Yang-Mills field configurations with critical source strengths

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(Received 25 January 1982)

A new class of type-I solutions of the Yang-Mills field equations is derived. This class of solutions exists only if the external source strengths exceed certain critical values although it has the same boundary conditions at the origin as at infinity.

I. INTRODUCTION

In the past few years, there has been considerable interest¹⁻⁸ in solutions of the classical non-Abelian Yang-Mills (YM) theories in the presence of static external sources, and some novel features have been discovered. In particular, the Abelian Coulomb solution¹ in the presence of a spherically symmetric external source is found to be unstable when gQ is large and is only gyroscopically stable when $gQ < \frac{3}{2}$, where g is the gauge-field coupling constant and Q is the external source strength. Furthermore, the nonlinear element present in the YM theories introduces a new class of solutions which exist only when the external source strength exceeds a certain critical value.^{3,4} It follows that the solutions of the YM theories with external sources can conveniently be classified into two categories: (A) solutions with sources of arbitrary strength and (B) solutions with sources of critical strength. The Abelian Coulomb solution and its time-dependent generalization — the so-called total-screening solutions² — as well as the non-Abelian Coulomb solution³ belong to class (A), whereas the magnetic dipole solution² and its generalization fall in class (B).

Recently Jackiw, Jacobs, and Rebbi³ obtained numerical solutions belonging, respectively, to class (A) and class (B) by specifying the external source in the radial gauge frame. Analytical forms of these solutions have also been derived.^{6,7} Two different sets of boundary conditions are prescribed for the gauge-field potentials. Solutions for which the gauge fields vanish at the origin as well as at infinity are called type-I solutions, whereas solutions for which the gauge fields vanish at the origin but approach a pure gauge at large distances are described as type-II solutions. The numerical

type-II solutions found in Ref. 3 possess two branches and are hence also known as bifurcating solutions. However, the analytic type-II solutions presented in Ref. 6 do not seem to bifurcate although the solutions require nonzero critical source strength to sustain themselves. Both the numerical and analytical type-I solutions^{3,6} obtained so far exist for any arbitrary value of the source strength, and in fact in the radial gauge frame they vanish with the source. Following Ref. 3 we shall call the type-I solutions with arbitrary source strength non-Abelian Coulomb solutions since in a suitable gauge frame the solutions appear similar to the Abelian Coulomb solution.

The purpose of the present paper is to point out that there exists a new class of type-I solutions which require critical source strength; that is, it is possible to construct a class of solutions which has the same boundary conditions at the origin as at infinity and which exists only if the external source strengths exceed certain critical values. This class of solutions escapes the numerical search of Ref. 3.

In the following section, we write down the pertinent equations and the boundary conditions. In Sec. III we present general arguments leading to the result that there are type-I solutions which exist only for nonzero external source strengths. The solutions vanish at the origin and at large distances. Explicit type-I solutions with critical source strength are then constructed in Sec. IV. We end with some remarks in the last section.

II. YANG-MILLS FIELD EQUATIONS WITH EXTERNAL SOURCES

The SU(2) YM field equations in the presence of an external static source are

$$(D_\mu F^{\mu\nu})_a = j_a^\nu = \delta_0^\nu \rho_a, \quad (1a)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A_\mu^b A_\nu^c \quad (1b)$$

and our metric is $g_{ii} = -g_{00} = 1$. Here ρ_a is the external charge density. We shall only consider solutions with finite total energy and finite total external charge. Following Ref. 3 we specify the external source in the radial frame and prescribe the Wu-Yang⁹ ansatz for the gauge-field potentials,

$$A_0^a = n^a f(y)/(gr), \quad n^a = x^a/r, \quad (2a)$$

$$A_i^a = \epsilon_{iaj} n^j [a(y) - 1]/(gr), \quad (2b)$$

$$\rho_a(\vec{r}) = n_a q(y)/(gr_0^3), \quad (2c)$$

where r_0 is a parameter of the dimension of length and $y = r/r_0$. The above ansatz simplifies Eq. (1) and yields the coupled nonlinear differential equations

$$-f'' + 2a^2 f/y^2 = yq, \quad (3a)$$

$$-a'' + (a^2 - 1 - f^2)a/y^2 = 0. \quad (3b)$$

The prime indicates differentiation with respect to y .

The non-Abelian electric and magnetic fields can be written as

$$E_a^i = F_a^{i0} = \frac{af}{gr_0^2 y^2} (n_a n^i - \delta_a^i) - \left[\frac{f}{y} \right]' \frac{n^i n_a}{gr_0^2}, \quad (4)$$

$$B_a^i = \frac{1}{2} \epsilon^{ijk} F_a^{jk} = \frac{a'}{gr_0^2 y} (\delta_a^i - n^i n_a) + \frac{a^2 - 1}{gr_0^2 y^2} n_a n^i. \quad (5)$$

The total energy of the system is given by

$$\begin{aligned} \xi &= \frac{1}{2} \int d^3r (\vec{E}_a^2 + \vec{B}_a^2) \\ &= \frac{4\pi}{g^2 r_0} \int_0^\infty dy \left[(a')^2 + \frac{1}{2y^2} (a^2 - 1)^2 \right. \\ &\quad \left. + \frac{1}{2} (f')^2 + (af/y)^2 \right]. \quad (6) \end{aligned}$$

The requirements that the energy ξ be finite, that $yq(y)$ vanish at the origin, and that $q(y)$ decrease faster than y^{-4} at large y impose the following conditions on the asymptotic behavior of the functions $a(y)$ and $f(y)$. For small y , $a(y)$ must tend to 1 and $f(y)$ must vanish at least as fast as³

$$a(y) \simeq 1 + a_2 y^2, \quad (7a)$$

$$f(y) \simeq f_2 y^2, \quad (7b)$$

where a_2 and f_2 are constants. For large y , there are two possible asymptotic forms for the function $a(y)$, depending on whether A_μ^a vanishes or tends to a nontrivial pure gauge at infinity. The type-I behavior is that the gauge-field potential A_μ^a vanishes at large distances, which implies the following behavior for $a(y)$ and $f(y)$:

$$a(y) \simeq 1 + a_{-1}^I y^{-1}, \quad (8a)$$

$$f(y) \simeq f_{-1}^I y^{-1}. \quad (8b)$$

For the type-II solution, A_μ^a approaches pure gauge at large y and we have

$$a(y) \simeq -1 + a_{-1}^{II} y^{-1}, \quad (9a)$$

$$f(y) \simeq f_{-1}^{II} y^{-1}. \quad (9b)$$

Here a_{-1}^I, f_{-1}^I and a_{-1}^{II}, f_{-1}^{II} are constants. For the prescribed behaviors (7a), (7b), (8a), (8b), (9a), and (9b) it is easy to establish

$$f_2 = 3^{1/2} a_2, \quad (10a)$$

$$f_{-1}^I = 3^{1/2} a_{-1}^I, \quad (10b)$$

$$f_{-1}^{II} = 3^{1/2} a_{-1}^{II}. \quad (10c)$$

The condition (7b) restricts the next order of approximation for $a(y)$ to be y^4 and not y^3 as y tends to zero.

III. TYPE-I CONFIGURATIONS WITH CRITICAL SOURCE STRENGTHS

We now argue generally that there exists a class of type-I solutions with critical source strengths and later on we shall exhibit explicit expressions for such class of solutions.

We first note that

$$a(y) = 1, \quad f(y) = 0 \quad (11)$$

for all y gives the trivial solution $A_\mu^a = 0$; and

$$a(y) = -1, \quad f(y) = 0 \quad (12)$$

leads to the pure gauge solution

$$\begin{aligned} A_i^a &= -\epsilon_{iaj} n^j \frac{2}{gr} = \text{pure gauge}, \\ A_0^a &= 0. \end{aligned} \quad (13)$$

Evidently for both solutions (11) and (12) the source strength is zero. Any solution which can be continuously deformed to solution (11) or solution

(12) for all y can therefore exist for any arbitrary source strength. On the other hand any solution which cannot be continuously deformed to solution (11) or solution (12) requires a critical source strength to support itself. The type-I solutions presented in Refs. 3, 6, and 7 can be continuously deformed to the trivial solution (11) for all y by varying the parameters of the solutions, indicating clearly that they can exist for any arbitrary value of the external source strength. The type-II solutions^{3,6} interpolate between the trivial solution (11) and the pure gauge solution (12) and consequently they cannot be deformed to either (11) or (12) for all y . Hence for the type-II solutions there must exist an interval of y such that $a \neq \pm 1$ and $f \neq 0$. In that interval, $q(y) \neq 0$, which implies that a minimum source strength is needed for the solutions to exist.

For the type-I solutions discussed in Refs. 3, 6, and 7, $a(y) \geq 1$ for all y . It follows from expression (7a) and (8a) that a_2 and a_{-1}^I are positive real numbers and that $y=0$ and $y=\infty$ are the two minimum points for $a(y)$. Suppose we now let a_2 and a_{-1}^I be negative real numbers, that is, $y=0$ and $y=\infty$ are the two maximum points of $a(y)$ and of course near these maxima $a(y) < 1$. As y starts to increase from zero, $a(y)$ decreases from 1 with $a''(y) < 0$. Now the minimum value of $a(y)$ cannot be positive for that would lead to a negative $f^2(y)$, which is unacceptable. Hence as y increases, $a(y)$ must continue to decrease until it intersects the y axis, and it is easy to see that the graph of $a(y)$ must cut the y axis at least twice before it reaches the maximum value again at $y=\infty$. To be more explicit, we consider the case when $a(y)$ intersects the positive y axis at only two points, namely at y_1 and at y_2 . In order for $f^2(y)$ to be positive and remembering that a_2 and a_{-1}^I are negative, we must have $a''(y) < 0$ for y in the intervals $(0, y_1)$ and (y_2, ∞) . This means that $a(y)$ takes values between zero and one for any y in the above intervals. At the intersecting points y_1 and y_2 , $a''(y)$ must vanish such that the ratio $a''(y)/a(y)$ is finite and negative. For y in the interval (y_1, y_2) , $a''(y) > 0$ and $a(y)$ assumes negative values. With the above behavior as prescribed for the function $a(y)$, one is led to a positive $f^2(y)$ for all y and hence an acceptable solution. However, because of the condition that the minimum of $a(y)$ must occur below the y axis it is impossible to deform this class of solutions either to the trivial solution (11) or the pure gauge solution (12). As a result, this class of solutions requires the support of critical source strengths.

The above argument does not rely on the fact that the constants a_2 and a_{-1}^I must be negative and indeed they can either be positive or negative. As long as there exists an interval $[\alpha, \beta]$ along the positive y axis such that $a(\alpha) = a(\beta) = k$, where $k \leq 1$ and for all y in $[\alpha, \beta]$ such that $a(y) \leq k$, then $a(y)$ must intersect the y axis twice in this interval. This can be seen as follows. Evidently there exists a point y_0 of the interval $[\alpha, \beta]$ at which $a(y)$ assumes a minimum value. Suppose $a(y)$ does not cut the y axis, that is, $a(y) > 0$ for all y in $[\alpha, \beta]$. Then at $y=y_0$, $-y^2 a''/a$ is negative and so is $(a^2 - 1)$. This yields a negative $f^2(y)$ at $y=y_0$ which is unacceptable. Hence $a(y)$ must intersect the y axis twice. Note that as $a''/a < 0$, $a(y)$ can oscillate. Consequently $a(y)$ may intersect the y axis any even number of times. Thus depending on whether the constants a_2 and a_{-1}^I are positive or negative, the acceptable forms of $a(y)$ are (a) maximum at $y=0$ and at $y=\infty$, (b) maximum at $y=0$ and minimum at $y=\infty$, (c) minimum at $y=0$ and maximum at $y=\infty$, and (d) minimum at $y=0$ and at $y=\infty$. These are illustrated in Fig. 1 for $a(y)$ with only two zeros. The graphs are deformable to each other. Incidentally, for the type-I solutions with arbitrary source strengths, only one form of $a(y)$ is acceptable, that is, $y=0$ and $y=\infty$ are the minimum point of $a(y)$.

IV. EXPLICIT SOLUTIONS

In this section we construct a class of explicit type-I solutions with critical source strengths. We consider only the case where $y=0$ and $y=\infty$ are the maximum points of the function $a(y)$. From our previous experience with analytic type-II solutions given in Ref. 6 we write down an expression for $a(y)$ which has a negative minimal value,

$$a(y) = 1 - \exp[-u(y)], \quad (14a)$$

$$u(y) = hy^{-2} + by^{-1} - d + ky + cy^2 + py^3, \quad (14b)$$

where h, b, d, k, c , and p are parameters to be determined. Suppose $a(y)$ intersects the y axis at two points y_1 and y_2 , then we must have

$$u(y_1) = 0, \quad (15a)$$

$$u(y_2) = 0. \quad (15b)$$

In order for $f(y)$ to be finite at y_1 and y_2 , clearly $a''(y)$ must also vanish at these two points. This implies

FIG. 1. The various forms of $a(y)$ with critical source strengths: (a) maximum at $y=0$ and at $y=\infty$; (b) maximum at $y=0$ and minimum at $y=\infty$; (c) minimum at $y=0$ and maximum at $y=\infty$; (d) minimum at $y=0$ and at $y=\infty$.

$$u''(y_1) = u'^2(y_1), \quad (16a)$$

$$u''(y_2) = u'^2(y_2). \quad (16b)$$

With the four conditions (15a), (15b), (16a), and (16b), we may solve for, say, h , b , d , k in terms of the other two parameters c and p , while y_1 and y_2 are kept fixed. However, it turns out that p can be exactly determined by the two intersecting points y_1 and y_2 and consequently we express h , b , d , and k in terms of c only. Thus with y_1 and y_2 being kept fixed, our solutions depend only on one parameter c .

Eliminating d from Eqs. (15a) and (15b) one easily obtains

$$k = h(y_1 + y_2)/(y_1 y_2)^2 + b/(y_1 y_2) - c(y_1 + y_2) - p(y_1^2 + y_1 y_2 + y_2^2). \quad (17)$$

Substituting (17) into (16a), we get

$$b^2 + 2b \left[S - \frac{y_1 y_2^2}{(y_1 - y_2)^2} \right] + S^2 - \left[\frac{y_1}{y_1 - y_2} \right]^2 (6h + 2cy_1^4 + 6py_1^5) = 0, \quad (18a)$$

$$S = h \frac{y_1 + 2y_2}{y_1 y_2} + cy_1^2 y_2 + py_1^2 y_2 (2y_1 + y_2).$$

The substitution of (17) into (16b) will give exactly the same equation as Eq. (18a) except with y_1 and y_2 being interchanged, and we call the resulting equation (18b). In order for conditions (16a) and (16b) to be fulfilled with the same values of b we require the roots of b in Eq. (18a) and Eq. (18b) to be identical. This is achieved by demanding that the coefficients of b^2 , b , and the terms in-

dependent of b be the same in Eqs. (18a) and (18b). This yields two equations from which the parameter h is eliminated. After some computation, we arrive at

$$p = \frac{3(y_1 + y_2)}{2(y_1 - y_2)^4}. \quad (19)$$

From now on it is then a simple matter to derive the following expressions:

$$h = (y_1 y_2)^2 \left[c + \frac{3(y_1 + y_2)^2}{(y_1 - y_2)^4} + (y_1 - y_2)^{-2} \right], \quad (20a)$$

$$b = \frac{-y_1 y_2}{(y_1 - y_2)^2} \left[(y_1 + y_2) \left[2c(y_1 - y_2)^2 + \frac{3(4y_1^2 + 7y_1 y_2 + 4y_2^2)}{2(y_1 - y_2)^2} + 1 \right] - G \right], \quad (20b)$$

$$k = -(y_1 + y_2) \left[2c + \frac{3(3y_1^2 + 4y_1 y_2 + 3y_2^2)}{2(y_1 - y_2)^4} \right] + G(y_1 - y_2)^{-2}, \quad (20c)$$

$$d = - \left[c(y_1^2 + 4y_1 y_2 + y_2^2) + 3(y_1^2 + 3y_1 y_2 + y_2^2) \frac{(y_1 + y_2)^2}{(y_1 - y_2)^4} + \frac{y_1 y_2}{(y_1 - y_2)^2} \right] + \frac{y_1 + y_2}{(y_1 - y_2)^2} G, \quad (20d)$$

where G is given by

$$G = \pm [2c(y_1 - y_2)^4 + 9(y_1 + y_2)^2 - 5y_1 y_2]^{1/2}. \quad (21)$$

Thus all the parameters in the expression (14) are expressed in terms of c and the intersecting points y_1 and y_2 . The solution $f(y)$ is evaluated from Eq. (3b),

$$f(y) = a^2 - 1 + y^2(1 - a)(u'^2 - u'')/a. \quad (22)$$

As y approaches zero, the behaviors of the solution $a(y)$ and $f(y)$ as well as the charge density $q(y)$ are

$$a(y) \simeq 1 - \exp(-h/y^2), \quad (23a)$$

$$f(y) \simeq 2hy^{-2} \exp[-h/(2y^2)], \quad (23b)$$

$$q(y) \simeq -2h^3 y^{-9} \exp[-h/(2y^2)], \quad (23c)$$

whereas at large distances we have

$$a(y) \simeq 1 - \exp(-py^3), \quad (24a)$$

$$f(y) \simeq 3py^3 \exp(-py^3/2), \quad (24b)$$

$$q(y) \simeq -\frac{27}{4}p^3 y^6 \exp(-py^3/2). \quad (24c)$$

Evidently it is necessary for the parameters h and p to assume positive values. That p is positive is already fulfilled by Eq. (19), and from expression (20a) h will be positive if

$$c > - \left[\frac{3(y_1 + y_2)^2}{(y_1 - y_2)^4} + (y_1 - y_2)^{-2} \right] = H. \quad (25)$$

We have plotted in Fig. 2 the solutions $a(y)$,

$f(y)$ and the source density $q(y)$ for $c = -27$ and $c = -22$, respectively, where we put $y_1 = 1$ and $y_2 = 2$ which yields $H = -28$. We also choose positive G from expression (21). The charge density distribution is a quickly damped wavy form in contrast to the δ function of Ref. 3. A gauge-invariant characterization of the total external charge can be defined as

$$Q = \int d^3r [\rho_a(\vec{r}) \rho_a(\vec{r})]^{1/2} \\ = \frac{4\pi}{g} \int_0^\infty dy y^2 |q(y)|.$$

It is not difficult to compute the total energy ξ and charge Q for the above solutions with $y_1 = 1$ and $y_2 = 2$. We find for $c = -27$ (-22), $\xi = [4\pi/(g^2 r_0)]$ 136.625 348 (215.700 218) and $Q = (4\pi/g)$ 154.888 140 (201.038 766). In fact it can be shown that with $y_1 = 1$ and $y_2 = 2$, the energy ξ and the total charge Q decrease as the parameter c tends to $H = -28$, and the respective minimum of ξ and Q occurs at the same value of c . This indicates that the solution (14) possesses a bifurcation point.¹⁰ Furthermore for a fixed c , ξ and Q increase as y_2 moves closer to y_1 and decrease when y_2 is away from y_1 .

V. COMMENTS

We now proceed to make some remarks.

(1) From the general arguments in Sec. III, the function $a(y)$ can intersect the y axis any even number of times. In the preceding section we merely manage to construct an expression for $a(y)$

FIG. 2. (a) The function $a(y)$ for the solution (14) with $y_1=1$ and $y_2=2$. (b) The function $f(y)$ for the solution (14) with $y_1=1$ and $y_2=2$. (c) The charge density $q(y)$ for the solution (14) with $y_1=1$ and $y_2=2$. In all three graphs the solid curve corresponds to $c = -27$ while the dashed curve corresponds to $c = -22$.

which cuts the y axis only twice, we are not yet able to find an explicit $a(y)$ with more than two zeros. We expect that the total energy of the system will increase with increasing number of zeros of $a(y)$. In passing we note that for the type-II solution,³ the $a(y)$ may intersect the y axis any odd

number of times.

(2) We have shown that for the finite-energy solutions with finite total source strengths, the function $a(y)$ and hence the gauge-field potential A_a^i can oscillate. This property is in fact also shared by solutions with sources of infinite extent.⁸

(3) The explicit solutions (14) given in Sec. IV contain six parameters and we have shown that the four conditions (15a), (15b), and (16a), and (16b) enables us to express four of the parameters in terms of the other two, in this case, c and p . It happens that p can be expressed in terms of y_1 and y_2 exactly and hence the four parameters are expressed finally in terms of only one parameter, namely c . There are also other ways to determine the parameters. For instance we may require in Eq. (18a) that b can only have one root. This additional condition together with the four conditions (15a), (15b), and (16a), and (16b) permit us to express all of the five parameters in the solutions (14) in terms of just one parameter, say c . That is, in this case, p also varies with c .

(4) From the asymptotic behavior (24) at large distances, one observes that the gauge-field potentials A_μ^a and hence the field strengths $F_{\mu\nu}^a$ vanish faster than the source density. For the analytic type-II solutions obtained in Ref. 6, the source density also decreases slower than the field strengths at large distances. Thus for these two types of solutions with critical source strengths, the YM fields are trapped inside the external source distribution. However, this is not always true for solutions with critical source strengths: for the numerical type-II solutions of Ref. 3, the field strengths tend to zero slower than the spherical shell source.

(5) For the type-II solutions, the critical source strength arises from the fact that the boundary conditions are different at the origin and at infinity. For the type-I solutions presented in Sec. IV, the boundary conditions are the same at the origin as at infinity and it is the reality of $f(y)$ or $A_a^0(x)$ which results in critical source strengths. It thus appears that the reason for the stability of the type-I solution with critical source strength, if it is stable at all, is not likely of topological origin but is related to the reality of $f(y)$.

(6) Finally, we note that for the magnetic dipole solutions² with critical source strengths,⁴ the gauge-field potentials also have the same boundary conditions; that is, A_μ^a vanish at origin and at infinity in the Abelian gauge frame. The critical source strength of the magnetic dipole solution arises because the function $\bar{a}(y)$ (Ref. 11) interpolates

between the trivial solutions $\bar{a}(y)=0$, $\phi(y)=0$ for all y and $\bar{a}(y)=1$, $\phi(y)=0$ for all y .

Note Added. It has been brought to our attention that type-I solutions have also been discussed by Jacobs and Wudka.¹²

ACKNOWLEDGMENT

One of us (Rosy Teh) wishes to thank CERN for financial support and the Theory Division for its hospitality.

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