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REDUCED LARGE N MODEL FOR CONTINUUM QCD

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ABSTRACT

Following the quenching prescription, a reduced model, equivalent to large N QCD, is presented. The model is in the continuum, is formulated in terms of compact $SU(N)$ group variables and includes fermions with no doubling problems. The formulation is expected to be useful for non-perturbative calculations, including analytic as well as Monte Carlo methods, as described.

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The observation of Eguchi and Kawai¹⁾ that the Schwinger-Dyson equations of the lattice gauge theory for $N \rightarrow \infty$ could be recovered from a reduced action has led to further developments that may help solve the $N \rightarrow \infty$ gauge theory. Following the quenching procedure of Bhanot, Heller and Neuberger²⁾, Parisi³⁾ and Gross and Kitazawa⁴⁾ have shown how to recover the perturbative planar diagrams of the continuum $N \rightarrow \infty$ theories, including gauge theories, from a reduced model. In this paper I propose a reduced model of the continuum gauge theory which I expect to be useful for non-perturbative calculations directly in the continuum.

Some time ago the gauge theory, including fermions, was reformulated^{5,6)} in terms of unitary matrices $B_{\mu\nu}(x)$ corresponding to path ordered exponential integrals of the form explained below and shown in fig. 1. The Lagrangian can be fully rewritten in terms of such gauge invariant unitary matrices $B_{\mu\nu}(x)$. This is done as follows in d dimensions:

Define d unitary $N \times N$ matrices $(B_\mu(x))_i^j$ ($\mu = 1, 2, \dots, d$; $i, j = 1, \dots, N$) which are related to the gauge potentials A_μ via ordinary derivatives as

$$\frac{i}{g} B_\mu^\dagger(x) \partial_\mu B_\mu(x) = A_\mu \quad \mu = 1, 2, \dots, d \quad (1)$$

For each μ , by multiplying by $B_\mu(x)$ from the left one obtains a one-dimensional differential equation from which $B_\mu(x)$ can be solved in terms of $A_\mu(x)$ in the form of a path ordered exponential integral,

$$B_\mu(x) = P \exp \left[-ig \int_{-\infty}^x dx'_\mu A_\mu(x') \right] \quad (2)$$

where the path is a straight line along the μ -direction, embedded in the d -dimensional volume, starting at $(-\infty)$ and ending at a point x_μ , as shown in fig. 1. The product of two such matrices at the point x_μ , as in fig. 1, defines the variables $B_{\mu\nu}(x)$

$$\begin{aligned} B_{\mu\nu}(x) &= B_\mu(x) B_\nu^\dagger(x) \\ B_{\nu\mu} &= (B_{\mu\nu})^\dagger = (B_{\mu\nu})^{-1} \end{aligned} \quad (3)$$

The $B_{\mu\nu}(x)$ are gauge invariant for local gauge transformations that vanish at infinity.

Using eq. (1) it is straightforward to compute the field strength $F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$ for each μ, ν

$$\begin{aligned} F_{\mu\nu} &= \frac{i}{g} B_\mu^\dagger \left[\partial_\mu (B_{\mu\nu} \partial_\nu B_{\nu\mu}) \right] B_\nu \\ \frac{1}{2} \text{Tr} (F_{\mu\nu} F_{\mu\nu}) &= -\frac{1}{2g^2} \text{Tr} \left[\partial_\mu (B_{\mu\nu} \partial_\nu B_{\nu\mu}) \right]^2 \end{aligned} \quad (4)$$

Note the similarity of this last expression to chiral models $\text{Tr}(U^\dagger \partial_\mu U)^2$, except for the extra derivative. One can also obtain the topological density in four dimensions $\text{Tr}(F_{\mu\nu} \tilde{F}_{\mu\nu})$ component by component. For example,

$$\text{Tr}(F_{12} F_{34}) = \text{Tr} [B_{31} \partial_1 (B_{12} \partial_2 B_{21}) B_{13} \partial_3 (B_{34} \partial_4 B_{43})] \quad (5)$$

Thus, in gauge invariant quantities only the gauge invariant variables $B_{\mu\nu}(x)$ appear.

Matter fields are easily incorporated. For example, for quarks in the fundamental representation, after using eq. (1) one obtains

$$\bar{\Psi} i \not{D} \Psi = \sum_{\mu=1}^d i [\bar{\Psi} B_\mu^\dagger \gamma_\mu \partial_\mu (B_\mu \Psi)] \quad (6)$$

Defining a gauge invariant fermion field $\psi'(x)$ via

$$\psi'(x) = B_1(x) \psi(x), \quad (7)$$

the fermion kinetic term takes the form

$$\bar{\Psi} i \not{D} \Psi - m \bar{\Psi} \Psi = \sum_{\mu=1}^d [\bar{\Psi}' B_\mu i \gamma_\mu \partial_\mu (B_\mu \psi')] - m \bar{\Psi}' \psi' \quad (8)$$

This last step essentially amounts to an axial gauge choice (see, e.g., last two articles in ref. 5).

In the path integral formulation, the measure can be computed by changing variables from ψ to ψ' and A_μ to B_μ . The differential dA_μ of A_μ can be related to the differential dB_μ of B_μ via eq. (1)

$$\begin{aligned} dA_\mu &= \frac{i}{g} d(B_\mu^\dagger \partial_\mu B_\mu) \\ &= -\frac{i}{g} B_\mu^\dagger [\partial_\mu (B_\mu dB_\mu^\dagger)] B_\mu \end{aligned} \quad (9)$$

Note that the $B_\mu^\dagger$ and B_μ on the left and right simply perform a unitary transformation, with unit Jacobian. Furthermore, if one regards ∂_μ as a linear transformation, its Jacobian ($\det \partial_\mu$) is irrelevant since it is field independent and drops out in any calculation. Then the volume element of (dA_μ) is simply replaced by the volume element of $(B_\mu dB_\mu^\dagger)$. However, this is nothing but the Haar measure for the unitary matrix B_μ at every x_μ . Since all terms in the action depend only on the combinations $B_{\mu\nu}$ [after eq. (7)], it is possible to integrate out (B_1) trivially, and because of the left and right invariance of the Haar measure, one can replace $(dB_{\mu 1})$ instead of (dB_μ) , leaving only $(d-1)$ matrices $B_{\mu 1}$ to integrate over. Note that we can write from eq. (3)

$$B_{\mu\nu} = B_{\mu}^{\dagger} B_1 B_1^{\dagger} B_{\nu} = B_{\mu 1} B_{1\nu} \quad (10)$$

so that the action actually depends only on $d - 1$ matrices. We emphasize again that in this form all the variables that enter in the theory are gauge invariant. The canonical formalism in the continuum⁵⁾ and on the lattice⁶⁾ was pursued in terms of this formulation. Now we seek the reduced model applicable to $N \rightarrow \infty$.

As shown in refs. 3 and 4, the large N limit of any $N \times N$ matrix theory can be recovered from a reduced model defined at only one point. Following their prescription, every derivative $i\partial_{\mu}$ is replaced with a $N \times N$ diagonal matrix P_{μ}

$$P_{\mu} = \begin{pmatrix} P_{1\mu} & & \\ & P_{2\mu} & \\ & & \ddots \\ & & & P_{N\mu} \end{pmatrix} \quad (11)$$

which occurs as a commutator with $N \times N$ matrices and just multiplies column or row matrices (such as ψ). To apply this procedure to the above reformulation of gauge theories, we define b_{μ} , $b_{\mu\nu}$, and χ as the matrices corresponding to $B_{\mu}(x)$, $B_{\mu\nu}(x)$, and $\psi'(x)$ at one point. The reduced theory is obtained by the following substitutions which are in accordance with eqs. (1-8):

$$g A_{\mu} = i B_{\mu}^{\dagger} \partial_{\mu} B_{\mu} \rightarrow b_{\mu}^{\dagger} [P_{\mu}, b_{\mu}] = b_{\mu}^{\dagger} P_{\mu} b_{\mu} - P_{\mu} \quad (12a)$$

$$i B_{\mu\nu} \partial_{\nu} B_{\nu\mu} \rightarrow b_{\mu\nu} [P_{\nu}, b_{\nu\mu}] = b_{\mu\nu} P_{\nu} b_{\nu\mu} - P_{\nu} \quad (12b)$$

$$i \partial_{\mu} [B_{\mu\nu} i \partial_{\nu} B_{\nu\mu}] \rightarrow [P_{\mu}, b_{\mu\nu} P_{\nu} b_{\nu\mu} - P_{\nu}] = [P_{\mu}, b_{\mu\nu} P_{\nu} b_{\nu\mu}] \quad (12c)$$

$$\frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} \rightarrow \frac{1}{g^2} \text{Tr} [P_{\mu}^2 b_{\mu\nu} P_{\nu}^2 b_{\nu\mu} - P_{\mu} b_{\mu\nu} P_{\nu} b_{\nu\mu} P_{\mu} b_{\mu\nu} P_{\nu} b_{\nu\mu}] \quad (12d)$$

$$\sum_{\mu=1}^d [\bar{\psi}' B_{1\mu} \gamma_{\mu} i \partial_{\mu} (B_{\mu 1} \psi')] - m \bar{\psi}' \psi' \rightarrow \sum_{\mu=1}^d [\bar{\chi} b_{1\mu} \gamma_{\mu} P_{\mu} b_{\mu 1} \chi] - m \bar{\chi} \chi \quad (12e)$$

The reduced theory has the same form as the field theory, but defined only at one point in terms of the variables $b_{\mu 1}$ and χ via eq. (12).

This reduction procedure may also be applied in the Hamiltonian formulation^{5,6)} of the theory, where only the matrix momenta with space indices P_{μ} , $\mu = 1, 2, \dots, (d - 1)$, replace $i\partial_{\mu}$, and time derivatives do not appear in the canonical formalism. Then the Hamiltonian is written in terms of the reduced variables

corresponding to $b_{\mu i}$, χ and their canonical conjugates^{5,6)}, plus the momenta $P_{\mu i}$ to be treated as constants as far as the quantum mechanics of these variables is concerned.

From these expressions it is seen that the quenching proposed in ref. 4 for gauge theories in fact amounts to using our string variables B_μ proposed in reformulating^{5,6)} gauge theories. Furthermore, our approach explains naturally the "diagonal constraints" on A_μ [eq. (12a)] which seemed unnatural but necessary in ref. 4. We have also gone one step further in writing the path integral completely in terms of the group variables $b_{\mu\nu}$, since it straightforwardly follows from the earlier path integral in the full field theory. This has the advantage of dealing with compact $SU(N)$ integrals, which can be important in practice, as described below. Furthermore, our variables $b_{\mu\nu}$ are appropriate for non-perturbative calculations as opposed to gA_μ . If the lattice version of our formulation is used in terms of the corner variables⁶⁾ $B_{\mu\nu}$ and the quenching procedure is applied, then the modified Eguchi-Kawai lattice model proposed in ref. 2 immediately follows. We emphasize that via eq. (12) we represent the $N \rightarrow \infty$ QCD in the continuum, not on the lattice, and that there are no fermion doubling problems. In ref. 5 $B_{\mu\nu}$ variables appropriate to the light cone and other metrics were also introduced. These could also be quenched according to the same procedure.

The action of the reduced theory is obtained from that of the field theory by making the substitutions of eq. (12) and replacing $\int d^d x \rightarrow (2\pi/\Lambda)^d$. The path integral is then reduced to the simpler integrals over the fermion χ and the unitary matrices $b_{\mu i}$ taken with the Haar measure. This integral must be performed with arbitrary values of the momenta $P_{\mu i}$. In the $N \rightarrow \infty$ limit, the free energy per unit volume ϵ follows from

$$e^{-N^2 W_N(P)} = \int (db_{\mu i}) d\chi d\bar{\chi} e^{-S(P, b, \chi)} \quad (13)$$

$$\epsilon(\Lambda, g(\Lambda)) = \lim_{N \rightarrow \infty} \int_\Lambda (dp) W_N(P)$$

where the integrals over all $P_{\mu i}$ are performed⁴⁾ with a cut-off Λ , and with a measure (dp) normalized to 1, $\int (dp) = 1$. Instead of performing the integral, it may be possible²⁻⁴⁾ to obtain the answer by quenching, that is by fixing the momenta p_μ^i in N -cells $i = 1, 2, \dots, N$ that cover the volume of (dp) integration densely and uniformly, and then letting $N \rightarrow \infty$. The $\Lambda \rightarrow \infty$ limit is taken after renormalization, defined via Green functions.

Correlation functions can be calculated by substituting the path integrals by reduced integrals. The x -dependence in Green functions is obtained via

$$B_{\mu\nu}(x) \longrightarrow e^{-iP \cdot x} b_{\mu\nu} e^{iP \cdot x} \quad (14)$$

For example, a rectangular Wilson loop in the 1-2 plane with corners at X_A, X_B, X_C, X_D as in fig. 2 can be written in terms of the $B_{\mu\nu}(x)$ variables as

$$\text{Tr} [B_{12}(X_A) B_{21}(X_B) B_{12}(X_C) B_{21}(X_D)] \quad (15)$$

In the reduced integral this is replaced by

$$\text{Tr} [e^{iP \cdot (X_D - X_A)} b_{12} e^{iP \cdot (X_A - X_B)} b_{21} e^{iP \cdot (X_B - X_C)} b_{12} e^{iP \cdot (X_C - X_D)} b_{21}] \quad (16)$$

The correlation function is obtained to all orders in $(g^2 N)$ by performing the integrals over χ and $b_{\mu i}$, multiplying by $e^{N^2 W_N(p)}$ and integrating over the momenta or quenching them.

The integrals over the fermions χ are, of course, trivially performed. The resulting determinants are $N \times N$ and could be explicitly written out. Thus, the effects of fermions are not neglected and one can attempt to compute meson propagators. Of course, we expect that their contribution for $N \rightarrow \infty$ is down by $1/N$ for each fermion loop, but the calculation described here is non-perturbative and does not involve keeping track of loops.

It is now possible to make a number of observations and suggest methods for analysing further this model:

- i) Since the integrals over $b_{\mu i}$ are compact group integrals, the result is surely finite for any given momenta $p_{\mu i}$. With a cut-off Λ the momentum integrals or quenching will yield a cut-off dependent result. We expect that the cut-off dependence can be removed by renormalization in the renormalizable four-dimensional theory $d = 4$ (and also $d < 4$). Therefore, any Green function is finite. This would establish (modulo possible problems in renormalization) that the $N \rightarrow \infty$ theory is indeed mathematically meaningful, confirming the expectations of 't Hooft⁷⁾.
- ii) The group integrals over $b_{\mu i}$ can be evaluated via Monte Carlo methods since they are compact variables. Computer programs are available⁸⁾ and it is expected that such integrals can be handled in practice up to $N \approx 20$. Combining Monte Carlo integration with quenching many interesting Green functions should be calculable with our formulation. It is of course much more worth while to apply Monte Carlo techniques to this problem which will produce results in the continuum not on the lattice, and which includes fermions.

- iii) After an appropriate rescaling by powers of N , it would be interesting to compare the numerical results of small N (such as $N = 3$) to those of large N and establish the rate of convergence. It would be amusing if the numerical estimates for $N = 3$ already are close to the $N \rightarrow \infty$ answer. In fact, it is even desirable to compute physical quantities for $N = 3$ with the reduced theory analytically or numerically and see how they compare to the real world, including baryons.
- iv) Performing the group integrals analytically over $b_{\mu 1}$ for any N may not be very easy, but for $N = 2$ almost certainly it can be done analytically, and perhaps for $N \rightarrow \infty$. Even $N = 2$ would provide an interesting laboratory for comparison to the full $N = 2$ field theory. Simpler group integrals that were attempted in the past⁹⁻¹¹⁾ for finite as well as infinite N will probably be useful as prototypes to develop the methodology.
- v) There are simpler models, which present similar difficulties and share many interesting properties as QCD. We have already mentioned, following eq. (4), the similarity of our formulation to the chiral $U(N)$ sigma model. This model is now essentially solved¹¹⁾ for $d = 1$ and $d = 2$ using the new reduction prescription for $N \rightarrow \infty$, and the results will be reported elsewhere. The $SU(N)$ group integrals encountered in this formulation of the sigma model happened to coincide with a known integral¹⁰⁾ in the case of $d = 1$, and for $d \geq 2$ have newly been obtained¹¹⁾ for any N . The $d = 1$ sigma model is particularly interesting because its field theoretical path integral is exactly solvable¹¹⁾ and provides a test case for the new reduced model approach. The two approaches give identical answers for $N \rightarrow \infty$ and they remain to be compared, by computer, for arbitrary N . We expect that the methods developed to solve the $d = 1, 2$ sigma models will provide the training grounds for a non-perturbative analytic assault on $N \rightarrow \infty$ QCD.

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Figure captions

Fig. 1 : Path ordered exponential integrals.

Fig. 2 : The path ordered exponential around the loop is given by the product of $B_{\mu\nu}$'s at the locations X_A, X_B, X_C, X_D .

FIG.1

FIG.2