RADIATION DAMPING AND LAGRANGE INVARIANTS

M. BELL AND J. S. Bell
CERN, 1211 Geneva 23, Switzerland

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A general formula is presented for the damping of small oscillations about closed orbits in classical mechanics by dissipative perturbations. It is based on the variation of Lagrange invariants. It is applied to rederive the standard results for the effects of classical radiation damping on storage-ring orbits.

1. INTRODUCTION

In the literature^{1–3} the discussion of classical radiation damping of storage-ring orbits has been along ad hoc—but perfectly sound!—lines. We will show here how the results can be obtained by straightforward calculation from a rather general formula.

The starting point is the constancy, for any Hamiltonian system, of the expression (related to ‘Lagrange brackets’,^{4,5})

$$\sum_n [(\delta_1 p_n)(\delta_2 q_n) - (\delta_1 q_n)(\delta_2 p_n)], \quad (1)$$

where q_n and p_n are canonical coordinates and momenta, the summation is over degrees of freedom, and δ_1 and δ_2 denote variations from a given solution of the equations of motion to nearby solutions. In what follows, the given solution will always be a closed orbit. It will be assumed that any dependence of the Hamiltonian on the independent variable—the distance s measured along the closed orbit—has the periodicity of that orbit.

For example, from a given variation $(\delta_1 q, \delta_1 p)$, a second can be obtained simply by looking one (or more than one) revolution further on

$$\begin{pmatrix} \delta_2 q_1(s) \\ \delta_2 p_1(s) \\ \vdots \end{pmatrix} = \begin{pmatrix} \delta_1 q_1(s + s_0) \\ \delta_1 p_1(s + s_0) \\ \vdots \end{pmatrix} = T(s) \begin{pmatrix} \delta_1 q_1(s) \\ \delta_1 p_1(s) \\ \vdots \end{pmatrix}. \quad (2)$$

Here $T(s)$ is the matrix which propagates small displacements one revolution around the machine, from the point s to $s + s_0$, where s_0 is the length of the closed orbit. Substitution of Eq. (2) into Eq. (1) gives a constant of the motion involving only the single displacement δ_1 . This constant of motion has been much used in accelerator theory,^{6–10} under various names or none. It has been applied for example to the question of adiabatic variation,⁶ and to the problem of ‘twist’ instabilities.^{7,11}

The constancy of (1) along the orbit is readily verified by differentiation with respect to the independent variable s (supposed here to be left unchanged by the variations δ_1 and δ_2) and invocation of the Hamilton equations

$$dq_n/ds = \partial K/\partial p_n, \quad dp_n/ds = -\partial K/\partial q_n,$$

where K is the Hamiltonian for independent variable s .^{4,7,12} Suppose now that these equations are perturbed to

$$\left. \begin{aligned} dq_n/ds &= \partial K/\partial p_n \\ dp_n/ds &= -\partial K/\partial q_n + \mathcal{F}_n \end{aligned} \right\}. \quad (3)$$

The $\mathcal{F}_n$ represent additional forces that we cannot, or do not wish to, incorporate into the Hamiltonian K . Then one readily finds

$$(d/ds) \sum_n [(\delta_1 p_n)(\delta_2 q_n) - (\delta_1 q_n)(\delta_2 p_n)] = \sum_n [(\delta_1 \mathcal{F}_n)(\delta_2 q_n) - (\delta_1 q_n)(\delta_2 \mathcal{F}_n)] \quad (4)$$

The validity of (4) depends only on the linearized equations of motion for small $(\delta q, \delta p)$. These remain valid when real solutions are combined to form complex solutions, as is often convenient. Let $(\delta_1 q, \delta_1 p)$ be identified with some complex solution $(\delta q, \delta p)$, and let $(\delta_2 q, \delta_2 p)$ be identified with the complex conjugate solution $(\delta q^*, \delta p^*)$. Then Eq. (4) becomes

$$(d/ds) \operatorname{Im} \sum_n (\delta p_n)^* \delta q_n = \operatorname{Im} \sum_n (\delta \mathcal{F}_n)^* \delta q_n. \quad (5)$$

Integrating round the ring, we have the change in one turn

$$\left[\operatorname{Im} \sum_n \delta p_n^* \delta q_n \right]_{s_0} - \left[\operatorname{Im} \sum_n \delta p_n^* \delta q_n \right]_0 = \oint ds \operatorname{Im} \sum_n \delta \mathcal{F}_n^* \delta q_n \quad (6)$$

Consider in particular one of the characteristic solutions which change only by an over-all factor in one revolution

$$[\delta q(s + s_0), \delta p(s + s_0)] = \exp(i\mu - d)[\delta q(s), \delta p(s)], \quad (7)$$

where μ and d are real. Using this in (6) and with the approximation

$$\exp(-2d) - 1 = -2d,$$

we have our main result

$$\begin{aligned} d &= N/D \\ N &= -(1/2) \oint ds \operatorname{Im} \sum_n \delta \mathcal{F}_n^* \delta q_n \\ D &= \operatorname{Im} \sum_n \delta p_n^* \delta q_n. \end{aligned} \quad (8)$$

The quantities on the right are evaluated along the unperturbed orbit; no particular argument s need then be specified for D , for it is independent of s .

2. THE RING

The notation will be essentially that of Sands.² The device is supposed to have a plane of symmetry, called 'horizontal' in what follows, and a closed orbit lying in that plane. As independent variable, we take the distance s measured along the closed orbit. As dependent variables, we take vertical displacement z , horizontal displacement x outward and perpendicular to the closed orbit, and time delay τ (opposite sign to Sands), with respect to the particle on the closed orbit at the given s . The corresponding canonical momenta are (Appendix A4)

$$\left. \begin{aligned} p_z &= m_0 \gamma z' / t' + e A_z \\ p_x &= m_0 \gamma x' / t' + e A_x \\ p_\tau &= -m_0 \gamma c^2 - e \phi = -E - e \phi \end{aligned} \right\}, \quad (9)$$

where m_0 is the particle rest mass; t is the time;

$$\gamma = (1 - v^2/c^2)^{-1/2},$$

where v is particle velocity and c the velocity of light; A and ϕ are electromagnetic potentials. A prime (') denotes differentiation with respect to s .

In what follows we consider for simplicity only the extreme relativistic approximation

$$\gamma \gg 1, \quad t' = dt/ds = (1 + x/\rho)c^{-1}, \quad (10)$$

where ρ is the radius of curvature of the closed orbit. Moreover we will work only to the first order in small deviations from the closed orbit. Then

$$\left. \begin{aligned} \delta z &= z \\ \delta x &= x \\ \delta \tau &= \tau \\ c \delta p_z &= E_0 \delta z' + \cdots \\ c \delta p_x &= E_0 \delta x' + \cdots \\ \delta p_\tau &= -\delta E + \cdots \end{aligned} \right\}, \quad (11)$$

where E_0 is the energy of the closed orbit, and $\cdots$ indicates potential terms which do not contribute to the Lagrange invariant (Appendix A), i.e., to D in Eq. (8).

3. RADIATION REACTION

The classical radiation reaction on a particle of charge e and velocity $\mathbf{v}$ in a magnetic field B is¹³

$$\mathbf{F} = -WE^2(v^2 B^2 - (\mathbf{v} \cdot \mathbf{B})^2)\mathbf{v}/c^3 \quad (12)$$

in the extreme relativistic case

$$\gamma \gg 1, \quad |v| \approx c,$$

with (in mks units)

$$W = e^4 / (6\pi\epsilon_0 m_0^4 c^6), \quad (13)$$

where ϵ_0 is the permittivity of free space.

In what follows we need $\mathbf{F}$ only to first order in small deviations from the closed orbit, where $\mathbf{v} \cdot \mathbf{B}$ is zero. So for our purposes

$$\mathbf{F} = -WE^2 B^2 \mathbf{v} / c. \quad (14)$$

Note that the energy loss per turn on the closed orbit is

$$U_0 = - \oint \mathbf{F} \cdot \mathbf{v} dt = WE_0^2 \oint B_0^2 ds \quad (15)$$

To first order in small quantities, the curvilinear components of $\mathbf{F}$ are (Appendix B16)

$$\left. \begin{aligned} \mathcal{F}_x &= -Wc^{-1} E_0^2 B_0^2 x' \\ \mathcal{F}_z &= -Wc^{-1} E_0^2 B_0^2 z' \\ \mathcal{F}_\tau &= WE^2 B^2 (1 + x/\rho) \end{aligned} \right\}. \quad (16)$$

In Eq. (8) then,

$$\left. \begin{aligned} c\delta\mathcal{F}_x &= -WE_0^2 B_0^2 \delta x', \\ c\delta\mathcal{F}_z &= -WE_0^2 B_0^2 \delta z' \\ \delta\mathcal{F}_\tau &= WE_0^2 B_0^2 (2\delta E/E + \delta x/\rho) \end{aligned} \right\} \quad (17)$$

where

$$\sigma^{-1} = \rho^{-1} + 2B_0^{-1} (\partial B / \partial x)|_{x=0}. \quad (18)$$

Note that Eq. (12) allows only for the magnetic guide field and not for the rf accelerating fields. The latter are regarded as of the same order as the radiation reaction itself, for which they have to compensate, and the corresponding terms in Eq. (12) would be a perturbation of higher order.

4. VERTICAL OSCILLATIONS

For the vertical betatron oscillations,

$$\delta x = \delta \tau = 0$$

The summations in Eq. (8) then reduce to single terms. From Eqs. (11) and (17),

$$cN = (1/2)WE_0^2 \oint ds B_0^2 \text{Im}(\delta z')^* \delta z \quad (19)$$

$$cD = E_0 \text{Im}(\delta z')^* \delta z. \quad (20)$$

Remembering that D is a constant of the motion, independent of s , the quotient is

$$\begin{aligned} d_z &= (1/2)WE_0 \oint ds B_0^2 \\ &= U_0/(2E_0). \end{aligned} \quad (21)$$

5. SYNCHROTRON OSCILLATIONS

The rf forces being supposed weak, a complete synchrotron oscillation requires many revolutions;

$$\mu_s \ll 1, \quad (22)$$

and over any single revolution the energy varies little;

$$\delta E = -\delta p_\tau = \text{constant}. \quad (23)$$

The energy shift δE induces a change in radial position

$$\delta x = \eta(s) \delta E/E_0, \quad (24)$$

where $\eta(s)$ is the 'off-energy function'.² At increased radius, the particle takes longer to traverse a given ds , and falls behind in time

$$c[\delta\tau(s + s_0) - \delta\tau(s)] = \oint ds (\eta(s)/\rho(s))(\delta E/E_0). \quad (25)$$

The left-hand side is also, for the characteristic solution,

$$c[\exp(i\mu_s) - 1]\delta\tau(s) \approx ci\mu_s \delta\tau(s), \quad (26)$$

so for small μ_s , and δE constant over one revolution,

$$c\delta\tau = (1/i\mu_s) (\delta E/E_0) \oint ds [\eta(s)/\rho(s)]. \quad (27)$$

Because of the large factor $(1/\mu_s)$ here, the δx terms can be neglected in comparison with the $\delta\tau$ terms in N and D of Eq. (8). [But in $\delta\mathcal{F}_\tau$, Eq. (17), one must not neglect the δx term in comparison with the δE term.]

Then from Eqs. (11) and (17),

$$\begin{aligned} N &= \left(\frac{1}{2}\right) \oint ds \text{Im}(\delta\mathcal{F}_\tau)^* \delta\tau \\ &= -\left(\frac{1}{2}\right) \oint ds WE_0^2 B_0^2 [2 + \eta(s)/\sigma(s)] \text{Im} \left[\left(\frac{\delta E}{E_0} \right)^* \delta\tau \right] \\ &= \text{Im}(\delta p_\tau)^* \delta\tau = -\text{Im}(\delta E)^* \delta\tau \end{aligned} \quad (29)$$

so

$$d_s = [U_0/(2E_0)](2 + \mathcal{D}), \quad (30)$$

where

$$\mathcal{D} = \left[\oint ds B_0^2 \left((2/B_0)(\partial B/\partial x) + (1/\rho)\eta(s) \right) \right] / \oint ds B_0^2 \quad (31)$$

6. HORIZONTAL BETATRON OSCILLATION

From the way in which radiation reaction perturbs Liouville's theorem, it is easily shown that¹

$$d_x + d_z + d_\tau = 2U_0/E_0. \quad (32)$$

It follows from Eqs. (21) and (30) that

$$d_s = (1/2)(U_0/E_0)(1 - \mathcal{D}). \quad (33)$$

However, it is perhaps of some interest to see how brutal application of Eq. (8) gives this result. We again have $\delta z = 0$ and again coupled oscillation of δx and $\delta \tau$. This coupling has to be followed in more detail than before, because μ_x is not supposed small. Once again

$$c[\delta\tau(s) - \delta\tau(0)] = \int_0^s d\bar{s} [\delta x(\bar{s})/\rho(\bar{s})]. \quad (34)$$

From

$$\delta\tau(s_0) = \exp(i\mu_x)\delta\tau(0)$$

follows

$$\delta\tau(0) = [-1 + \exp(i\mu_x)]^{-1} \int_0^{s_0} d\bar{s} (1/c) [\delta x(\bar{s})/\rho(\bar{s})], \quad (35)$$

and then

$$\delta\tau(s) = [-1 + \exp(i\mu_x)]^{-1} \left\{ \int_0^{s_0} + [-1 + \exp(i\mu_x)] \int_0^s \right\} d\bar{s} (1/c) [\delta x(\bar{s})/\rho(\bar{s})] \quad (36)$$

$$= [-1 + \exp(i\mu_x)]^{-1} \left\{ \int_s^{s_0} + \int_{s_0}^{s_0+s} \right\} d\bar{s} (1/c) [\delta x(\bar{s})/\rho(\bar{s})], \quad (37)$$

where we have used

$$\exp(i\mu_x) \delta x(s) = \delta x(s + s_0).$$

We introduce now the standard form¹⁴ (with $a \equiv \epsilon_x^{1/2}$)

$$\delta x(s) = a\beta_x^{1/2}(s) \exp\{i[\psi(s) + \delta]\}, \quad (38)$$

where a and δ are s -independent amplitude and phase, the function $\beta_x(s)$ is characteristic of the focusing system, and

$$\psi'(s) = 1/\beta_x(s). \quad (39)$$

Then

$$c\delta\tau(s) = [\delta x(s)/\{2i \sin(\mu_x/2)\exp(i\mu_x/2)\}] \int_{s_0}^{s_0+s} d\bar{s} [\delta x(\bar{s})/\delta x(s)] [1/\rho(\bar{s})],$$

where μ_x is the phase change per revolution, or

$$c\delta\tau(s) = [(\delta x(s)/\beta_x(s))][\lambda(s) - i\eta(s)] \quad (40)$$

where

$$\eta(s) = \{\beta_x^{1/2}(s)/[2 \sin(\mu_x/2)] \int_s^{s_0+s} d\bar{s} [\beta_x^{1/2}(\bar{s})/\rho(\bar{s})] \cos[\psi(\bar{s}) - \psi(s) - \mu_x/2] \quad (41)$$

and $\lambda(s)$ is defined likewise with the $\cos$ under the integral sign replaced by $\sin$. It is important to recognize [e.g., Eq. (3.6) in Ref. 8], that this $\eta(s)$ is the same 'off-energy function' already used in (24).

Strictly speaking, the time variation $\delta\tau$, in conjunction with the rf field, implies an energy variation δE . But since we regard the rf field as of the same order of magnitude as the radiation reaction for the zero-order trajectories we have

$$\delta E = 0. \quad (42)$$

Then in Eq. (8)

$$\begin{aligned} cN &= -(1/2) \operatorname{Im} \oint ds (\delta \mathcal{F}_x^* \delta x + \delta \mathcal{F}_\tau^* \delta \tau) c \\ &= (1/2) W E_0^2 \operatorname{Im} \oint ds B_0^{-2} \{ (\delta x')^* \delta x - \sigma^{-1} (\delta x)^* \delta x (\lambda - i\eta) \beta_x^{-1} \} \\ &= (1/2) a^2 W E_0^2 \oint ds B_0^{-2} (-1 + \eta(s)/\sigma) \end{aligned} \quad (43)$$

[using Eq. (15)]

$$= -(1/2) a^2 U_0 (1 - \mathcal{D}). \quad (44)$$

The denominator is, from Eqs. (8) and (11),

$$\begin{aligned} cD &= \operatorname{Im} E_0 (\delta x')^* \delta x \\ &= -a^2 E_0 \end{aligned} \quad (45)$$

Dividing Eq. (44) by Eq. (45) gives Eq. (33), as expected.

7. CONCLUDING REMARKS

We think it is already of some interest to see how the familiar results fit into the more general framework related to the formula (8). If it were necessary to improve on the extreme relativistic approximation, or (more likely) on the weak rf approximation, the present method would probably be less painful than those in the literature.

In comparing the present treatment with others, it might be thought strange that the discussion of vertical oscillation damping in §4 makes no explicit reference to the accelerating cavity. In other approaches,² it is said sometimes that the damping occurs 'at the cavity' as a result of the acceleration. But this is dependent on the variable that is considered. The slope z' is not changed directly by the radiation reaction, which does not change directly the direction of the particle. This slope changes at the rf cavity when the total momentum E/c changes but the transverse momentum Ez'/c does not. On the other hand, this transverse momentum is changed by the radiation reaction—which changes E . This last picture, considering transverse momentum rather than slope, is the more closely related to our considerations.

APPENDIX A

Role of the Potentials

The canonical momenta are not just velocities, but contain also the vector and scalar potentials. However, we will see that the potentials do not contribute to the relevant Lagrange invariants—in the present problem. The Cartesian components of velocity are

$$\begin{aligned} v_s &= \dot{s}(1 + x/\rho) = (1 + x/\rho)/t' \\ v_x &= \dot{x} = x'/t' \\ v_z &= \dot{z} = z'/t'. \end{aligned} \quad (\text{A1})$$

Then from (B1) and (B7)

$$\begin{aligned} M &= -m_0 c^2 \{ t'^2 - [x'^2 + z'^2 + (1 + x/\rho)^2]/c^2 \}^{1/2} \\ &\quad + e \{ -\phi t' + x' A_x + z' A_z + (1 + x/\rho) A_s \}. \end{aligned} \quad (\text{A2})$$

Write in this

$$t = t_0(s) + \tau, \quad t' = t'_0(s) + \tau', \quad (\text{A3})$$

where $\tau = 0$ for the reference orbit. Then

$$\left. \begin{aligned} p_z &= \partial M / \partial z' = m_0 \gamma z' / t' + e A_z \\ p_x &= \partial M / \partial x' = m_0 \gamma x' / t' + e A_x \\ p_\tau &= \partial M / \partial \tau' = -m_0 \gamma c^2 - e \phi \end{aligned} \right\}, \quad (\text{A4})$$

where

$$\gamma = \{ t'^2 - [x'^2 + z'^2 + (1 + x/\rho)^2]/c^2 \}^{-1/2} t' \quad (\text{A5})$$

$$= (1 - v^2/c^2)^{-1/2}. \quad (\text{A6})$$

Thus (as is well known) p_z and p_x are the usual canonical momenta, and p_τ is the negative of the total energy.

Consider now how the potential parts of the p 's contribute to the Lagrange invariant

$$\sum (\delta_1 p_n \delta_2 q_n - \delta_1 q_n \delta_2 p_n), \quad (\text{A7})$$

where the summation is over q_n ($= z, x, \tau$).

The simplest case is that of purely vertical oscillation. The only potential contribution is

$$(\partial A_z / \partial z) \delta_1 z \delta_2 z - \delta_1 z (\partial A_z / \partial z) \delta_2 z = 0. \quad (\text{A8})$$

Quite generally contributions from $(\partial A_x / \partial x)$ and $(\partial \phi / \partial \tau)$ cancel out in this same way. The remaining contributions involve the combinations

$$\begin{aligned} (\partial A_z / \partial \tau) + (\partial \phi / \partial z) &= E_z \\ (\partial A_x / \partial \tau) + (\partial \phi / \partial x) &= E_x \\ (\partial A_x / \partial z) - (\partial A_z / \partial x) &= \pm B_s. \end{aligned} \quad (\text{A9})$$

[The last sign depending on whether the (x, s, z) system is right- or left-handed.]

In this paper we assume the reference orbit to be a plane of symmetry. Then E_z and B_s are zero on the reference orbit (where the derivatives in question have to be evaluated). There could be a horizontal electric field E_x (in the accelerating cavity)—but it will be assumed negligible (for the cavity is designed to accelerate rather than deflect the particle).

APPENDIX B

Forces with s as Independent Variable

With the usual Lagrangian

$$L = -e\phi + e \mathbf{A} \cdot \mathbf{u} - m_0 c^2 (1 - u^2/c^2)^{1/2}, \quad (\text{B1})$$

where ϕ and $\mathbf{A}$ are the potentials of the applied fields, and $\mathbf{u} = \mathbf{r}$ is the velocity, the Cartesian-co-ordinate equations of motion are

$$(d/dt)(\partial L / \partial \dot{\mathbf{r}}) - (\partial L / \partial \mathbf{r}) = \mathbf{F}, \quad (\text{B2})$$

where $\mathbf{F}$ is the radiation reaction force. This implies that for variations δ , away from a solution of the equations, restricted to a finite part of the orbit,

$$\delta \int L dt = - \int \mathbf{F} \cdot \delta \mathbf{r}|_t dt \quad (\text{B3})$$

The variations $\delta \mathbf{r}|_t$ are at fixed time. The variation $\delta \mathbf{r}|_s$ at fixed value of some other variable s is

$$\delta \mathbf{r}|_s = \delta \mathbf{r}|_t + \mathbf{u} \delta t|_s. \quad (\text{B4})$$

Then an equivalent variational principle is

$$\delta \int L (dt/ds) ds = - \int ds (dt/ds) (\mathbf{F} \cdot \delta \mathbf{r}|_s - \mathbf{F} \cdot \mathbf{u} \delta t|_s) \quad (\text{B5})$$

or

$$\delta \int M ds = - \int ds \sum_i \mathcal{F}_i \delta q_i|_s \quad (\text{B6})$$

with

$$M = L (dt/ds) \quad (\text{B7})$$

$$\mathcal{F}_i = (dt/ds) (\mathbf{F} \cdot (\partial \mathbf{r} / \partial q_i)|_s - \mathbf{F} \cdot \mathbf{u} (\partial t / \partial q_i)|_s), \quad (\text{B8})$$

where the q_i are a set of arbitrary curvilinear coordinates including time but excluding the new independent variable s . The new Lagrangian equations of motion are

$$(d/ds) (\partial M / \partial q_i') - (\partial M / \partial q_i) = \mathcal{F}_i,$$

where a prime denotes differentiation with respect to s

$$q_i' = (dq_i/ds),$$

Or equivalently, in Hamiltonian form,

$$\left. \begin{aligned} p_i' &= -\partial K / \partial q_i + \mathcal{F}_i \\ q_i' &= \partial K / \partial p_i \end{aligned} \right\}, \quad (\text{B9})$$

where the new Hamiltonian is

$$K(p, q) = \sum_i p_i q_i' - M \quad (\text{B10})$$

and

$$p_i = \partial M(q', q) / \partial q_i' \quad (\text{B11})$$

We take for q_i the radial and vertical displacements x and z from the reference orbit, and time-delay τ with respect to the reference particle, all at given s , where s is the distance measured along the reference orbit.

Then from Eq. (B8)

$$\begin{aligned} \mathcal{F}_x &= (dt/ds) \mathbf{F} \cdot (\partial \mathbf{r} / \partial x)_s \\ \mathcal{F}_z &= (dt/ds) \mathbf{F} \cdot (\partial \mathbf{r} / \partial z)_s \\ \mathcal{F}_\tau &= -(dt/ds) \mathbf{F} \cdot \mathbf{u} \end{aligned} \quad (\text{B12})$$

In the text we consider explicitly only the extreme relativistic limit. Then

$$(dt/ds) = c^{-1}(1 + x/\rho_0), \quad (\text{B13})$$

where ρ_0 is the reference orbit radius of curvature. On its actual trajectory, the velocity is c , but with s measured along the reference orbit we have the extra factor $(1 + x/\rho_0)$. Moreover, we work only to first order in x, z, τ .

Then with

$$\mathbf{F} = -WE^2B^2\mathbf{u}/c, \quad (\text{B14})$$

where

$$W = e4/(6\pi\epsilon_0 c^6 m_0^4) \quad (\text{B15})$$

we have finally

$$\begin{aligned} \mathcal{F}_x &= -(W/c)E^2B^2x' \\ \mathcal{F}_z &= -(W/c)E_0^2B_0^2z' \\ \mathcal{F}_\tau &= WE^2B^2(1 + x/\rho) \end{aligned} \quad (\text{B16})$$

where E is the particle energy, B is the applied magnetic field, and E_0 and B_0 refer to the reference orbit.

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