REF.TH.3336-CERN

MEAN-FIELD ANALYSIS OF $Z(N)$ LATTICE GAUGE THEORIES

V. Alessandrini^{*)}
CERN -- Geneva

ABSTRACT

The phase diagram of $Z(N)$ lattice gauge theories is examined in the mean-field approach. It is shown how the one-loop corrections to the mean-field approximation provide strong evidence for the Coulomb to Higgs phase transition, which is absent in the zeroth order result.

^{*)} Permanent address: LPTHE (Laboratoire associé au CNRS) Univ. de Paris XI, 91405 Orsay, France.

REF.TH.3336-CERN

17 June 1982

It is by now well established that $Z(N)$ lattice gauge theories in four space-time dimensions exhibit, for N not too small, three distinct phases: a strong-coupling electric confining phase, a massless Coulomb phase in the weak-coupling region and a magnetic confining (Higgs) phase at very weak coupling¹⁾⁻³⁾. The purpose of this paper is to discuss how mean-field techniques⁴⁾⁻⁷⁾, which predict a first-order deconfining phase transition when going from strong to weak coupling, can also strongly suggest the existence of the second (Coulomb to Higgs) phase transition that occurs in $Z(N)$ lattice gauge theories.

Both in the Coulomb and in the Higgs phases there is a non-vanishing mean-field and the Wilson loops obey the perimeter law. The order parameters that would neatly discriminate between both phases are the dual loops introduced by 't Hooft⁸⁾. Since the mean-field approximation is, in some sense to be discussed below, a semiclassical approximation for Wilson loops, one would expect the behaviour of the dual loops to be entirely controlled by the fluctuations around the mean-field result. Indeed, we shall see that in the case of $Z(N)$ lattice gauge theories, the mean-field approximation alone shows almost no sign of the Coulomb to Higgs phase transition. However, strong indications of its existence are obtained when one goes beyond this approximation and computes the effects of corrections to the mean-field result. In particular, the photon mass obtained at the one-loop level exhibits a dramatic change of behaviour in going from the weak to the superweak coupling region, which strongly hints at the transition from a Coulomb to a Higgs regime.

The mean-field approach has recently been formulated by Drouffe⁶⁾ and Brézin and Drouffe⁷⁾ as a saddle-point evaluation of a properly rewritten version of the lattice gauge theory path integral. In the case of the $Z(N)$ gauge group, as well as in the case of $U(1)$, the fluctuating degrees of freedom are a complex mean-field V_ℓ and a complex external field A_ℓ at each link. For a translational invariant field configuration where V_ℓ and A_ℓ take the same value at each link, the mean-field free energy per link is given by

$$F_N = -\bar{\beta} (\lambda^2 + \xi^2)^2 + (\lambda x + \xi y) - \omega_N(x, y) \quad (1)$$

where we have used the notation

$$V = \lambda + i\xi = \rho e^{i\theta} ; \quad A = x + iy = \alpha e^{i\phi} \quad (2)$$

and $\bar{\beta} = \beta(d-1)$, β being the inverse of the gauge coupling constant. In this paper we use the same conventions as in Ref. 9). In Eq. (1), $\omega_N(x, y)$ is the logarithm of the one-link integral for $Z(N)$:

$$\omega_N = \ln f^{(N)} ; \quad f^{(N)} = \frac{1}{N} \sum_{n=0}^{N-1} \exp \left[\alpha \cos \left(\frac{2\pi n}{N} + \phi \right) \right] \quad (3)$$

and the values of the mean-field V and the external field A are to be determined as functions of β by solving the saddle-point equations:

$$\lambda = \frac{f_x^{(N)}}{f^{(N)}} ; \quad \xi = \frac{f_y^{(N)}}{f^{(N)}} \quad (4)$$

$$x = 4\beta\lambda(\lambda^2 + \xi^2) ; \quad y = 4\beta\xi(\lambda^2 + \xi^2) \quad (5)$$

so as to minimize F_N .

We first establish a result that will play an important role in what follows. Since $f^{(N)}(\alpha, \phi)$ is a periodic function of ϕ with period $2\pi/N$, we can compute its Fourier coefficients. By using the well-known expansion of $\exp[\alpha \cos(2\pi n/N + \phi)]$ in terms of Bessel functions, both the integration over ϕ and the finite sum over n can be explicitly carried out. We then obtain

$$f^{(N)}(\alpha, \phi) = I_0(\alpha) + 2 \sum_{k=1}^{\infty} I_{kN}(\alpha) \cos(kN\phi) \quad (6)$$

where $I_\nu(\alpha)$ are the modified Bessel functions of order ν . The first term in the right-hand side of Eq. (6) is just the $U(1)$ one-link integral. This result shows that when N grows at fixed values of the external field A , the $Z(N)$ one-link integral is given by the $U(1)$ result plus correction terms that are exponentially small in N since they behave as $1/N!$

We next determine the saddle-point values of the phases θ and ϕ . The ratio of Eqs. (5) tells us that they are equal modulus π . From this result and Eqs. (4) we obtain:

$$y f_x^{(N)} - x f_y^{(N)} = \frac{1}{N} \sum_{n=0}^{N-1} \sin \left(\frac{2\pi n}{N} + \phi \right) \exp \left[\alpha \cos \left(\frac{2\pi n}{N} + \phi \right) \right] = 0 \quad (7)$$

This is an odd periodic function of ϕ with period $2\pi/N$, and it therefore vanishes at the $Z(N)$ phases $\phi_n = 2\pi n/N$. Other roots of Eq. (7) are the phases in between the $Z(N)$ ones, $\phi'_n = (2n+1)\pi/N$. It follows by inspection that, both

for $\phi = \phi_n$ and $\phi = \phi'_n$, either the terms in the sum are zero or they cancel by pairs. Numerical analysis indicates that there are no other roots of Eq. (7), but we have not proved this fact in detail because we do not need to. Indeed, going back to Eq. (14) for the free energy and using Eq. (6) for $f^{(N)}$ together with the positivity properties of modified Bessel functions, it follows that, at fixed values of α , the $Z(N)$ phases ϕ_n are absolute minima of F_N . Because of $Z(N)$ gauge invariance, all those minima are degenerate, so we can restrict ourselves to $\phi = \theta = 0$, in which case the saddle-point equations simplify to

$$x = 4\bar{\beta} \lambda^3 \quad (8)$$

$$\lambda = \frac{f_x^{(\omega)}}{f^{(\omega)}} \Big|_{y=0} \quad (9)$$

It is easy to convince oneself that λ is a monotonic function of x that tends to one as $x \rightarrow \infty$. Therefore, the solutions of Eqs. (8) and (9) have the standard properties of the mean-field approximation. There is always a trivial solution $x = \lambda = 0$ which describes at strong coupling the electric confining phase. At some critical value $\bar{\beta}_c$ a second non-trivial minimum of the free energy takes over and there is a first-order phase transition to an unconfined phase. No third minimum that could be associated with the transition from the Coulomb to the Higgs phase ever occurs. Explicit numerical calculations carried out for $N = 16$ show that the deconfining phase transition takes place at $\bar{\beta} = 1.82$. This is the same critical value as for $U(1)$ lattice gauge theory since in this region ω_N differs from the $U(1)$ one-link integral only in one part in 10^6 . As one proceeds further into the very weak-coupling region, the free energy is a rather smooth function of $\bar{\beta}$.

The smoothness of F_N as a function of $\bar{\beta}$ can be easily understood. In the weak-coupling limit, λ is of order one and $x \approx 4\bar{\beta}$. Let us first consider the region $N \gg x$. Here we have, up to exponentially small corrections in N , a $U(1)$ -like behaviour. The saddle-point equations (8) and (9) can be solved near $\lambda = 1$ and one finds

$$F_{U(1)} = -\bar{\beta} + \frac{1}{2} \ln(2\pi \bar{\beta}) + O(1/\bar{\beta}) \quad (10)$$

On the other hand, Eqs. (8) and (9) can also be solved in the Higgs-like region $x \gg N$. In this case an accurate approximation to $f^{(N)}$ can be obtained directly from Eq. (3):

$$f^{(N)}(x, y) = \frac{e^x}{N} \left[1 + 2e^{-ax} \cosh(by) + \dots \right] \quad (11)$$

where $a = 1 - \cos(2\pi n/N)$ and $b = \sin(2\pi n/N)$. We then obtain for the free energy:

$$F_N = -\bar{\beta} + \ln N + O(e^{-4a\bar{\beta}}) \quad (12)$$

which shows that, in the weak-coupling limit, the difference between both regimes shows up only in the non-leading contributions to F_N .

Nevertheless, it is clear that there is, at large N , a change of regime from a weak-coupling region where $x \ll N$ and the system is $U(1)$ -like to a very weak coupling region where $x \gg N$ and the system behaves in a way quite different from $U(1)$. This can be seen if one computes the value of the mean-field in the presence of an external source. When the magnitude of the external source grows, the modulus ρ of the mean-field tends to 1. However, the phase θ of the mean-field, which follows first the phase of the external source, ultimately departs from it and approaches the nearest $Z(N)$ phase ϕ_n . However, numerical analysis of $N = 16$ shows that this change of behaviour is very smooth.

In order to exhibit more sharply the transition region between Coulomb and Higgs behaviour, we have computed the one-loop corrections to the mean-field approximation. The calculation follows the methods discussed in detail in Ref. 9), except that in our case there are no complications arising from gauge zero modes. The mean-field action is expanded around the saddle-point values of V_ℓ and A_ℓ and the quadratic fluctuations in A_ℓ are integrated first. This integration adds a contribution to the quadratic fluctuations in V_ℓ which arise from the expansion of the Wilson action. The real and imaginary fluctuations of V_ℓ are decoupled, and the resulting quadratic forms are diagonalized in momentum space. We define the masses of the elementary excitations around the mean-field as the value of the eigenvalues of the quadratic forms at zero momentum. We find, for the imaginary fluctuations, which are clearly related to phase fluctuations of V_ℓ ,

$$m = \frac{f^{(\omega)}}{2f_{yy}^{(\omega)}} \Big|_{y=0} - \frac{x}{2\lambda} \quad (13)$$

and for the real fluctuations, related to length fluctuations of V_ℓ ,

$$M = \frac{f^{(N)}}{2f_{xx}^{(N)}} \Big|_{y=0} - \frac{x}{2\lambda} = \frac{1}{2} \frac{2m + (x/\lambda)}{2m + (x/\lambda) - 1} - \frac{x}{2\lambda} \quad (14)$$

where we have used the relation $f_{xx}^{(N)} = f^{(N)} - f_{yy}^{(N)}$ (valid at $y = 0$) to express M in terms of m . We shall refer to m as the photon mass, since this is what the corresponding excitation becomes in the $U(1)$ limit where m vanishes. In fact, it is quite easy to check by using Eq. (9) for λ that any function $f^{(N)}$ which depends on α but not on ϕ gives $m = 0$. As far as M is concerned, it does not vanish in the $U(1)$ limit and has a very smooth behaviour in the intermediate coupling region.

The photon mass is exponentially small at large N in the $U(1)$ -like region. Keeping only the leading $k = 1$ term in Eq. (6) for $f^{(N)}$ we obtain

$$m \simeq \frac{N^2}{x} \frac{I_N(x)}{I_1(x)} \simeq \frac{1}{2} \left(\frac{x}{2}\right)^{N-1} \frac{1}{(N-2)! I_1(x)} \quad (15)$$

On the other hand, in the Higgs region $x \gg N$ we obtain from Eq. (11)

$$m \simeq \frac{e^{ax}}{4bz} \quad ; \quad M \simeq \frac{e^{ax}}{4a^2} \quad (x \simeq 4\bar{\beta}) \quad (16)$$

Both masses diverge exponentially in the same way because in the super weak-coupling limit the mean field is frozen at a $Z(N)$ phase on the unit circle and the restoring forces in both orthogonal directions must be comparable. It then follows from Eqs. (15) and (16) that, contrary to the free energy or to the mean-field itself, the photon mass undergoes a dramatic change of behaviour in passing from the region $x \ll N$ to the region $x \gg N$.

More relevant than the photon mass itself is the mass ratio (m/M) , which tends to a constant limit of order N^{-2} when the system is well inside the Higgs phase. We have plotted in the Figure the behaviour of (ma^2/Mb^2) as a function of $\bar{\beta}$ for $N = 16$. For $\bar{\beta} < 1.82$, the $Z(N)$ system is in its electric confining phase. At $\bar{\beta} = 1.82$, the normalized mass ratio is of order 10^{-6} . The transition to the Higgs regime starts to take place at $\bar{\beta} \simeq 5$. Creutz, Jacobs and Rebbi¹⁰⁾ have fitted their Monte Carlo data in four space-time dimensions with an expression for the critical coupling for the Coulomb to Higgs transition which, with

our definition of β , reads $\beta_c = 0.38[1 - \cos(2\pi/N)]$. This would place the critical value of β for $N = 16$ at $\beta_c = 15.37$, which is well after the transition region of the Figure, where the mass ratio starts to flatten. The transition region predicted by mean-field theory is not sharp enough to allow for an accurate determination of β_c .

To sum up, we want to emphasize that the Coulomb to Higgs phase transition of $Z(N)$ lattice gauge theories has to be understood, in the standard mean-field approach, through radiative corrections to the mean-field result. The fact that the mass gap changes from an exponentially small value for large enough N to a value of order N^{-2} in the very weak-coupling region strongly indicates that a Coulomb to Higgs phase transition wants to take place. Whether or not there is a sharp phase transition when the number of space-time dimensions d is large, needs further study. This issue can be settled by computing radiative corrections to the photon mass itself. One would then expect them to sharpen the behaviour exhibited in the Figure. However, if these corrections vanish as $1/d$, one would be forced to conclude that no sharp phase transition occurs at large enough d . This problem is currently under investigation.

ACKNOWLEDGMENTS

It is a pleasure to thank B. Lautrup and S. Yankielowicz for discussions, and the CERN Theory Division for its hospitality.

REFERENCES

- 1) S. Elitzur, R.B. Pearson and J. Shigemitsu, Phys. Rev. D19 (1978) 3698.
- 2) A. Guth, A.K. Ukawa and P. Windey, Phys. Rev. D21 (1980) 1013.
- 3) D. Horn, M. Weinstein and S. Yankielowicz, Phys. Rev. D19 (1979) 3715.
- 4) K.G. Wilson, Phys. Rev. D10 (1974) 2445.
- 5) R. Balian, J.M. Drouffe and C. Itzykson, Phys. Rev. D11 (1975) 2104.
- 6) J.M. Drouffe, Nucl. Phys. B170 (FS1) (1980) 211.
- 7) E. Brézin and J.M. Drouffe, Nucl. Phys. B200 (FS4) (1982) 93.
- 8) G. 't Hooft, Nucl. Phys. B138 (1978) 1.
- 9) V. Alessandrini, V. Hakim and A. Krzywicki, CERN preprint TH-3305, submitted to Nucl. Phys. B.
- 10) M. Creutz, L. Jacobs and C. Rebbi, Phys. Rev. D20 (1979) 1915.

FIGURE CAPTION

Behaviour of the photon mass m as a function of $\bar{\beta} = \beta(d-1)$ for $N = 16$.
The normalization used is explained in the text.