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SYMMETRY ALGEBRAS OF THE HEISENBERG MODEL AND THE
NON-LINEAR SCHRÖDINGER EQUATION

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ABSTRACT

The Lax pairs of the Heisenberg model and the non-linear Schrödinger equation are each shown to give rise to a representation of the infinite dimensional graded Lie algebra $su(2) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$.

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1. - It has been known for several years that the stationary, axially symmetric Einstein equations possess an infinite dimensional algebra of infinitesimal symmetry transformations¹⁾ which is isomorphic to the $\mathbb{Z}$ graded Lie algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$ of $sl(2, \mathbb{R})$ valued Laurent polynomials in the undetermined λ . Recently, analogous results have been established²⁾ for the two-dimensional chiral field theories [a class of models with non-Abelian global symmetries which can be integrated by applying the Riemann transform method to the associated Lax pair of linear differential equations³⁾] and also for the self-dual sector of the four-dimensional $SU(N)$ Yang-Mills theory.

In this note, the symmetry algebra of another two-dimensional classical field theory, the Heisenberg model, is constructed. It is no surprise to find an algebra isomorphic to $su(2) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$ due to the global $SU(2)$ invariance and the integrability of the model. Somewhat less expected might be the fact that this symmetry algebra carries over to the non-linear Schrödinger equation (NLS), the latter theory being related to the former by the principle of gauge equivalence of Lax pairs⁴⁾. The NLS equation does not have a global non-Abelian symmetry; nevertheless, it has an infinite dimensional non-Abelian algebra of symmetry transformations the $su(2)$ subalgebra of which acts trivially on the NLS field.

2. - The Heisenberg model involves a field

$$S(t, x) = \sum_{i=1}^3 S_i(t, x) \sigma_i \quad \text{with} \quad S^2 = \mathbf{1},$$

where σ_i $i = 1, 2, 3$ denote the Pauli matrices. Its equation of motion

$$\partial_t S = \frac{1}{2i} \partial_x [S, \partial_x S] \quad (1)$$

is the consistency condition of the Lax pair

$$\partial_x R^{(\lambda)} = -i\lambda S R^{(\lambda)} \quad (2)$$

$$\partial_t R^{(\lambda)} = \left(-\frac{\lambda}{2} [S, \partial_x S] + 2i\lambda^2 S\right) R^{(\lambda)}.$$

For real λ , the $R^{(\lambda)}$ is an $SU(2)$ valued function of t and x . We are concerned with the Lie algebra

$$\mathfrak{g} = su(2) \otimes \mathbb{R}[\lambda, \lambda^{-1}];$$

its elements can be represented as

$$X \in \mathfrak{g} : X(\lambda) = \sum_{j=-\infty}^{+\infty} \lambda^{-j} X^{(j)}$$

where $X^{(j)} \in su(2)$,

$$[X, Y] = \sum_{j_1, j_2 = -\infty}^{+\infty} \lambda^{-(j_1 + j_2)} [X^{(j_1)}, Y^{(j_2)}]$$

and the sum is taken to be finite.

Consider now the space of all $SU(2)$ valued functions $R^{(\lambda)}$ such that

- i) $R^{(\lambda)}$ solves (2) for some smooth field S ,
- ii) $R^{(0)}(t, x) = 1$.

This is the space on which the infinitesimal symmetries act:

$$R^{(\lambda)} \longrightarrow (1 + \delta M_X) R^{(\lambda)}, \quad X \in \mathcal{G},$$

where δ is the infinitesimal parameter. M_X is defined by

$$M_X R^{(\lambda)} = - \frac{1}{2\pi i} \int_{C_{0,\lambda}} \frac{d\mu}{\mu} \frac{\lambda}{\lambda - \mu} V_X(\mu) R^{(\lambda)}, \quad (4)$$

$$V_X(\mu) = R^{(\mu)} X(\mu) R^{(\mu)^{-1}}. \quad (5)$$

The corresponding transformations of the Lax pair coefficients are given by

$$M_X S = - \frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\mu} [V_X(\mu), S] \quad (6)$$

$$M_X [S, \partial_X S] = - \frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\mu} [V_X(\mu), [S, \partial_X S] - 4i\mu S] \quad (7)$$

In all these formulae, $C_{a_1, \dots, a_n}$ denotes a circle in the complex μ plane with the centre at $\mu = 0$ surrounding the points $a_1, \dots, a_n$ once counterclockwise. The formulation of the symmetries in terms of contour integrals⁵⁾ is a by-product of the Riemann transform method³⁾ and makes the verification of the commutation relations very simple. A straightforward computation shows that $(1 + \delta M_X) R^{(\lambda)}$ solves Eq. (2) upto terms of order δ^2 if one replaces S and $[S, \partial_X S]$ by $(1 + \delta M_X)S$ and $(1 + \delta M_X)[S, \partial_X S]$. Because obviously $M_X R^{(0)} = 0$ and $(M_X R^{(\lambda)}) R^{(\lambda)^{-1}} \in su(2)$ for real λ , the function space defined above is, to first order in δ , mapped into itself by $(1 + \delta M_X)$.

To prove that M_X actually provides a representation of $\mathfrak{G}$, i.e.,

$$[M_X, M_Y] = M_{[X, Y]}, \quad (8)$$

we observe

$$\begin{aligned} M_X(V_Y(\mu)) &= M_X(R^{(\mu)} Y(\mu) R^{(\mu)-1}) \\ &= [(M_X R^{(\mu)}) R^{(\mu)-1}, V_Y(\mu)] \\ &= -\frac{1}{2\pi i} \int_{C_{0,\mu}} \frac{d\lambda}{\lambda} \frac{\mu}{\mu-\lambda} [V_X(\lambda), V_Y(\mu)], \end{aligned} \quad (9)$$

so that we can write

$$\begin{aligned} ([M_X, M_Y] R^{(\lambda)}) R^{(\lambda)-1} &= \\ &= \frac{1}{2\pi i} \int_{C_{0,\lambda}} \frac{d\mu}{\mu} \frac{\lambda}{\lambda-\mu} \frac{1}{2\pi i} \int_{C_{0,\lambda,\mu}} \frac{d\lambda}{\lambda} \left\{ \frac{\mu}{\mu-\lambda} ([V_X(\lambda), V_Y(\mu)] - [V_Y(\lambda), V_X(\mu)]) + \frac{\lambda}{\lambda-\mu} [V_Y(\lambda), V_X(\mu)] \right\} \\ &= \frac{1}{2\pi i} \int_{C_{0,\lambda}} \frac{d\mu}{\mu} \frac{1}{2\pi i} \int_{C_{0,\lambda,\mu}} d\lambda \left\{ \frac{\lambda}{\lambda-\mu} \frac{\mu}{\mu(\mu-\lambda)} [V_X(\lambda), V_Y(\mu)] - \frac{\lambda}{(\lambda-\mu)(\mu-\lambda)} [V_Y(\lambda), V_X(\mu)] \right\}. \end{aligned} \quad (10)$$

In the first term of (10), the contour $C_{0,\lambda,\mu}$ can be deformed into a contour C'_0 , lying inside $C_{0,\lambda}$, plus a contribution from the pole at $\lambda = \mu$. In the second term, the $C_{0,\lambda}$ contour can be deformed into C'_0 (the integrand being analytic at $\mu = \lambda$); then $C_{0,\lambda,\mu}$ can be deformed into $C_{0,\lambda}$. So we are left with

$$\begin{aligned} ([M_X, M_Y] R^{(\lambda)}) R^{(\lambda)-1} &= -\frac{1}{2\pi i} \int_{C_{0,\lambda}} \frac{d\mu}{\mu} \frac{\lambda}{\lambda-\mu} [V_X(\mu), V_Y(\mu)] \\ &\quad + \frac{1}{2\pi i} \int_{C_{0,\lambda}} d\mu \frac{1}{2\pi i} \int_{C'_0} d\lambda \frac{\lambda}{\lambda(\lambda-\mu)(\mu-\lambda)} [V_X(\lambda), V_Y(\mu)] \\ &\quad - \frac{1}{2\pi i} \int_{C'_0} d\mu \frac{1}{2\pi i} \int_{C_{0,\lambda}} d\lambda \frac{\lambda}{\mu(\lambda-\mu)(\mu-\lambda)} [V_Y(\lambda), V_X(\mu)]. \end{aligned}$$

Interchanging μ and λ in the last term, we see that it compensates for the second one, thus

$$\begin{aligned} [M_x, M_y] R^{(\lambda)} &= -\frac{1}{2\pi i} \int \frac{d\mu}{\mu} \frac{\lambda}{\lambda-\mu} V_{[x,y]}(\mu) R^{(\lambda)} \\ &= M_{[x,y]}^{C_{0,\lambda}} R^{(\lambda)}. \end{aligned}$$

In terms of a basis

$$X_a^{(n)} : X_a^{(n)}(\lambda) = \lambda^{-n} \frac{\sigma_a}{2i}, \quad M_a^{(n)} = M_{X_a^{(n)}},$$

Eq. (8) reads

$$[M_a^{(n)}, M_b^{(m)}] = \varepsilon_{abc} M_c^{(n+m)}.$$

Because (4) is an infinitesimal symmetry of the Lax pair, (6) is an infinitesimal symmetry of the equations of motion (1), the latter being the consistency condition of the Lax pair.

3. - The non-linear Schrödinger equation

$$i\partial_t \psi + \partial_x^2 \psi + 2|\psi|^2 \psi = 0 \quad (11)$$

for the complex field $\psi(t,x)$ is the consistency condition of the Lax pair

$$\partial_x \phi^{(\lambda)} = \left\{ \begin{pmatrix} 0 & \bar{\psi} \\ -\psi & 0 \end{pmatrix} - i\lambda \sigma_3 \right\} \phi^{(\lambda)} \quad (12)$$

$$\partial_t \phi^{(\lambda)} = \left\{ \frac{1}{i} \begin{pmatrix} |\psi|^2 & \partial_x \bar{\psi} \\ \partial_x \psi & -|\psi|^2 \end{pmatrix} - 2\lambda \begin{pmatrix} 0 & \bar{\psi} \\ -\psi & 0 \end{pmatrix} + 2i\lambda^2 \sigma_3 \right\} \phi^{(\lambda)}.$$

The system (11), (12) is known to be gauge equivalent⁴⁾ to (1), (2). We write the field S in the form

$$S = g \sigma_3 g^{-1} \quad \text{with } \text{diag}(g^{-1} \partial_x g) = 0, \quad g(t,x) \in SU(2). \quad (13)$$

Then (11), (12) and (1), (2) are related by

$$\mathcal{R}^{(\lambda)} = g \phi^{(\lambda)}, \quad g^{-1} \partial_x g = \begin{pmatrix} & -\bar{\psi} \\ \psi & \end{pmatrix}, \quad g^{-1} \partial_t g = i \begin{pmatrix} |\psi|^2 & \partial_x \bar{\psi} \\ \partial_x \psi & -|\psi|^2 \end{pmatrix}, \quad (14)$$

in particular, solutions of (1) are mapped to solutions of (11), and vice versa.

The g field is determined by S up to $g \rightarrow g \exp i\sigma_3 \alpha$, α constant. This amounts to a phase transformation $\psi \rightarrow \exp(2i\alpha)\psi$.

The formulae (13) and (14) allow us to carry the symmetry (4) - (6) over to the NLS. The action of M_X on g is

$$M_X g = -\frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\mu} V_X(\mu) g = -\frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\mu} g W_X(\mu) \quad (15)$$

with $W_X(\mu) = g^{-1} V_X(\mu) g = \phi^{(\mu)} X(\mu) \phi^{(\mu)^{-1}}$. To see that (15) in fact implies the correct transformation of S , compute

$$M_X S = [(M_X g) g^{-1}, S] = -\frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\mu} [V_X(\mu), S]$$

as in (6). From (14), (4) and (15) we find

$$\begin{aligned} M_X \phi^{(\lambda)} &= g^{-1} (M_X R^{(\lambda)}) - g^{-1} (M_X g) \phi^{(\lambda)} \\ &= \left\{ -\frac{1}{2\pi i} \int_{C_{0,\lambda}} \frac{d\mu}{\mu} \frac{\lambda}{\lambda-\mu} W_X(\mu) + \frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\mu} W_X(\mu) \right\} \phi^{(\lambda)}, \end{aligned}$$

so that

$$M_X \phi^{(\lambda)} = \phi^{(\lambda)} X(\lambda) - \frac{1}{2\pi i} \int_{C_0} \frac{d\mu}{\lambda-\mu} W_X(\mu) \phi^{(\lambda)}, \quad (16)$$

$$M_X (g^{-1} \partial_X g) = -\frac{1}{2\pi i} \int_{C_0} d\mu [W_X(\mu), i\sigma_3], \quad (17)$$

thus $\text{diag}(M_X (g^{-1} \partial_X g)) = 0$,

$$M_X (g^{-1} \partial_t g) = -\frac{1}{2\pi i} \int_{C_0} d\mu [2g^{-1} \partial_X g + 2i\mu \sigma_3, W_X(\mu)] \quad (18)$$

and, in particular,

$$M_X \psi = -\frac{1}{2\pi i} \int_{C_0} d\mu \text{tr} \left\{ (i\sigma_1 - \sigma_2) W_X(\mu) \right\}. \quad (19)$$

In (16) the point λ lies outside C_0 .

As above, one can easily verify that Eqs. (16) - (18) define a representation of $\text{su}(2) \otimes \mathbb{R}[\lambda, \lambda^{-1}]$ acting on the $\phi^{(\lambda)}$ which solve (12) for some smooth field ψ . Because of $g = (\phi^{(0)})^{-1}$, however, there is no normalization

condition on $\phi^{(0)}$: it transforms non-trivially since g does so. Equation (18) shows that the $SU(2)$ rotations act trivially on ψ : For $X(\mu) = \mu^0 X^{(0)}$, $M_X \psi = 0$. In contrast to this, the transformation $M_a^{(1)}$ has already a very complicated form

$$M_a^{(1)} \psi = -tr \left\{ (i\sigma_1 - \sigma_2) g^{-1} \frac{\sigma_a}{2i} g \right\} ,$$

and $g = \phi^{(0)-1}$ is a path-ordered exponential of matrices involving ψ and $\partial_x \psi$.

The constant phase transformation $\psi \rightarrow \exp i\alpha\psi$ commutes with all the M_X ; if one includes $\psi \rightarrow (1+i\alpha)\psi$ in the symmetry algebra, the resulting algebra has a one-dimensional centre.

One notes that on the fields S and ψ only the "positive" subalgebra (generated by the $M_a^{(n)}$, $n \geq 0$) acts non-trivially. The role of the "negative" subalgebra (the $M_a^{(n)}$, $n < 0$) is to multiply $R^{(\lambda)}$ and $\phi^{(\lambda)}$ from the right by the tensor-product representation of $\mathfrak{g}$. Since the Lax pair is a system of linear differential equations, this might seem somewhat trivial. Non-trivial, however, is the fact that the commutator of "positive" with "negative" transformations fits to the structure of $su(2) \otimes R[\lambda, \lambda^{-1}]$.

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