec{k}$, is associated with electric effects. This question will be discussed in detail in section 5 with the aid of linear response theory. On a more intuitive level, the association can also be seen by considering the fluctuations of the fields [i.e., thermal averages of the type $\langle A(t)A(0) \rangle$]. Inserting from eq. (2.3) the fields to leading order in g , the fluctuations are in the temporal axial gauge

$$\begin{aligned}\langle E^2 \rangle &\sim \langle \partial_0 A_i \partial_0 A_i \rangle \sim k_0^2 \mathcal{D}_{ii} = k_0^2 \mathcal{D}_{nn} \\ &= 2k_0^2 \alpha + k^2 \beta, \\ \langle B^2 \rangle &\sim \langle \vec{\nabla} \times \vec{A} \cdot \vec{\nabla} \times \vec{A} \rangle \sim \langle \vec{k} \times \vec{A} \cdot \vec{k} \times \vec{A} \rangle \\ &= \vec{k}^2 \langle A_i A_i \rangle - k_i k_j \langle A_i A_j \rangle \\ &= \vec{k}^2 \mathcal{D}_{nn} - k_n k_\nu \mathcal{D}_{n\nu} \\ &= 2\vec{k}^2 \alpha,\end{aligned}\tag{2.35}$$

(the fluctuation-dissipation theorem relates fluctuations to retarded Green functions). In the static limit $k_0 = 0$

$$\begin{aligned} \langle E^2 \rangle &\sim \vec{k}^2 \beta = \frac{\vec{k}^2}{\vec{k}^2 + F(0, \vec{k})}, \\ \langle B^2 \rangle &\sim 2 \vec{k}^2 \beta = \frac{2 \vec{k}^2}{\vec{k}^2 + G(0, \vec{k})}, \end{aligned} \quad (2.36)$$

which exhibit the association of G and F with static electric and magnetic effects, respectively.

A similar intuitive association can also be made in the Landau gauge. Writing in eqs. (2.30) and (2.32) $a = k^2 + G_L$, $b = k^2 + F_L$, $c \rightarrow (k^2 + F_L)$, we have

$$\begin{aligned} \mathcal{D}_{\mu\nu} &= \frac{1}{k^2 + G_L} A_{\mu\nu} + \frac{1}{k^2 + F_L} B_{\mu\nu}, \\ \Pi_{\mu\nu} &= G_L A_{\mu\nu} + F_L B_{\mu\nu} + c(k^2 + F_L) C_{\mu\nu} + \frac{1}{2} c^2 (k^2 + F_L) D_{\mu\nu}. \end{aligned} \quad (2.37)$$

Thus one expects $G_L(0, \vec{k} \rightarrow 0)$ and $F_L(0, \vec{k} \rightarrow 0)$ to be associated with magnetic and electric screening. To verify this one has to carry out the linear response analysis of section 5 for the Landau gauge. No such simple interpretation is possible in arbitrary covariant gauges.

Finally, the Feynman rules we use are defined as follows. The propagator is given by eqs. (2.9) and (2.10), or eqs. (2.20) and (2.21) in four-vector notation. The bare and full propagators are shown in fig. 2 along with the bare three and four-point vertices. There is a factor of $(2\pi)^3 \delta(\vec{p}_{\text{in}} - \vec{p}_{\text{out}})$ at each vertex for momentum conservation and a factor of $\delta_{p_{\text{in}}^0, p_{\text{out}}^0} / T$ at each vertex for energy conservation. In this paper, unless otherwise stated, we will be working in Euclidean space. The metric is +---, $p^2 = p_0^2 + \vec{p}^2$, and $p_0 = 2\pi nT$. The main exception is section 3 on perturbative results. Then it is more convenient to work in Minkowski space, with metric +---. One goes from Euclidean to Minkowski space by the rules

$$\begin{aligned}
 p_0^2 &\rightarrow -p_0^2 \\
 p^2 &\rightarrow -p^2 \\
 \pi_{00} &\rightarrow -\pi_{00} \\
 \pi_{ii} &\rightarrow \pi_{ii} \\
 F &\rightarrow F \\
 G &\rightarrow G
 \end{aligned}
 \tag{2.38}$$

3. TEMPORAL AXIAL GAUGE GLUON PROPAGATOR IN PERTURBATION THEORY

In this section, we shall calculate the gluon propagator in lowest order perturbation theory in the temporal axial gauge. The results serve to illustrate the relationship of this gauge to other covariant gauges and, in particular, the inapplicability of perturbation theory to correctly obtain the infrared limit of the propagator. As shown in section 4, this requires more powerful self-consistent and non-perturbative methods. It is interesting to note how complicated the structure of the gluon propagator in the infrared limit is both at $T = 0$ (where the limit is relevant for quark confinement) and at very large T .

3.1 One loop gluon self energy

In lowest order perturbation theory the gluon propagator is obtained by calculating the gluon self-energy from the diagrams of fig. 3 by using the Feynman rules of $T > 0$ field theory. It is sufficient to calculate Π_{00} and Π_{ii} ; F , G , $\Pi_{\mu\nu}$ and $\mathcal{Q}_{\mu\nu}$ are then given by eqs. (2.25), (2.26) and (2.10). The notation becomes slightly more economical if we do not separate the i from the $k_0 = 2\pi T n i$ in the temperature Feynman rules. As we shall also discuss the continuation of k_0 from the imaginary axis to the complex k_0 plane, we shall in this section not transform to the Euclidian metric.

After a considerable amount of algebra, one obtains the following results

$$\begin{aligned} \Pi_{00}(k_0, \vec{k}) = & \frac{1}{2} g^2 N T \sum_n \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{2}{q_0^2} + \frac{4}{q^2} \right. \\ & \left. - (p_0 - q_0)^2 \left[\frac{1}{q^2 p^2} + \frac{1}{q^2 p_0^2} + \frac{1}{q_0^2 p^2} + \frac{(\vec{p} \cdot \vec{q})^2}{p^2 q^2 p_0^2 q_0^2} \right] \right\}, \quad (3.1) \end{aligned}$$

$$\begin{aligned} \Pi_{ii}(k_0, \vec{k}) = & -\frac{1}{2} g^2 N T \sum_n \int \frac{d^3 q}{(2\pi)^3} \left\{ \frac{4}{q_0^2} + \frac{8}{q^2} \right. \\ & + \frac{1}{p^2 q^2} \left[q \vec{k}^2 - 20 \vec{p} \cdot \vec{q} - 7 \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} + 12 \frac{(\vec{p} \cdot \vec{q})^3}{\vec{p}^2 \vec{q}^2} \right] \\ & + \frac{1}{p_0^2 q^2} \left[\vec{k}^2 + 4 \vec{p} \cdot \vec{q} + 6 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2} + \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \\ & + \frac{1}{p^2 q_0^2} \left[\vec{k}^2 + 4 \vec{p} \cdot \vec{q} + 6 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{q}^2} + \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \\ & \left. + \frac{1}{p_0^2 q_0^2} \vec{k}^2 \left[1 - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \right\}, \quad (3.2) \end{aligned}$$

where $\vec{k} + \vec{p} + \vec{q} = 0$, $k_0 + p_0 + q_0 = 0$, $q_0 = 2\pi n T i$, $n = 0, \pm 1, \dots$, similarly for k_0 and p_0 . Note also that the colour factor δ_{ab} has been deleted. The sum over n in eq. (3.2) is carried out by using the formula

$$T \sum_{n=-\infty}^{\infty} f(q_0 = 2\pi n T i) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dq_0 f(q_0) + \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dq_0 \frac{f(q_0) + f(-q_0)}{\exp(q_0/T) - 1}. \quad (3.3)$$

In lowest order perturbation theory the first term here gives the vacuum contribution and the second, which is evaluated by encircling the poles of the integrand in the right half-plane, gives the matter contribution. The latter will always contain the Bose-Einstein distribution for gluons:

$$n(\beta q) = \frac{1}{\exp(\beta q) - 1} \quad (3.4)$$

In higher orders, as in section 4, even the integrand in the first term may be T-dependent.

3.2 Vacuum contribution

Let us first consider the vacuum contribution, given by the first term in (3.3). Since [see eq. (2.25) , in Minkowski space $A_{\mu\nu} + B_{\mu\nu} = -g_{\mu\nu} + k_\mu k_\nu / k^2$]

$$\Pi_{\mu\nu} = G(-g_{\mu\nu} + k_\mu k_\nu / k^2) + (F - G) B_{\mu\nu} \quad (3.5)$$

we expect, because of the renormalizability of the theory, that the divergent parts of F and G are equal. It is illustrative to first calculate the imaginary parts of G and F-G. Using Cutkosky's rule in eqs. (3.1) and (3.2) and noting that, according to the prescriptions appropriate for the temporal axial gauge, the terms $1/q_0^2$ or $1/p_0^2$ do not contribute to the imaginary part, one finds that for k_0 real, $k_0^2 - \vec{k}^2 \geq 0$,

$$\begin{aligned} \text{Im } \Pi_{00}(k_0, \vec{k}) &= \frac{-g^2 N}{32\pi |\vec{k}|} \int_{q_-}^{q_+} dq (|\vec{p}| - |\vec{q}|)^2 \left[1 + \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right], \\ \text{Im } \Pi_{ii}(k_0, \vec{k}) &= -\frac{g^2 N}{32\pi |\vec{k}|} \int_{q_-}^{q_+} dq \left[q \vec{k}^2 - 20 \vec{p} \cdot \vec{q} - 7 \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} + 12 \frac{(\vec{p} \cdot \vec{q})^3}{\vec{p}^2 \vec{q}^2} \right], \end{aligned} \quad (3.6)$$

where $q_{\pm} = \frac{1}{2}(k_0 \pm |\vec{k}|)$ are the kinematic limits of the momentum of one of the gluons from the decay of an off-shell gluon with energy k_0 and momentum $|\vec{k}|$ to two real gluons. Integrating explicitly and inserting to eq. (2.26) one obtains

$$\begin{aligned} \text{Im} \frac{G}{k^2} &= \frac{g^2 N}{32\pi} \left[\frac{34}{3} + \frac{3}{a^2} + a^2 - \frac{3 + 11a^2 + a^4 + a^6}{2a^3} \ln\left(\frac{1+a}{1-a}\right) \right] \\ \text{Im} \frac{F-G}{k^2} &= -\frac{g^2 N}{32\pi} [3 + a^2] \left[\frac{3}{a^2} + 1 - \frac{3+a^4}{2a^3} \ln\left(\frac{1+a}{1-a}\right) \right], \quad (3.7) \\ a &= \frac{|\vec{k}|}{k_0} = \frac{|\vec{k}|}{\sqrt{\vec{k}^2 + k^2}}. \end{aligned}$$

The $k^2 \rightarrow \infty$ or $a \rightarrow 0$ limits of the brackets in eq. (3.7) are $-2/3$ and $-8a^2/5$, respectively. Thus the dispersion relation for G/k^2 requires one subtraction while that for $F-G$ requires none. In view of eq. (3.5) this is as expected. It is natural to subtract at $k_0 = 0$ and write

$$\begin{aligned} G(k_0, \vec{k}) &= -\frac{k^2}{\vec{k}^2} G(0, \vec{k}) + \frac{k^2 k_0^2}{\pi} \int_0^\infty dk'^2 \frac{A(a = |\vec{k}|/\sqrt{\vec{k}^2 + k'^2})}{(k'^2 - k^2)(\vec{k}^2 + k'^2)}, \\ F(k_0, \vec{k}) - G(k_0, \vec{k}) &= \frac{k^2}{\pi} \int_0^\infty dk'^2 \frac{B(a = |\vec{k}|/\sqrt{\vec{k}^2 + k'^2})}{k'^2 - k^2}, \quad (3.8) \end{aligned}$$

where A and B denote the right hand sides of eq. (3.7).

To find the temporal axial gauge renormalization constant Z_A we first note that, in contrast to covariant gauges, it is not related to the behaviour of A in eq. (3.8) at $k'^2 \rightarrow \infty$ [15-16]. This is due to the $1/q_0^2$ and $1/p_0^2$ terms in eqs. (3.1)-(3.2) which did not contribute to the real part. Since Π_{00} is simpler than Π_{ii} we first calculate from the second of eqs. (3.8) that

$$G(0, \vec{k}) = F(0, \vec{k}) + \frac{g^2 N}{48\pi^2} \vec{k}^2 (20 - 8 \ln 2) \quad (3.9)$$

and then $F(0, \vec{k}) = -\Pi_{00}(0, \vec{k})$ from (3.1):

$$\begin{aligned} F(0, \vec{k}) &= - \frac{g^2 N}{8\pi^3} \int d^3 q \left[\frac{1}{q+p} + \frac{1}{p} - \frac{(\vec{p} \cdot \vec{q})^2 (p^2 + pq + q^2)}{p^3 q^3 (q+p)} \right] \\ &= - \frac{g^2 N}{48\pi^2} \vec{k}^2 \left[11 \ln(\Lambda_c^2 / \vec{k}^2) + 22 \ln 2 - 2 \right] \end{aligned} \quad (3.10)$$

where $q = |\vec{q}|$, $p = |\vec{q} + \vec{k}|$ and Λ_c is the UV cut-off in q . Renormalization is now carried out by writing

$$\begin{aligned} \mathcal{D}_{\mu\nu}^{-1}(k_0, \vec{k}) &= Z_A \mathcal{D}_0^{-1}{}_{\mu\nu}(k_0, \vec{k}) - \Pi_{\mu\nu} \\ &= \left[Z_A - \frac{G(k_0, \vec{k})}{k^2} \right] \mathcal{D}_0^{-1}{}_{\mu\nu}(k_0, \vec{k}) - (F - G) B_{\mu\nu}. \end{aligned} \quad (3.11)$$

Different renormalization prescriptions are obtained depending on how the constant terms in Z_A are fixed [18-19]. In temporal axial gauge the propagator depends separately on k^2 and k_0 . We shall define an axial gauge MOM scheme by demanding that the coefficient of the $\mathcal{D}_0^{-1}$ term in (3.11) vanish when $k_0 = 0$ and $k^2 = -\mu^2 = -\vec{k}^2$. Combining eqs. (3.9)-(3.11) then gives

$$Z_A = 1 + \frac{g^2 N}{48\pi^2} \left[11 \ln(\Lambda_c^2 / \mu^2) + 30 \ln 2 - 22 \right]. \quad (3.12)$$

Using the methods of ref. [18] this can be translated to

$$\frac{\Lambda_{\text{MOM}}^{\text{axial}}}{\Lambda_{\overline{\text{MS}}}} = \exp\left(\frac{1}{2}\gamma - \frac{1}{2} \ln 4\pi + \frac{15}{11} \ln 2 - 1\right) = 0.356 \quad (3.13)$$

which further gives

$$\frac{\Lambda_{\text{MOM}}^{\text{axial}}}{\Lambda_{\text{MOM}}^{\text{Feynman}, n_f=0}} = 0.123. \quad (3.14)$$

3.3 Matter contribution

In order to calculate the matter contribution to Π_{00} and Π_{ii} we first perform the sums over n in eqs. (3.1) and (3.2) by using the formula (3.3). The poles at $q_0 = 0$ or at $p_0 = q_0 + k_0 = 0$ lie on the imaginary q_0 axis and remain outside the path of integration in the second term of eq. (3.3) when it is moved to the right. They thus do not contribute to the matter part. One obtains the following results for the sums [recall eq. (3.4)]:

$$\begin{aligned} T \sum_n \frac{1}{q^2} &= - \frac{n(\beta q)}{q} - \frac{1}{2q} \\ T \sum_n \frac{1}{q^2 p^2} &= - \text{Re} \frac{1}{(q+k_0)^2 - p^2} \frac{n(\beta q)}{q} + (p \leftrightarrow q) + \text{vacuum} \\ T \sum_n \frac{1}{q^2 p_0^2} &= - \text{Re} \frac{1}{(q+k_0)^2} \frac{n(\beta q)}{q} + \text{vacuum} \\ T \sum_n \frac{(p_0 - q_0)^2}{q^2 p^2} &= - \text{Re} \frac{(2q+k_0)^2}{(q+k_0)^2 - p^2} \frac{n(\beta q)}{q} + (p \leftrightarrow q) + \text{vacuum} \quad (3.15) \\ T \sum_n \frac{(p_0 - q_0)^2}{q^2 p_0^2} &= - \text{Re} \frac{(2q+k_0)^2}{(q+k_0)^2} \frac{n(\beta q)}{q} \\ T \sum_n \frac{(p_0 - q_0)^2}{q^2 p^2 q_0^2 p_0^2} &= - \text{Re} \frac{(2q+k_0)^2}{[(q+k_0)^2 - p^2](q+k_0)^2 q^2} \frac{n(\beta q)}{q} + (p \leftrightarrow q) \\ &\quad + \text{vacuum} \end{aligned}$$

where $k_0 + q_0 + p_0 = 0$, $\vec{k} + \vec{q} + \vec{p} = 0$, on the left hand-side $q^2 = q_0^2 - \vec{q}^2$, $p^2 = p_0^2 - \vec{p}^2$ while on the right hand side and otherwise from now on $q = |\vec{q}|$. In the eqs. (3.15) Re is short for $\text{Re } f(k_0) = \frac{1}{2}[f(k_0) + f(-k_0)]$, although this operation only gives a real number when k_0 is pure imaginary, i.e. before analytic continuation. Integrating further over the angles of $\vec{q}$ relative to $\vec{k}$ one meets with integrals of the type

$$\begin{aligned} \int d^3q \frac{n(\beta q)}{q} \frac{1}{p^2 - (q+k_0)^2} &= \frac{\pi}{k} \int_0^\infty dq n(\beta q) \ln \left[\frac{k^2 - k_0^2 - 2qk_0 + 2qk}{k^2 - k_0^2 - 2qk_0 - 2qk} \right] \\ &= \frac{\pi}{2k} \int_0^\infty dq n(\beta q) \left\{ \ln \left[\frac{(k+2q)^2 - k_0^2}{(k-2q)^2 - k_0^2} \right] + \ln \frac{(k-k_0)^2 [(k+k_0)^2 - 4q^2]}{(k+k_0)^2 [(k-k_0)^2 - 4q^2]} \right\} \end{aligned} \quad (3.16)$$

where the logarithm has been split up into even and odd parts in k_0 , i.e. real and imaginary parts for $k_0 = 2\pi i T n$. The final results are (here and in the following the expressions refer only to the matter part)

$$\begin{aligned} \Pi_{00}(k_0, \vec{k}) &= -\frac{g^2 N}{4\pi^2} \int_0^\infty dq q n(\beta q) \text{Re} \left\{ 4 - \frac{(k^2 - 2qk_0 - k_0^2)(2q+k_0)^2}{2q^2 (q+k_0)^2} \right. \\ &+ \frac{(2q+k_0)^2}{2qk} \left[1 + \frac{(q^2 + (q+k_0)^2 - k^2)^2}{4q^2 (q+k_0)^2} \right] \\ &\times \ln \left[\frac{k^2 - 2qk_0 - k_0^2 + 2qk}{k^2 - 2qk_0 - k_0^2 - 2qk} \right] \Bigg\}, \end{aligned} \quad (3.17)$$

$$\begin{aligned}
 \pi_{ii}(k_0, \vec{k}) = & -\frac{g^2 N}{4\pi^2} \int_0^\infty dq q n(\beta q) \operatorname{Re} \left\{ 12 - \right. \\
 & \frac{2}{(q+k_0)^2} \left[8q^2 - k^2 - \frac{k^4}{4q^2} + 9k_0(k_0+2q) - \frac{5k^2}{4q^2} k_0(k_0+2q) \right. \\
 & + \frac{3k_0^2}{2q^2} (k_0+2q)^2 \left. \right] + \frac{1}{2qk} \left[-k^2 + 10q^2 + 10(q+k_0)^2 \right. \\
 & - \frac{1}{2q^2(q+k_0)^2} \left(q^2 + (q+k_0)^2 - k^2 \right)^2 \left(3q^2 + 3(q+k_0)^2 + \frac{1}{2} k^2 \right) \left. \right] \\
 & \times \ln \left[\frac{k^2 - 2qk_0 - k_0^2 + 2qk}{k^2 - 2qk_0 - k_0^2 - 2qk} \right] \left. \right\}, \quad (3.18)
 \end{aligned}$$

where $k = |\vec{k}|$. The properties of these rather untransparent expressions should become clearer when we discuss their various limits and applications. Note that the factors multiplying the logarithms in eqs. (3.17) and (3.18) are essentially the same as in the integrands in eq. (3.6). For comparison, the results in Feynman gauge are [5, 20]

$$\begin{aligned}
 \pi_{00}(k_0, \vec{k}) = & -\frac{g^2 N}{4\pi^2} \int_0^\infty dq q n(\beta q) \operatorname{Re} \left\{ 4 + \frac{1}{qk} \left[(k_0+2q)^2 - 2k^2 \right] \ln \left[\right] \right\}, \\
 \pi_{ii}(k_0, \vec{k}) = & \frac{g^2 N}{4\pi^2} \int_0^\infty dq q n(\beta q) \operatorname{Re} \left\{ 4 + \frac{1}{qk} \left[4k_0^2 - 4qk_0 - 4q^2 - 3k^2 \right] \ln \left[\right] \right\}, \quad (3.19)
 \end{aligned}$$

where the logarithms are as in eq. (3.16).

Let us first consider the continuation of eqs. (3.17) and (3.18) from $k_0 = 2\pi nTi$ to the complex to the plane. This has to be carefully defined near the real k_0 axis, where a cut exists for all k_0 . One finds [21-23] that the replacement $2\pi nTi = \omega + i\eta$ gives the retarded Green function. More precisely if $A(t)$ and $B(t)$ are operators, the Fourier transforms $G_R(\omega)$ and $G_B(2\pi nTi)$ of the retarded Green function

$\theta(t)\langle A(t)B(0) - B(0)A(t) \rangle$ and the temperature Green function $\langle T[A(\tau)B(0)] \rangle$ ($\tau = i\tau$) have the spectral representations

$$G_R(\omega) = i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega - \omega' + i\eta}$$

$$G_\beta(2\pi n T i) = i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{2\pi n T i - \omega'} \quad (3.20)$$

with the same spectral function $\rho(\omega) = (1 - e^{-\beta\omega})J(\omega)$.

$$J(\omega) = Z^{-1} \sum \exp(-\beta E_m) \langle n | B(0) | m \rangle \langle m | A(0) | n \rangle 2\pi \delta(\omega + E_m - E_n).$$

With this analytic continuation, one finds that for $k_0 = \omega + i\eta$, $\omega > 0$, the logarithms in eqs. (3.17)-(3.19) have an imaginary part $+i\pi$ for

$$\frac{|\omega - k|}{2} \leq q \leq \frac{\omega + q}{2}.$$

The sign of this imaginary part will reflect itself in the sign of the damping constant of plasma oscillations, as studied in section 5.

3.4 Static infrared limit in perturbation theory

As a first special case of Π_{00} and Π_{ii} , consider the static limit, $k_0 = 0$. This limit is relevant for screening (section 5) and both Π_{00} and Π_{ii} are then purely real. One finds that, in the temporal axial gauge

$$\Pi_{00}(0, \vec{k}) = -\frac{g^2 N}{\pi^2} \int_0^\infty dq q n(\beta q) \left[1 - \frac{k^2}{2q^2} + \left(\frac{q}{k} - \frac{k}{2q} + \frac{k^3}{8q^3} \right) \ln \left| \frac{k+2q}{k-2q} \right| \right]$$

$$= \begin{cases} -g^2 N \left(\frac{1}{3} T^2 - \frac{1}{4} k T \right) + \frac{g^2 N}{48\pi^2} 11 k^2 \ln(k^2/T^2) + \mathcal{O}(k^2) & k \ll T \\ \frac{1}{18} g^2 N T^2 - 11 \left(\frac{2\pi}{15} \right)^2 g^2 N \frac{T^4}{k^2} + \mathcal{O}\left(\frac{T^6}{k^4}\right) & k \gg T, \end{cases} \quad (3.21)$$

$$\begin{aligned}
 \Pi_{ii}(0, \vec{k}) &= \frac{g^2 N}{\pi^2} \int_0^\infty dq q n(\beta q) \left[1 - \frac{k^2}{2q^2} - \frac{k^4}{8q^4} + \left(-\frac{q}{k} - \frac{5k}{4q} + \frac{k^3}{4q^3} + \frac{k^5}{32q^5} \right) \right. \\
 &\quad \left. \times \ln \left| \frac{k+2q}{k-2q} \right| \right] \\
 &= \begin{cases} -\frac{5}{8} g^2 N k T & k \ll T \\ -\frac{17}{45} g^2 N T^2 - \frac{166}{7} \left(\frac{2\pi}{15} \right)^2 g^2 N \frac{T^4}{k^2} + O\left(\frac{T^6}{k^4}\right) & k \gg T. \end{cases} \quad (3.22)
 \end{aligned}$$

For comparison, the results in the Feynman gauge are

$$\begin{aligned}
 \Pi_{00}(0, \vec{k}) &= -\frac{g^2 N}{\pi^2} \int_0^\infty dq q n(\beta q) \left[1 + \left(\frac{q}{k} - \frac{k}{2q} \right) \ln \left| \frac{k+2q}{k-2q} \right| \right] \\
 &= \begin{cases} -g^2 N \left(\frac{1}{3} T^2 - \frac{1}{4} k T \right) & k \ll T \\ \frac{1}{6} g^2 N T^2 - 5 \left(\frac{2\pi}{15} \right)^2 g^2 N \frac{T^4}{k^2} & k \gg T, \end{cases} \quad (3.23)
 \end{aligned}$$

$$\begin{aligned}
 \Pi_{ii}(0, \vec{k}) &= \frac{g^2 N}{\pi^2} \int_0^\infty dq q n(\beta q) \left[1 + \left(-\frac{q}{k} - \frac{3k}{4q} \right) \ln \left| \frac{k+2q}{k-2q} \right| \right] \\
 &= -\frac{3}{8} g^2 N k T \quad k \ll T. \quad (3.24)
 \end{aligned}$$

In an arbitrary covariant gauge specified by the gauge parameter ξ the leading infrared term for $-\Pi_{00}$ is, of course, the gauge independent inverse electric screening length squared $\frac{1}{3}g^2 N T^2$, while that for Π_{ii} is [24]

$$\Pi_{ii}(0, \vec{k}) = -\frac{1}{32} (9 + 2\xi + \xi^2) g^2 N k T \quad k \ll T. \quad (3.25)$$

In these infrared limits, note that the correction to the leading T^2 term is kT and not k^2 . The reason for this is that for bosons $\eta(\beta q) \approx T/q$ for small q . For fermions (with $\mu = 0$) $n_F(\beta q) = \frac{1}{2}$ for small q and the correction factor would be $O(k^2)$.

Using the results of section 5, eqs. (3.21) and (3.22) predict that in lowest order perturbation theory the screening of electric and magnetic fields is essentially determined by the factors [cf. eqs. (5.19) and (5.27)]

$$\vec{k}^2 - \Pi_{00}(0, \vec{k}) \approx k^2 + \frac{1}{3} g^2 N T^2 - \frac{1}{4} g^2 N k T, \quad (3.26)$$

$$\vec{k}^2 + \frac{1}{2} \Pi_{ii}(0, \vec{k}) \approx k^2 - \frac{5}{16} g^2 N k T. \quad (3.27)$$

These expressions imply [8] that, at least for magnetic screening, the perturbation expansion fails for $k \leq g^2 T$. What really happens in the theory in this limit has been the topic of a considerable amount of theoretical research [2, 8, 25]. This work has been based on the conjecture that the sum over all orders in perturbation theory is able to generate a magnetic mass of order $g^2 T$:

$$\lim_{k \rightarrow 0} \left[-\frac{5}{16} g^2 N k T + a_2 g^4 T^2 + a_3 g^6 \frac{T^3}{k} + \dots \right] = m_{m0}^2(T) \approx g^4 T^2. \quad (3.28)$$

This in turn would imply that the thermodynamic potential is perturbatively calculable only up to order g^6 , as discussed in the introduction [see eq. (1.4)].

3.5 The long wave-length limit $k \ll |k_0|$

The limit $k \ll |k_0|$ will be needed in the study of long wave-length plasma oscillations. These modes will be seen to exist when $|k_0| \geq gT$. To begin with, $\Pi_{00}(k_0 \neq 0, 0) = 0$, although $\Pi_{00}(0, k \rightarrow 0) = -\frac{1}{3} g^2 N T^2$. Expanding further in powers of $\vec{k}^2/k_0^2$ one finds that

$$\begin{aligned} \Pi_{00}(k_0, \vec{k}) = & \frac{g^2 N}{\pi^2} \int_0^\infty dq q n(\beta q) \left\{ \left[1 - \frac{4}{3} \frac{q^2}{4q^2 - k_0^2} + \frac{(3q^2 - k_0^2)k_0^2}{3(q^2 - k_0^2)^2} \right] \frac{\vec{k}^2}{k_0^2} \right. \\ & + \left[\frac{4q^2(4q^2 + k_0^2)}{(4q^2 - k_0^2)^2} - \frac{16}{5} \frac{q^4(12q^2 + k_0^2)}{(4q^2 - k_0^2)^3} - \frac{k_0^2}{4q^2 - k_0^2} \right. \\ & \left. \left. - \frac{1}{3} \frac{k_0^2(q^2 + k_0^2)}{(q^2 - k_0^2)^2} + \frac{4}{15} \frac{k_0^2 q^2}{(q^2 - k_0^2)^2} - \frac{5q^2 + k_0^2}{(4q^2 - k_0^2)} \right] \frac{(\vec{k}^2)^2}{k_0^4} + \dots \right\}. \end{aligned} \quad (3.29)$$

If one expands Π_{00} in Feynman gauge [eq. (3.19)] similarly, one notices that the expansion coefficients differ only in those terms which are proportional to k_0^2 . As shown in section 5, the contribution of these terms is down by g^2 relative to the leading terms in the dispersion relation, which thus is gauge independent to leading order.

The other independent component $\Pi_{ii}(k_0, \vec{k})$ can be similarly expanded for $k \ll |k_0|$. The result and its gauge dependence is similar to eq. (3.29) and we give the result only when k_0^2 is neglected relative to q^2 in the coefficients. Then

$$\Pi_{ii}(k_0, \vec{k}) = \frac{1}{3} g^2 N T^2 \left[1 + \frac{1}{3} \frac{\vec{k}^2}{k_0^2} + \dots \right]. \quad (3.30)$$

4. NON-PERTURBATIVE CALCULATION OF THE INFRARED LIMIT OF THE GLUON PROPAGATOR

Following the discussions of section 1 and section 3 we know that perturbation theory is not adequate to calculate the static infrared limit of gluon Green functions. More precisely, for the gluon propagator the perturbative expansion in powers of g^2 breaks down when $k_0 = 0$ and $k \leq g^2 T$. See eq. (3.27). This is true despite the fact that the effective coupling $g^2(T) \sim 1/\ln(T/\Lambda)$ goes to zero as T goes to infinity. Therefore one might hope that weak coupling, but non-perturbative, methods could resolve this problem. With this in mind we now turn to a discussion of the Schwinger-Dyson equation for the gluon propagator.

4.1 Schwinger-Dyson equations

The graphical representation of the Schwinger-Dyson equations at finite temperature take the same form as in the vacuum. The equation for the gluon self-energy, $\Pi = \mathcal{D}^{-1} - \mathcal{D}_0^{-1}$, is shown in fig. 4. This is not a closed equation since it involves the 3-point and 4-point gluon Green functions. One can also write down the Schwinger-Dyson equations for the 3-point and 4-point functions. Unfortunately this set of equations is still not closed, since an n-point function is always related to the n+1-point and n+2-point functions, ad infinitum. This is what makes quantum field theories so difficult to solve.

For several years Baker, Ball and Zachariasen and coworkers have been studying the infrared limit of the vacuum gluon propagator in the axial gauges and in a truncated version of the full Schwinger-Dyson equations. (See ref. [27] and references contained therein.) They made the ansatz, and subsequently verified, that the dominant infrared singularity in the propagator occurs in only one of the two independent scalar functions. In the temporal axial gauge this is equivalent to setting $F = G$ in eq. (2.10). It then follows that the full propagator equals the bare propagator times some as yet undetermined dimensionless scalar function, $\mathcal{D}(k) = Z(k)\mathcal{D}_0(k)$. When determining $Z(k)$ from fig. 4 the four-gluon terms drop out. The main assumption is that the remaining full three-gluon vertex can be replaced by its longitudinal part, which in turn can be expressed in terms of the inverse gluon propagator via a Slavnov-Taylor-Ward identity. This leads to a closed equation for the gluon propagator as shown in fig. 5. It leads to an infrared behaviour of the desired kind, namely $\mathcal{D} \sim M^2/k^4$, where M is some energy scale.

Unfortunately this equation is not adequate for our purpose, namely the infrared limit at high temperature as opposed to zero temperature. First, and most obviously, it has been reduced via a verifiable ansatz (at zero temperature anyway) to $F = G$. This means that the difference between electric mass and magnetic mass has been lost, which is just the effect we are interested in investigating. Second, it was pointed out [27] that the assumption that the longitudinal piece of the 3-gluon vertex dominates the transverse piece is equivalent, in QED, to the

observation that magnetic forces fall off more rapidly with distance than do electric forces. From perturbation theory, anyway, we expect just the opposite situation in QCD at high temperature (magnetic mass less than electric mass). Finally, the above equation is gauge dependent everywhere except in the infrared limit. In particular M is gauge dependent. We note that work has been done to relax the restrictions which cause these problems [28].

A rather similar truncated Schwinger-Dyson equation has been studied by Mandelstam in the covariant gauges [29]. This is shown in fig. 6. The idea here is that confinement in the vacuum is caused by an enhancement of the low frequency gluon modes. In a naive sense one has real, bare gluons interacting with a self-consistently generated, full gluon field which plays a role analogous to an external field. It was shown that this equation does provide confinement in the sense of an infrared behaviour of k^{-4} for the gluon propagator.

However, Mandelstam's approximation scheme does not allow one to calculate higher order corrections which converge to the exact propagator, since these higher order corrections are not defined. Furthermore the approximation scheme explicitly violates the gauge symmetry.

The equation we shall study at high temperature is shown in fig. 7 [9]. This is just a one-loop, self-consistent approximation to the full Schwinger-Dyson equation shown in fig. 4. It is obtained by expanding in terms of the exact propagator and bare vertices and keeping only the one-loop terms. Clearly a self-consistent equation is necessary to study the non-perturbative effects we are interested in. The one-loop approximation ought to be sufficient since, after all, the effective coupling is decreasing to zero as the temperature goes to infinity. Whether the one-loop approximation really is sufficient to determine the leading infrared behaviour can only be justified a posteriori. An important point is that the one-loop approximation does not violate the gauge symmetry. As emphasized by Coleman and Weinberg [30] a loop expansion is an expansion in powers of the Lagrangian. If one does a loop expansion and keeps all terms up to a fixed number of loops, then one will not violate any of the symmetries of the Lagrangian. What

this means for the gluon propagator which, except possibly for certain limits, is in general gauge dependent, must be carefully stated. Suppose we calculate the gluon propagator in two different gauges, and denote the results by $\mathcal{D}_a$ and $\mathcal{D}_b$. The gauge transformation which connects these two gauges is denoted by Ω_{ab} . If the calculation scheme we employ to calculate $\mathcal{D}$ respects the gauge symmetry then, knowing $\mathcal{D}_a$, we can apply Ω_{ab} to obtain the propagator in the b gauge, and this agrees with $\mathcal{D}_b$. If the approximation scheme violates the gauge symmetry, then the quantity we get by gauge transforming $\mathcal{D}_a$ will not be the same as $\mathcal{D}_b$.

The unfortunate aspect of the equation we study is that it cannot lead to a confining vacuum. Therefore we cannot study the phase transition with this truncated Schwinger-Dyson equation. If we replaced one of the full propagators in the first diagram of fig. 7 with the bare propagator, then we would be led to Mandelstam's equation, fig. 6, in the temporal axial gauge. This would lead to confinement but it explicitly violates the gauge symmetry.

4.2 Weak coupling non-perturbative solutions

Recall that in the temporal axial gauge the gluon propagator takes the form given in eqs. (2.9) and (2.10). The diagrams in fig. 7 may be evaluated with the full propagator. After some quite lengthy algebra we arrive at

$$\begin{aligned}
 \Pi_{00}(k_0, \vec{k}) = & g^2 N T \sum_n \int \frac{d^3 p}{(2\pi)^3} \left\{ 2 \alpha(p) + \frac{p^2}{p_0^2} \beta(p) \right. \\
 & - \frac{1}{2} (p_0 - q_0)^2 \left[\alpha(p) \alpha(q) + \frac{q^2}{q_0^2} \alpha(p) \beta(q) + \frac{p^2}{p_0^2} \alpha(q) \beta(p) \right. \\
 & \left. \left. + \left(\alpha(p) - \frac{p^2}{p_0^2} \beta(p) \right) \left(\alpha(q) - \frac{q^2}{q_0^2} \beta(q) \right) \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \right\},
 \end{aligned} \tag{4.1}$$

$$\begin{aligned}
\frac{1}{2} \Pi_{ii}(k_0, \vec{k}) &= g^2 N T \sum_n \int \frac{d^3 p}{(2\pi)^3} \left\{ 2 \alpha(p) + \frac{p^2}{p_0^2} \beta(p) \right. \\
&- \frac{1}{4} \beta(p) \beta(q) \frac{p^2 q^2}{p_0^2 q_0^2} \vec{k}^2 \left[1 - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \\
&- \frac{1}{4} \alpha(p) \beta(q) \frac{q^2}{q_0^2} \left[\vec{k}^2 + 4 \vec{p} \cdot \vec{q} + 6 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{q}^2} + \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \\
&- \frac{1}{4} \alpha(q) \beta(p) \frac{p^2}{p_0^2} \left[\vec{k}^2 + 4 \vec{p} \cdot \vec{q} + 6 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2} + \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} \right] \\
&- \frac{1}{4} \alpha(p) \alpha(q) \left[q \vec{k}^2 - 20 \vec{p} \cdot \vec{q} - 7 \vec{k}^2 \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2 \vec{q}^2} + 12 \frac{(\vec{p} \cdot \vec{q})^3}{\vec{p}^2 \vec{q}^2} \right] \Bigg\}. \tag{4.2}
\end{aligned}$$

Here we use the more compact notation of eq. (2.21). Also $p_\mu + q_\mu + k_\mu = 0$, with $k_0 = 2\pi nT$, $p_0 = 2\pi mT$; m and n are integers. One may check that eqs. (4.1) and (4.2) reduce to the correct perturbative results, eqs. (3.1) and (3.2), in the limit $F = G = 0$.

The eqs. (4.1)-(4.2) are a pair of coupled integral equations for two unknown scalar functions each of which depends on two variables. These equations also need to be regularized. Consequently they are quite difficult to solve. Fortunately it is not difficult to extract the static infrared limits of F and G in the weak coupling domain.

Taking the static infrared limit, $k_0 = 0$ and then $|\vec{k}| \rightarrow 0$, we find

$$m_{el}^2 \equiv F(0,0) = \pi_{00}(0,0) = g^2 N T \sum_n \int \frac{d^3 p}{(2\pi)^3} \left[\frac{p^2}{p_0^2 (p^2 + F(p))} + \frac{2}{p^2 + G(p)} - \frac{4 p_0^2}{(p^2 + G(p))^2} - \frac{2}{p_0^2} \left(\frac{p^2}{p^2 + F(p)} \right)^2 \right], \quad (4.3)$$

$$m_{mag}^2 \equiv G(0,0) = \frac{1}{2} \pi_{ii}(0,0) = g^2 N T \sum_n \int \frac{d^3 p}{(2\pi)^3} \left[\frac{p^2}{p_0^2 (p^2 + F(p))} + \frac{2}{p^2 + G(p)} - \frac{2 \vec{p}^2}{(p^2 + G(p))^2} - \frac{1}{p_0^2} \frac{p^2}{p^2 + F(p)} \frac{\vec{p}^2}{p^2 + G(p)} \right]. \quad (4.4)$$

At this point we simply define the electric mass and the magnetic mass in the above fashion. In section 5 we will show that indeed they provide for the screening of static electric and magnetic fields, respectively.

Consider the lowest order perturbative contributions to m_{el}^2 and m_{mag}^2 . They are obtained by setting $F = G = 0$ in the right hand sides of eqs. (4.3) and (4.4). In this one-loop calculation there are two contributions, a finite temperature-dependent part and an infinite temperature-independent part. As discussed in section 3, dimensional regularization renders the temperature-independent part zero, which is consistent with gauge symmetry. The remaining finite parts are [eqs. (3.26) and (3.27)]

$$m_{el}^2 = \frac{1}{3} g^2 N T^2, \quad (4.5)$$

$$m_{mag}^2 = 0. \quad (4.6)$$

These are the same results one obtains in the Feynman gauge [5, 2]. They are precisely analogous to what one finds in a non-relativistic electron gas [31] and in an ultra-relativistic electron gas [32]. In fact it is known that in QED plasma the magnetic mass is identically zero.

One would at first think that corrections to these values of m_{el}^2 and m_{mag}^2 would be of order $g^4 T^2$ and would be obtained by expanding the integrands in eqs. (4.3) and (4.4) to first order in F and G . One would insert the lowest order g^2 results for F and G including the full momentum dependence. In this way one would obtain an iterative solution to eqs. (4.1)-(4.2) and also to eqs. (4.3)-(4.4). However one cannot obtain iteratively such a solution because of an infrared divergence. If one attempts the expansion

$$\frac{1}{p^2 + F(p)} = \frac{1}{p^2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{F(p)}{p^2} \right]^n \quad (4.7)$$

then one finds an increasingly worse infrared divergence at increasing values of n since $F(0,0) \neq 0$. The situation with respect to G is better since $G(0,0) = 0$ in lowest order. The situation is similar to what happens when one tries to calculate the thermodynamic potential to order g^4 . There one encounters the infrared divergence by expanding $\ln(1+F(p)/p^2)$. See the introduction and ref. [5].

To alleviate this problem and to pick out the dominant non-perturbative effect as $g \rightarrow 0$, we write

$$\frac{1}{p^2 + F(p, \vec{p})} = \frac{1}{p^2 + F(0,0)} + \left[\frac{1}{p^2 + F(p, \vec{p})} - \frac{1}{p^2 + F(0,0)} \right]. \quad (4.8)$$

The first term on the right hand side picks out the dominant infrared behaviour. The second contributes to higher order. We should insert eq. (4.8), together with $G(p) = 0$, into eq. (4.3) and evaluate the sum. Recall that the sum can be converted to contour integrals by the formula eq. (3.4). The first integral, the vacuum

integral, does not depend on the temperature explicitly but implicitly through $F(0,0) \sim \frac{1}{3} g^2 N T^2$. However, as is easily seen, it is less infrared singular than the second integral, the matter integral, by one power of the energy. Thus it is less important than the second integral by a factor of g or possibly even $g \ln g^2$. Therefore we may neglect it. The matter integral contour may be closed and the residues at the poles picked up to yield

$$m_{el}^2 = \frac{g^2 N}{2\pi^2} \int_0^\infty dp \left[-\frac{m_{el}^2}{\omega_{el}^5} \frac{3p^4 - 2p^2 m_{el}^2 + m_{el}^4}{\exp(\omega_{el}/T) - 1} + \frac{4p}{\exp(p/T) - 1} \right]. \quad (4.9)$$

Here $\omega_{el}^2 = p^2 + m_{el}^2$.

To leading order we may set $m_{el}^2 = \frac{1}{3} g^2 N T^2$ on the right hand side of eq. (4.9). In general the integral cannot be evaluated in closed form. However, since $m_{el} \sim gT \ll T$, we may employ the methods of ref. [21] to find the expansion in the small quantity $m_{el}/T \equiv \bar{m}$. The integrals we need are

$$\begin{aligned} \int_0^\infty \frac{dx}{y} \frac{1}{\exp(y) - 1} &= \frac{\pi}{2\bar{m}} + \frac{1}{2} \ln \frac{\bar{m}}{4\pi} + \frac{1}{2} C_{Euler} + \dots \\ \int_0^\infty \frac{dx}{y^3} \frac{1}{\exp(y) - 1} &= \frac{\pi}{4\bar{m}^3} - \frac{1}{2\bar{m}^2} + \frac{\pi}{24\bar{m}} + \dots \\ \int_0^\infty \frac{dx}{y^5} \frac{1}{\exp(y) - 1} &= \frac{3\pi}{16\bar{m}^5} - \frac{1}{3\bar{m}^4} + \frac{\pi}{48\bar{m}^3} + \dots \end{aligned} \quad (4.10)$$

$$y^2 = x^2 + \bar{m}^2.$$

The final result is

$$m_{el}^2 = \frac{1}{3} g^2 N T^2 - \frac{15}{16\pi} \left(\frac{1}{3} g^2 N \right)^{3/2} T^2. \quad (4.11)$$

The analysis of the equation for the magnetic mass follows in the same way with one exception. The last term in eq. (4.4) corresponds to a coupling between longitudinal and transverse oscillations of the gluon field. For this term one cannot neglect the momentum dependence of F and G because they are different. It behaves as $-\vec{p}^2/p_0^2 p^2$ and not as $-\vec{p}^2/p_0^2 [p^2 + F(0,0)]$. To see this note that it can be written as

$$-\frac{\vec{p}^2}{p_0^2} \frac{1}{p^2 + G(p_0, \vec{p})} + \frac{\vec{p}^2}{p_0^2} \frac{F(p_0, \vec{p})}{F(p_0, \vec{p}) - G(p_0, \vec{p})} \left[\frac{1}{p^2 + G(p_0, \vec{p})} - \frac{1}{p^2 + F(p_0, \vec{p})} \right]. \quad (4.12)$$

This is convenient in order to separate the product of two different pole contributions. We may safely expand the pole $1/[p^2 + G(p_0, \vec{p})]$ and keep only the zeroth order term as before. Now, however, we may also expand the pole $1/(p^2 + m_{e1}^2)$ and keep the first two terms to obtain

$$-\frac{\vec{p}^2}{p_0^2 p^2} + \frac{\vec{p}^2}{p_0^2} R(p_0, \vec{p}) \frac{m_{e1}^2}{(p^2)^2}, \quad (4.13)$$

where $R = F/(F - G)$. From eqs. (2.26), (3.17) and (3.18) it is easy to show that, in Minkowski space, $R(p_0 = |\vec{p}|) = 0$ for any value of $|\vec{p}|$. Therefore, when performing the matter contour integral, this term will not be sufficiently infrared singular to contribute in the order g^3 . It is possible though that this term will contribute in the orders g^4 or $g^4 \ln g^2$, either because of the cuts in the self-energy, or because of the contribution from the vacuum integral.

With this knowledge we find that eq. (4.4) reduces in lowest non-trivial order to

$$m_{mag}^2 = \frac{g^2 N}{2\pi^2} \int_0^\infty dp p^2 \frac{m_{e1}^2}{\omega_{e1}^3} \frac{1}{\exp(\omega_{e1}/T) - 1}, \quad (4.14)$$

and hence to

$$m_{mag}^2 = \frac{3}{8\pi} \left(\frac{1}{3} g^2 N \right)^{3/2} T^2. \quad (4.15)$$

The results obtained here, eqs. (4.11) and (4.15), are remarkable. The order g^3 contributions are essentially non-perturbative in origin since the self-energy has a formal perturbative expansion in powers of g^2 . Such a magnetic mass does not appear in a QED plasma. As we will show in section 5, it corresponds to the screening of static magnetic fields, which is called the Meissner effect in superconductivity. A magnetic mass is generated in a QED superconductor. A theorem due to Schafroth [33] shows that in QED the Meissner effect cannot be obtained by summing the electron-phonon interaction to any finite order in the fine structure constant $e^2/4\pi$. The analogy with the gluon plasma is striking.

One might wonder whether there are any other diagrams which would contribute to order g^3 and which are not included in fig. 7. The answer seems to be no. The non-perturbative effect arises from the self-energy insertions calculated in order g^2 , just as the plasmon effect in order g^3 in the thermodynamic potential arises from the same self-energy insertions. Iteration of the one-loop equation yields this dominant g^3 effect. For the static, three dimensional version of the theory in a covariant gauge, it has been shown explicitly that indeed the infrared divergences in the perturbative two-loop diagrams all arise from the self-energy insertions [34].

4.3 Other approaches for calculating the infrared limit

The static, infrared cut-off for magnetic fields, or the transverse part of the propagator, was expected to be of order $g^2 T$ [8]. See also refs. [2, 35]. However, the coefficient was predicted to be uncalculable in the normal sense, since an infinite number of diagrams were expected to contribute. As shown in the last subsection this infinite number of diagrams is readily classified and summed to give the unexpected result $cg^{3/2}T$. Despite this pessimism a number of approaches were used to try to get at the infrared limit. We wish to quickly summarize them here.

Based on a truncated Schwinger-Dyson equation similar to that of Baker, Ball and Zachariasen, and Mandelstam, an infrared cut-off of order $g^2 T$ was determined in an arbitrary covariant gauge [36]. However, since the coefficient depended upon the gauge fixing parameter ξ , and could even vanish for $\xi = -1$, the significance of this result is not clear.

At least in some gauges, it was suggested that one could study the static infrared sector of the full four-dimensional theory by considering only the $n = 0$ modes and in three dimensions [2, 25, 34, 37]. The relationship of that approach to ours is not clear. Certainly there would be problems to do that in the temporal axial gauge. Anyway it has not yet proven possible to calculate the magnetic mass in this manner.

An estimate was made of the possible instanton contribution to the magnetic mass but was found to be quite negligible [38].

The first real evidence that the magnetic infrared cut-off really existed was provided by Monte Carlo simulations on the lattice [39, 40]. Although these results were shown to be consistent with a $g^2 T$ behaviour, we would like to show that they are also consistent with a $g^{3/2} T$ behaviour.

The first paper calculated the free energy $T\mu$ of one unit of magnetic flux in a box of volume $(N_s a)^3$, where a is the lattice spacing. The expectation is that this quantity falls off exponentially with the cross sectional area perpendicular to the direction of flux:

$$\mu = c \exp \left[- m_{mag}^2 (T) (N_s a)^2 \right]. \quad (4.16)$$

Here c is a normalization constant (parameter) and $N_s = 4$ was the number of spatial lattice sites in one linear dimension. The temperature is $T = 1/N_t a$, where $N_t = 2$ was used for the number of temporal lattice sites. Defining (γ is a second parameter)

$$b(T) = \frac{4}{g^2(T)} = \frac{11}{3\pi^2} \ln(\gamma T/\Lambda), \quad (4.17)$$

we have

$$b(T) \simeq b_0 - \frac{11}{3\pi^2} \ln(57.5 N_t / \gamma). \quad (4.18)$$

Then $b_0 = 4/g_0^2$, where g_0 is the bare coupling constant. The Monte Carlo procedure actually was able to calculate only the derivative $\partial\mu/\partial b_0$. Assuming that $m_{\text{mag}}^2(T) = \kappa g^2(T)T^2$ and $\gamma = 4$ [5] the data were fit with $\kappa = 0.24$ and $c = 1.53$. Nevertheless the data may equally well be fit by assuming eq. (4.15) and fitting the parameters c and γ . This is alright since the parameter γ cannot be determined unambiguously anyway without including the two-loop correction to eq. (4.17). Thus we fit the data in fig. 8 with the formulas

$$\frac{\partial\mu}{\partial b_0} = c \frac{12}{\pi} \sqrt{\frac{2}{3}} b^{-5/2}(T) \exp\left[-\frac{8}{\pi} \sqrt{\frac{2}{3}} b^{-3/2}(T)\right],$$

$$b(T) = b_0 - \frac{11}{3\pi^2} \ln(115/\gamma). \quad (4.19)$$

We find that $c = 1.65$ and $\gamma = 1.6$. The uncertainty in γ is $+0.8, -0.5$. This fit is as good as the original. Of course the reason the Monte Carlo data cannot distinguish between the fits is that the difference in temperature dependence in $m_{\text{mag}}^2(T)$ is $\sqrt{\ln(\gamma T/\Lambda)}$. That is a very weak dependence indeed! We should also point out that $g^2(T)$, for the range of T used in this Monte Carlo study, is really not so small that corrections to eq. (4.15) are negligible.

The second paper [40] worked at only one temperature but varied the spatial size of the lattice to show magnetic screening more directly. Their result is

$$m_{mag} = (0.79 \pm 0.08) T \quad (4.20)$$

at $T = 1/2a$ and $b_0 = 2.6$. Since

$$T = \frac{1}{2} \frac{\Lambda}{57.5} \exp(3\pi^2 b_0/11) \quad (4.21)$$

a fit of eq. (4.15) to eq. (4.20) using eq. (4.17) yields $\gamma = 1.14 (+0.5, -0.3)$.

This is consistent with the other Monte Carlo calculations.

Finally we note that D'Hoker has been studying the infrared limit in the pure gauge theory in two space and one time dimension [41, 42]. It would be interesting to extend our results from fig. 7 to one less dimension for comparison.

5. LINEAR RESPONSE THEORY OF SCREENING AND OF PLASMA OSCILLATIONS

In this section we shall be interested in probing the structure of the gluon plasma using linear response theory. (See for example ref. [22]). This formalism allows one to calculate the effective electric and magnetic fields induced in the plasma by some externally supplied fields or by inhomogeneities. In particular we will show that both static electric and static magnetic fields are screened. We will also study the frequency spectrum and damping of color plasma oscillations.

One might wonder about the physical significance of these things since the colour electric and magnetic fields are not gauge invariant quantities. As long as one works in a physical gauge without spurious degrees of freedom and which admits a Hamiltonian formalism, we think that the screening and plasma oscillations are of physical significance. For example, one might insert colored and electrified test charges, heavy quarks, into the gluon plasma. Their motion under the influence of the effective electric and magnetic fields could be monitored by observing the photons emitted by the quark when it is accelerated. If one likes one could even arrange for the net assembly of test charges to be color neutral.

We are less certain of the physical significance in the low temperature phase where quarks are confined. At the moment we are not able to do reliable calculations in that phase anyway.

5.1 Linear response theory

Suppose that we couple some external field to the gluon plasma via the coupling Hamiltonian $H^{\text{ext}}(t)$. The goal of linear response theory is to calculate the change in any desired operator $Y(\vec{x}, t)$ caused by this coupling Hamiltonian to first order in the applied field. Since

$$\frac{\partial}{\partial t} Y(\vec{x}, t) = i [H(t), Y(\vec{x}, t)], \quad (5.1)$$

we can determine the change in Y in any given state $|jN\rangle$ to first order by

$$\delta \langle jN | Y(\vec{x}, t) | jN \rangle = i \int_{t_0}^t dt' \langle jN | [H^{\text{ext}}(t'), Y(\vec{x}, t)] | jN \rangle. \quad (5.2)$$

The external field is turned on at time t_0 . Here $|jN\rangle$ labels an eigenstate of the full Hamiltonian and color charge operator of the system in the absence of H^{ext} .

Taking the ensemble average of eq. (5.2) gives

$$\begin{aligned} \delta \langle Y(\vec{x}, t) \rangle &= \sum_{jN} \exp[\beta(\Omega - E_j(N))] \delta \langle jN | Y(\vec{x}, t) | jN \rangle \\ &= i \int_{t_0}^t dt' \text{Tr} \left\{ \hat{\rho}_G [H^{\text{ext}}(t'), Y(\vec{x}, t)] \right\}. \end{aligned} \quad (5.3)$$

Notice the trace over the density matrix $\hat{\rho}_G$.

Equation (5.3) is quite general and useful. We are interested in applying it to the gluon plasma in the temporal axial gauge $A_0^a(\vec{x}, t) = 0$. The basic equations of the Hamiltonian formalism in this gauge were given in section 2, eqs. (2.3)-(2.7).

5.2 Screening of static electric fields

Suppose a static electric field $\mathbf{E}_1^a(\vec{x})$, but no magnetic field, is applied to the system. The coupling Hamiltonian is

$$H^{\text{ext}}(t) = \int d^3x \mathcal{H}^{\text{ext}}(\vec{x}, t),$$

$$\mathcal{H}^{\text{ext}}(\vec{x}, t) = E_j^a(\vec{x}, t) \mathcal{E}_j^a(\vec{x}). \quad (5.4)$$

From eq. (5.3) the electric field induced in the medium is

$$\delta \langle E_i^a(\vec{x}, t) \rangle = i \int_{t_0}^t dt' \int d^3x' \mathcal{E}_j^b(\vec{x}') \text{Tr} \left\{ \hat{\rho}_G [E_j^b(\vec{x}', t'), E_i^a(\vec{x}, t)] \right\}. \quad (5.5)$$

The retarded real-time Green function is

$$i D_{ij}^{abR}(\vec{x}, t; \vec{x}', t') = \text{Tr} \left\{ \hat{\rho}_G [A_i^a(\vec{x}, t), A_j^b(\vec{x}', t')] \right\} \Theta(t - t'). \quad (5.6)$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial}{\partial t'} i D_{ij}^{abR}(\vec{x}, t; \vec{x}', t') &= \Theta(t - t') \text{Tr} \left\{ \hat{\rho}_G [E_i^a(\vec{x}, t), E_j^b(\vec{x}', t')] \right\} \\ &+ \delta(t - t') \text{Tr} \left\{ \hat{\rho}_G [E_i^a(\vec{x}, t), A_j^b(\vec{x}', t')] \right\} \\ &- \delta(t - t') \text{Tr} \left\{ \hat{\rho}_G [A_i^a(\vec{x}, t), E_j^b(\vec{x}', t')] \right\} \\ &- \delta'(t - t') \text{Tr} \left\{ \hat{\rho}_G [A_i^a(\vec{x}, t), A_j^b(\vec{x}', t')] \right\}, \end{aligned} \quad (5.7)$$

since $E_i^a(\vec{x}, t) = -\partial A_i^a(\vec{x}, t)/\partial t$, etc. In eq. (5.5) we can let $t_0 \rightarrow -\infty$. Then we insert eq. (5.7) into eq. (5.5) and use the canonical commutation relations

$$[A_i^a(\vec{x}, t), E_j^b(\vec{x}', t)] = -i \delta_{ij} \delta^{ab} \delta(\vec{x} - \vec{x}') \quad (5.8)$$

to obtain

$$\delta \langle E_i^a(\vec{x}, t) \rangle = -E_i^a(\vec{x}) + \int_{-\infty}^{\infty} dt' \int d^3x' E_j^b(\vec{x}') \frac{\partial}{\partial t} \frac{\partial}{\partial t'} D_{ij}^{abR}(\vec{x}, t; \vec{x}', t'). \quad (5.9)$$

From translation invariance the retarded Green function depends only on the differences $\vec{x} - \vec{x}'$ and $t - t'$. We introduce the Fourier transforms

$$D_{ij}^{abR}(\vec{x} - \vec{x}', t - t') = \frac{1}{(2\pi)^4} \int d^3q d\omega e^{i\vec{q} \cdot (\vec{x} - \vec{x}') - i\omega(t - t')} D_{ij}^{abR}(\omega, \vec{q}). \quad (5.10)$$

and

$$E_j^b(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3p e^{i\vec{p} \cdot \vec{x}} E_j^b(\vec{p}). \quad (5.11)$$

The net effective electric field in the plasma is then

$$E_i^a(\vec{x})_{\text{eff}} = \frac{1}{(2\pi)^3} \int d^3q e^{i\vec{q} \cdot \vec{x}} E_j^b(\vec{q}) \lim_{\omega \rightarrow 0} [\omega^2 D_{ij}^{abR}(\omega, \vec{q})]. \quad (5.12)$$

The retarded real-time Green function is simply related to the temperature Green function by analytic continuation. As shown in section 3 we must analytically continue in the variable $\omega_n = 2\pi nT$ in the following way:

$$D_{ij}^{abR}(\omega, \vec{q}) = \mathcal{D}_{ij}^{ab}(\omega_n \rightarrow \omega + i\eta, \vec{q}), \quad \eta \rightarrow 0^+. \quad (5.13)$$

Making use of the structure of the temperature Green function from eqs. (2.9)-(2.10) we obtain finally

$$\varepsilon_i^a(\vec{q})_{\text{eff}} = \frac{\varepsilon_i^a(\vec{q})}{1 + \Pi_{00}(0, q)/q^2} \quad (5.14)$$

We emphasize that this is an exact result to be used with the exact expression for Π_{00} or any approximation to it. The only assumption is that the applied field $\varepsilon_i^a(\vec{q})$ is weak enough to justify the linearity approximation.

The proportionality factor in eq. (5.14) is the screening factor. In the limit of no interactions, $g \rightarrow 0$, $\Pi_{00} \rightarrow 0$, and the effective field equals the applied field. In general Π_{00} can be split into a temperature independent vacuum part and a temperature dependent matter part. In the limit $q \rightarrow \infty$ the vacuum part dominates,

$$\Pi_{00}(0, q)/q^2 \xrightarrow{q \rightarrow \infty} \frac{11}{48\pi^2} g^2 N \ln(q^2/M^2). \quad (5.15)$$

This is the lowest order result in g given by eq. (3.10) after renormalization, which introduces the renormalization energy scale M . In the limit $q \rightarrow 0$ the matter part dominates,

$$\Pi_{00}(0, q)/q^2 \xrightarrow{q \rightarrow 0} m_{e1}^2/q^2, \quad (5.16)$$

where $m_{e1}^2 \equiv \Pi_{00}(0,0)$. This justifies the definition of electric mass-squared as $\Pi_{00}(0,0)$. Approximately, then, the screening factor can be written as

$$1 + \frac{11}{48\pi^2} g^2 N \ln(q^2/M^2) + m_{e1}^2 / q^2 \quad (5.17)$$

This factor exhibits the screening of electric fields in the ultraviolet due to asymptotic freedom, and the screening of electric fields in the infrared due to thermal fluctuations.

5.3 Screening of static magnetic fields

Suppose a static magnetic field $\mathcal{B}_i^a(\vec{x})$, but no electric field, is applied to the system. Then the coupling Hamiltonian density is

$$\mathcal{H}^{\text{ext}}(\vec{x}, t) = B_j^b(\vec{x}, t) \mathcal{B}_j^b(\vec{x}). \quad (5.18)$$

From eq. (5.3) the magnetic field induced in the medium is

$$\delta \langle B_i^a(\vec{x}, t) \rangle = i \int_{t_0}^t dt' \int d^3x' \mathcal{B}_j^b(\vec{x}') \text{Tr} \left\{ \hat{\rho}_0 [B_j^b(\vec{x}', t'), B_i^a(\vec{x}, t)] \right\}. \quad (5.19)$$

Recall that

$$B_i^a = \epsilon_{ijk} \left(\nabla_j A_k^a + \frac{1}{2} g f^{abc} A_j^b A_k^c \right). \quad (5.20)$$

If we were to use eq. (5.20) in eq. (5.19) we would need to know the three and four-point gluon vertices! Since that is outside the scope of this paper we shall calculate only the lowest order contribution. We use $B_i^a = \epsilon_{ijk} \nabla_j A_k^a$. Note that we do not need to make this approximation for the applied magnetic field $\mathcal{B}$.

After carrying out some manipulations similar to those in the previous subsection we arrive at

$$\begin{aligned} \mathcal{B}_i^a(\vec{x})_{\text{eff}} &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}} \mathcal{B}_j^b(\vec{q}) \\ &\times \lim_{\omega \rightarrow 0} \left[\varepsilon_{ilm} \varepsilon_{jkn} q_l q_k D_{mn}^{abR}(\omega, \vec{q}) \right]. \end{aligned} \quad (5.21)$$

We relate the real-time Green function to the temperature Green function, perform the sum over spatial indices, and then take the limit $\omega \rightarrow 0$. This leads to the effective magnetic field

$$\mathcal{B}_i^a(\vec{q})_{\text{eff}} = \frac{\delta_{ij} - q_i q_j / q^2}{1 + \frac{1}{2} \pi_{ll}(0, q) / q^2} \mathcal{B}_j^a(\vec{q}). \quad (5.22)$$

In contrast to eq. (5.14) this result assumes not only that the applied field is weak but also that the coupling constant is small.

The proportionality factor in eq. (5.22) is easy to understand. We assumed that $\vec{B} = \vec{\nabla} \times \vec{A}$, so that $\vec{\nabla} \cdot \vec{\mathcal{B}}_{\text{eff}} = 0$ even if $\vec{\nabla} \cdot \vec{\mathcal{B}} \neq 0$. In the limit that $g \rightarrow 0$ then of course $\vec{\mathcal{B}}$ can always be expressed as $\vec{\nabla} \times \vec{A}$. Also $\pi_{ll} = 0$ when $g = 0$, so that $\vec{\mathcal{B}}_{\text{eff}} = \vec{\mathcal{B}}$.

In general the proportionality factor in eq. (5.22) is the screening factor. The self-energy π_{ll} can be split into a temperature independent vacuum part and a temperature dependent matter part. In the limit $q \rightarrow \infty$ the vacuum part dominates,

$$\frac{1}{2} \pi_{ll}(0, q) / q^2 \xrightarrow{q \rightarrow \infty} \frac{11}{48\pi^2} g^2 N \ln(q^2 / M^2), \quad (5.23)$$

which is the lowest order result in g . In the limit $q \rightarrow 0$ the matter part dominates

$$\frac{1}{2} \pi_{\ell\ell}(0, q) / q^2 \xrightarrow{q \rightarrow 0} m_{\text{mag}}^2 / q^2, \quad (5.24)$$

where $m_{\text{mag}}^2 = \frac{1}{2} \pi_{\ell\ell}(0, 0)$. This justifies the definition of magnetic mass-squared as $\frac{1}{2} \pi_{\ell\ell}(0, 0)$. Approximately, then, the screening factor can be written as

$$\frac{1}{1 + \frac{11}{48\pi^2} g^2 N \ln(q^2/M^2) + m_{\text{mag}}^2 / q^2} \quad (5.25)$$

Like eq. (5.17), this factor exhibits the screening of magnetic fields in the ultra-violet due to asymptotic freedom, and the screening of magnetic fields in the infrared due to thermal fluctuations.

5.4 Effective charges and beta function at finite temperature

The problem of how to define and then calculate an effective coupling constant or renormalization group beta function at given four-momentum transfer at finite temperature or density is an interesting one [43-45]. However, it is clear from eqs. (5.14) and (5.22) that such a unique effective coupling constant does not exist. The precise definition of a temperature or density dependent effective coupling depends upon the physical circumstances, for instance whether one is speaking of static electric or static magnetic interactions. The existence of an infrared cut-off on static electric interactions does not necessarily prove the absence of all infrared instabilities in the system, since it may have turned out that the magnetic mass was zero and that magnetic interactions were long range. Furthermore what happens in time dependent situations?

For the purposes of this paper let us restrict ourselves to the static limit and discuss a little the renormalization group properties of the effective electric

and magnetic couplings defined by eqs. (5.14) and (5.22). They can be written approximately as [see eqs. (5.17) and (5.25)]

$$\bar{g}^2(q, M, T, g(M)) = \frac{g^2(M)}{1 + \frac{11}{48\pi^2} g^2(M) N \ln\left(\frac{q^2}{M^2}\right) + \frac{m^2(g^2(M), T)}{q^2}} \quad (5.26)$$

Here $m^2 = m_{el}^2$ or m_{mag}^2 , which in either case is proportional to T^2 . On the right side of eq. (5.26) $g^2(M)$ is the renormalized coupling constant at the energy scale M . Although M is arbitrary and $g^2(M)$ is arbitrary, $g^2(M)$ actually runs or depends on M in a specific way predicted by the renormalization group. To see this in a restricted sense, multiply the numerator and the denominator of $\bar{g}^2$ by $1/g^2(M)$.

$$\frac{1}{\bar{g}^2} = \frac{1}{g^2(M)} + \frac{11}{48\pi^2} N \ln(q^2/M^2) + m^2(g^2(M), T)/g^2(M) q^2. \quad (5.27)$$

Define a new parameter Λ by

$$\frac{1}{g^2(M)} = \frac{11}{48\pi^2} N \ln(M^2/\Lambda^2). \quad (5.28)$$

Then

$$\frac{1}{\bar{g}^2} = \frac{11}{48\pi^2} N \ln(q^2/\Lambda^2) + m^2(g^2(M), T)/g^2(M) q^2. \quad (5.29)$$

Now, as usual, the convenient choice to make for M is γT , where γ is of order unity. This choice suppresses dangerous logarithms of M^2/T^2 which occur in higher orders of a weak coupling expansion. Putting everything together we find

$$\bar{g}^2(q, \Lambda, T) = \frac{1}{\frac{11}{48\pi^2} N \ln(q^2/\Lambda^2) + m^2(g^2(T), T)/g^2(T) q^2}, \quad (5.30)$$

where

$$g^2(T) = \frac{1}{\frac{11}{48\pi^2} N \ln(T^2/\Lambda^2)}. \quad (5.31)$$

So, for instance,

$$m_{e_1}^2(g^2(T), T) = \frac{1}{3} g^2(T) N T^2 + \dots \quad (5.32)$$

We can change T , and we can change q , but the one parameter which is fixed and over which we have no control is Λ . This sets the intrinsic energy scale of the theory. In principle it may be deduced from the string tension at zero temperature, or from deep inelastic e - N scattering, or from e^+e^- annihilation experiments. Note, however, that the numerical value of Λ depends on the gauge and on the renormalization scheme in a computable way.

Since $\bar{g}^2(q, \Lambda, 0)$ is the well-known running coupling constant in the vacuum, one may interpret eq. (5.30) as the running coupling constant at finite temperature. The effective coupling runs differently in the electric and magnetic directions.

5.5 Plasma oscillations

Suppose that instead of applying static electric or magnetic fields to the plasma we hit it with an impulsive perturbation of the form $\delta(t) \exp(i\vec{q} \cdot \vec{x})$. Then the poles of the propagator tell us the spectrum of oscillation frequencies. This can be shown in the usual way using linear response theory. To be precise the

poles of the retarded real-time Green function give the oscillation frequencies of the colour density waves. At least this is the case in the temporal axial gauge which admits a Hamiltonian formulation. Although we shall quote some results from the covariant gauges, we are not certain of their physical interpretation.

Recalling the most general form of the gluon propagator at finite temperature, eqs. (2.9)-(2.10), we see that the poles occur at

$$k_0^2 - \vec{k}^2 - F(k_0, \vec{k}) = 0, \quad (5.32)$$

$$k_0^2 - \vec{k}^2 - G(k_0, \vec{k}) = 0. \quad (5.33)$$

This is in Minkowski space. The former corresponds to longitudinal oscillations, the latter to transverse oscillations. In general these equations have complex solutions which we write as $k_0 = \omega - i\gamma$. Then ω is the real oscillation frequency and $\gamma > 0$ is the damping factor. We shall assume that $\gamma \ll \omega$. If this is not true then the oscillations will be overdamped and will not propagate. Let us examine each of these oscillations in turn.

From the definition of F eq. (5.32) corresponds to

$$\begin{aligned} \vec{k}^2 &= \text{Re } \Pi_{00}(\omega(\vec{k}), \vec{k}), \\ \gamma &\simeq \frac{\text{Im } \Pi_{00}(\omega(\vec{k}), \vec{k})}{2 \text{Re } \Pi_{00} / 2\omega}. \end{aligned} \quad (5.34)$$

From here on k shall refer to $|\vec{k}|$. First consider the long wavelength oscillations with $k \ll \omega$. From eq. (3.30) one easily obtains

$$\text{Re } \Pi_{00}(\omega \gg k) = \omega_p^2 \frac{k^2}{\omega^2} \left(1 + \frac{3}{5} \frac{k^2}{\omega^2} + \dots \right). \quad (5.35)$$

Here we define the plasma frequency [46] $\omega_p^2 = g^2 N T^2 / 9$. The frequency spectrum of long wavelength oscillations is

$$\omega_L^2 = \omega_p^2 + \frac{3}{5} k^2 + \dots \quad (5.36)$$

To obtain the correct damping factor in this limit some care must be exercised regarding the correct analytic continuation for the retarded Green function in terms of the temperature Green function. See subsection 3.3. In particular, in eq. (3.30) one obtains an imaginary part from the terms in $1/(4q^2 - k_0^2)$, but not from the terms in $1/(q^2 - k_0^2)$. This is because the former comes from the expansion of a logarithm and contributes an imaginary part corresponding to particle production, whereas the latter comes from the temporal axial gauge pole at $n \cdot p = 0$ and so the imaginary part is removed by the principal value prescription. One obtains

$$\text{Im } \Pi_{00} = -\frac{1}{24} g^2 N k^2 n(\beta \omega/2) \simeq -\frac{1}{12\pi} g^2 N \frac{k^2 T}{\omega}. \quad (5.37)$$

Finally

$$\gamma_L \simeq \frac{1}{24\pi} g^2 N T; \quad k \ll \omega_p. \quad (5.38)$$

It turns out that there is no solution of eq. (5.34) when one assumes that $\omega \simeq k \gg T$. The reason is that the assumption $\gamma \ll \omega$ is no longer valid: the oscillations are overdamped and do not propagate. This is physically reasonable. At short distances the temperature effects are unimportant. These density oscillations then correspond to essentially free gluons. Longitudinal gluons do not propagate.

For comparison we note that one obtains the same first two terms in eq. (5.36) in the Feynman gauge, so that this result is gauge independent. In fact one obtains

that same result in any covariant gauge [24]. One might suspect that at least the coefficients of the leading order g^2 terms are the same. However, in the Feynman gauge the damping γ is a factor of 5 larger. As noted earlier, this is of no consequence since one can safely speak of plasma oscillations only in a physical gauge.

Turning now to the long wavelength transverse oscillations we recall eq. (3.31)

$$\text{Re } \pi_{ii}(\omega \gg k) = -3\omega_p^2 \left(1 + \frac{1}{3} \frac{k^2}{\omega^2} + \dots \right). \quad (5.39)$$

Then from eq. (5.39) and the definition of G we find

$$\omega_T^2 = \omega_p^2 + \frac{6}{5} k^2 + \dots; \quad k \ll \omega_p. \quad (5.40)$$

For the imaginary part of π_{ii} it is readily found that $\text{Im } \pi_{ii} = 3\omega^2 \cdot \text{Im } \pi_{00} / k^2$.

Then

$$\gamma_T \simeq \frac{1}{24\pi} g^2 N T; \quad k \ll \omega_p. \quad (5.41)$$

Notice that $\gamma_L = \gamma_T$ in the long wavelength limit. We also note that one obtains the first two terms in eq. (5.40) in any covariant gauge, but that the damping is again a factor of 5 larger. See the remarks above.

For short wavelength transverse oscillations, $\omega \gtrsim k \gg T$, we find that

$$\omega_T^2 = k^2 + \frac{3}{2} \omega_p^2 + \dots; \quad k \gg T. \quad (5.42)$$

Furthermore, to order g^2 , these waves are undamped. The physical interpretation is obvious: These oscillations are just transversely propagating real gluons. We note that, again, the first two terms of eq. (5.42) are also obtained in the covariant gauges.

The study of plasma oscillations is a very gratifying one. The relativistic many-body theory is able to unify the highly collective long wavelength color

density oscillations with the short wavelength propagation of real free gluons. The plasma frequency in mass ω_p is not the same as the electric screening mass m_{el} or the magnetic screening mass m_{mag} . The fact that various limits of the gluon propagator control such diverse physical phenomena as electric and magnetic screening and plasma oscillations is a highly non-trivial consequence of linear response theory, the prerequisite for which is the existence of a Hamiltonian formalism.

6. CONCLUSIONS

We have in this paper discussed the behaviour of a gluon plasma at very high temperatures, $T \gg T_c \approx \Lambda_{QCD}$. Although, due to asymptotic freedom, the coupling constant approaches zero when T increases and the system approaches the seemingly trivial system of an ideal gas of gluons, it still exhibits a considerable amount of interesting and highly non-trivial dynamic structure. A particularly noteworthy fact is that in some cases the correct structure can be revealed only by applying self-consistent and non-perturbative methods.

The techniques applied in this paper were those of field theory at non-zero temperatures. These techniques have been extremely useful in condensed matter physics in the eV-domain. Here the temperatures have been at least of the order of GeV and the system considered has been a system of zero-mass bosons with specific and complicated self-interactions. Also in the eV-domain systems of interacting bosons have been notoriously difficult to solve, even with experimental results as a guidance. We find the close relation between the QCD problem treated here and usual condensed matter physics one more example of the beautiful unity of physics.

Of course, one cannot be sure that the results presented here describe the state of affairs in nature until they are verified by terrestrial experiments or celestial observations. This will not be simple. The first step will be to observe that matter really can exist in a state in which colour is non-confined. After that one can proceed to more complicated diagnostic measurements of the properties of the quark-gluon plasma. We note the usefulness and uniqueness of heavy ion

collisions at very high energies for diagnostics of quark-gluon plasma and hope that the results of this paper will serve to enhance the motivation for them.

On the theoretical side there are several problems in the physics of gluon plasmas one should proceed to discuss:

The screening of colour electric and magnetic fields should be computed in the Landau gauge using linear response theory and solving the electric and magnetic masses self-consistently as explained above. In the Landau gauge one more function, corresponding to the exact ghost propagator, enters the problem (section 2). This now involves the self-consistent Schwinger-Dyson equation for the ghost. One more step further in complexity would be the Feynman gauge, for which a fourth independent function enters the problem.

The g^4 contribution to the thermodynamic potential Ω , which is already known for $T = 0$ quark matter, should be computed for gluon plasma. This would shed light on the convergence of the perturbation expansion and the choice of Λ .

One should attempt to find a unified approximation to the exact Schwinger-Dyson equation, which would permit one also to investigate the properties of gluon matter at temperatures close to the colour de-confining phase transition. At present the approximations valid at $T = 0$ and $T \gg T_c$ are unconnected. That many interesting phenomena take place near $T = T_c$ is clear. For example, the negative sign of the $O(g^3)$ correction to m_{el}^2 in eq. (4.11) may indicate that electric screening disappears near $T = T_c$.

Linear response to further types of applied perturbations should be investigated. A particularly interesting problem would be the calculation of colour electric conductivity. Since colour magnetic forces are screened in a gluon plasma one may expect, in analogy with ordinary superconductors, that the gluon plasma would also be a colour electric superconductor. Further, if the phase transition is of the percolation type, it would in the phase transition region behave as a colour conductor and finally, in the colour confining phase, it would be a colour insulator. The calculation of colour conductivity in a microscopic theory requires

the knowledge of the gluon three- and four-point functions and approximation techniques going beyond those presented in this paper would thus have to be developed.

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Figure captions

- Fig. 1 : Plasmon contribution to the thermodynamic potential in a covariant gauge. The lines correspond to: solid-quarks, dashed-ghosts, wavy-gluons.
- Fig. 2 : Feynman rules in temporal axial gauge in Euclidean space.
- Fig. 3 : Gluon self-energy in lowest order perturbation theory in the temporal axial gauge.
- Fig. 4 : The exact Schwinger-Dyson equation for the gluon self-energy. There is an additional diagram if the gauge is not ghost-free.
- Fig. 5 : The approximate Schwinger-Dyson equation studied by Baker, Ball and Zachariasen and coworkers for the vacuum gluon propagator.
- Fig. 6 : The approximate Schwinger-Dyson equation studied by Mandelstam for the vacuum gluon propagator.
- Fig. 7 : The self-consistent one-loop Schwinger-Dyson equation studied in this paper for the gluon self-energy at high temperature.
- Fig. 8 : Plot of $\partial u / \partial b_0$ from the Monte Carlo study of ref. [39]. The curve corresponds to eq. (4.19) with $c = 1.65$ and $\gamma = 1.60$.

$$\begin{aligned}
& - \frac{1}{2} \left[\frac{1}{2} \text{ (loop with 2 } \pi \text{)} - \frac{1}{3} \text{ (loop with 3 } \pi \text{)} + \dots \right] \\
\\
& \text{wavy line} \text{---} (\pi) \text{---} \text{wavy line} = \text{wavy line} \text{---} \text{circle with arrows} \text{---} \text{wavy line} + \text{wavy line} \text{---} \text{circle with arrows} \text{---} \text{wavy line} \\
& \quad - \frac{1}{2} \text{wavy line} \text{---} \text{blob} \text{---} \text{wavy line} - \frac{1}{2} \text{wavy line} \text{---} \text{blob} \text{---} \text{wavy line}
\end{aligned}$$

Fig. 1

$$\mathfrak{D}_{0\mu\nu}^{ab} = \frac{\delta^{ab}}{k^2} \left[A_{\mu\nu} + B_{\mu\nu} - \sqrt{2} \frac{|\vec{k}|}{k_0} C_{\mu\nu} + \frac{\vec{k}^2}{k_0^2} D_{\mu\nu} \right]$$

$$\mathfrak{D}_{\mu\nu}^{ab} = \frac{\delta^{ab}}{k^2 + G} A_{\mu\nu} + \frac{\delta^{ab}}{k^2 + F} \left[B_{\mu\nu} - \sqrt{2} \frac{|\vec{k}|}{k_0} C_{\mu\nu} + \frac{\vec{k}^2}{k_0^2} D_{\mu\nu} \right]$$

$$\Gamma_0^{(3)} = -igf_{abc} \left[\delta_{\beta\gamma} (r-q)_\alpha + \delta_{\alpha\beta} (k-r)_\gamma + \delta_{\gamma\alpha} (q-k)_\beta \right]$$

$$\begin{aligned} \Gamma_0^{(4)} = & -g^2 \left[f_{ade} f_{ebc} (\delta_{\alpha\beta} \delta_{\delta\gamma} - \delta_{\alpha\gamma} \delta_{\delta\beta}) \right. \\ & + f_{abe} f_{edc} (\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\gamma} \delta_{\delta\beta}) \\ & \left. + f_{ace} f_{edb} (\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\beta} \delta_{\delta\gamma}) \right] \end{aligned}$$

Fig. 2

$$\pi = -\frac{1}{2} \text{ [diagram: wavy line to a circle to a wavy line]} - \frac{1}{2} \text{ [diagram: wavy line to a circle on a wavy line]}$$

Fig. 3

$$\begin{aligned} \pi = & -\frac{1}{2} \text{ [diagram: wavy line to a diamond with 4 shaded circles to a wavy line]} - \frac{1}{2} \text{ [diagram: wavy line to a circle with 1 shaded circle on top]} \\ & -\frac{1}{6} \text{ [diagram: wavy line to a diamond with 3 shaded circles on the left, middle, and right]} - \frac{1}{2} \text{ [diagram: wavy line to a diamond with 6 shaded circles on the left, middle, and right]} \end{aligned}$$

+ possible ghost

Fig. 4

$$\pi = -\frac{1}{2} \left[\text{Diagram 1} \right] - \frac{1}{2} \left[\text{Diagram 2} \right]$$

Diagram 1: A diamond-shaped loop with a wavy line on the left and a wavy line on the right. The top and bottom vertices of the diamond are marked with shaded circles. A circle labeled 'L' is attached to the right vertex.

Diagram 2: A circular loop with a wavy line on the left and a wavy line on the right. A shaded circle is attached to the top of the loop.

Fig. 5

$$\pi = -\frac{1}{2} \left[\text{Diagram 3} \right] - \frac{1}{2} \left[\text{Diagram 4} \right]$$

Diagram 3: A circular loop with a wavy line on the left and a wavy line on the right. A shaded circle is attached to the top of the loop.

Diagram 4: A circular loop with a wavy line on the left and a wavy line on the right. A shaded circle is attached to the top of the loop.

+ ghost

Fig. 6

$$\pi = -\frac{1}{2} \left[\text{Diagram 5} \right] - \frac{1}{2} \left[\text{Diagram 6} \right]$$

Diagram 5: A diamond-shaped loop with a wavy line on the left and a wavy line on the right. The top and bottom vertices of the diamond are marked with shaded circles.

Diagram 6: A circular loop with a wavy line on the left and a wavy line on the right. A shaded circle is attached to the top of the loop.

Fig. 7

Fig. 8