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ON THE FUNCTIONAL INTEGRATION MEASURE
OF SUPERSYMMETRIC YANG-MILLS THEORIES

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ABSTRACT

For the simplest $N = 1$ supersymmetric Yang-Mills theory, we construct a bosonic field transformation which linearizes the Yang-Mills action in each topological sector and whose Jacobian equals the product of Fadeev-Popov and Matthews-Salam determinants up to cubic order at least. Some possible implications of this result for the maximally extended $N = 4$ theory are discussed.

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The special status of supersymmetric gauge theories¹⁾ is generally appreciated. While the perturbative aspects of supersymmetric models have been extensively studied and appear to be well understood, at least in the case of $N = 1$ rigid supersymmetry²⁾, much less is known about their non-perturbative properties. For instance, superspace perturbation theory greatly facilitates complex calculations and renders cancellations manifest. Can one hope, in a similar vein, that supersymmetry could make non-perturbative calculations easier? The answer to this question deserves some attention, especially with regard to the problem of supersymmetry breaking. Since no boson fermion degeneracy has been found in Nature so far, this is a central issue. However, supersymmetry is notoriously difficult to break³⁾. For example, spontaneous breaking requires an adjustment of parameters which is natural in a technical sense at best, and explicit breaking, however "soft", is rather unattractive. Possible resolutions based on dynamical breaking³⁾ come closer to one's aesthetic preconceptions, but they require a better understanding of the global structure of these theories.

It is not clear whether the superspace approach⁴⁾ is adequate in this respect, and therefore independent methods should be developed. There is one alternative characterization of supersymmetric theories directly in terms of the functional measure of the associated quantum field theory⁵⁾ which, furthermore, does not employ the concept of anticommuting variables. Namely, it can be shown that for any rigidly supersymmetric theory, there exists a non-linear and generally non-local mapping of the bosonic variables which rotates the full functional measure of the theory [including Matthews-Salam⁶⁾ and Faddeev-Popov⁷⁾ determinants] into that of a Gaussian, i.e., free theory. The reason is the vanishing of the vacuum energy in supersymmetric theories⁸⁾ or, more precisely, the adiabatic invariance of its vacuum functional (up to jump discontinuities) under variations of the coupling parameters. The proof of this statement, as given in Ref. 5), is based on an iterative procedure and is therefore perturbative. As such, it does not take into account non-perturbative effects which could appear in the form of exponentially small ($\sim \exp[-K/g^2]$) contributions to the vacuum energy. In the latter case, supersymmetry and adiabatic invariance with respect to small changes in g would be violated, and the theorem clearly requires modification.

In this paper we reconsider this theorem in the context of $N = 1$ supersymmetric Yang-Mills theory in four dimensions¹⁾ (= SYM for brevity). We explicitly construct a transformation of the bosonic fields which "almost" solves the problem in the following sense: it linearizes the Yang-Mills action while taking into account the topologically inequivalent sectors of the theory, and its Jacobian determinant is equal to the product of the Faddeev-Popov and the

Matthews-Salam determinant up to third order and exactly equal up to zero modes at the stationary points of the Yang-Mills action.

The possible relevance of such a bosonic field transformation was pointed out in Ref. 9) where it was shown that the winding number of this mapping can be identified with Witten's index³⁾. It is evident that an explicit construction [as opposed to the iterative construction of Ref. 5)] would provide us with a very powerful tool for studying these theories and that even a partial construction could be helpful. Firstly, perturbation theory would be considerably simplified. In contradistinction to the superspace approach where many auxiliary degrees of freedom must be introduced, one here strips down the theory to its bosonic physical degrees of freedom: this has been called the infra-diagrammatic approach¹⁰⁾. Secondly, it is well known that the ordinary perturbation expansion diverges even after UV regularization because of the factorial growth of its coefficients¹¹⁾. These combinatorial divergences are produced by functional integration according to Wick's theorem. Since the power series expansion for the field transformation is done *before* functional integration, this expansion is likely to have a fine radius of convergence. Thirdly, and most importantly, one may be able to compute non-perturbative effects more reliably.

Somewhat surprisingly, there are indeed examples where the mapping can be constructed explicitly, namely supersymmetric quantum mechanics^{3),12)} and the $N = 2$ Wess-Zumino model in two space-time dimensions^{10),13)}. In both cases, one may directly verify that the bosonic action becomes a free action after substitution of the new variables, and that the Jacobian of the transformation equals the Matthews-Salam determinant obtained upon integrating out the fermions. However, no such examples are yet known in the case of four dimensions, and this paper represents a first attempt in this direction. We will not discuss the lower dimensional examples in any detail here, but it is nonetheless useful to keep in mind the close analogy of the considerations below with the results obtained there, particularly for supersymmetric quantum mechanics¹²⁾.

In a Euclidean space-time, the Lagrangian of the $N = 1$ SYM theory is given by^{*)}

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} \bar{\lambda}^a \gamma^\mu (D_\mu \lambda)^a \quad (1)$$

^{*)} We follow the notation and conventions of Ref. 5).

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \quad (2)$$

$$(D_\mu \lambda)^a = \partial_\mu \lambda^a + g f^{abc} A_\mu^b \lambda^c \quad (3)$$

and λ^a is a Euclidean Majorana spinor as defined in Ref. 14). Of course, the Lagrangian (1) cannot be Hermitean in the conventional sense. However, in Euclidean field theory, the relevant notion is not hermiticity but rather Osterwalder-Schrader positivity¹⁵⁾ which guarantees that the resulting theory may be analytically continued back to Minkowski space. It can be shown that Eq. (1) gives rise to an action which formally has this property.

As (1) contains superfluous degrees of freedom, a gauge must be fixed. We therefore choose to work in the axial gauge which is defined by the condition

$$A_0^a(x) = 0 \quad (4)$$

so, after the gauge choice (4), the theory only contains the spatial components $A_i^a(x)$. We shall also need the notion of topological charge Q associated with a gauge field configuration $A_\mu^a(x)$ ¹⁶⁾ which is given by

$$\begin{aligned} Q(A) &= \frac{1}{4} \int \epsilon^{\mu\nu\sigma\tau} F_{\mu\nu}^a(A) F_{\sigma\tau}^a(A) d^4x = \\ &= \int \partial^\mu \eta_\mu d^4x \end{aligned} \quad (5)$$

with the topological current¹⁶⁾

$$\eta^\mu = \epsilon^{\mu\nu\sigma\tau} \left(A_\nu^a \partial_\sigma A_\tau^a + \frac{1}{3} g f^{abc} A_\nu^a A_\sigma^b A_\tau^c \right) \quad (6)$$

It can be shown that the topological charge Q is quantized¹⁶⁾

$$Q = \frac{8\pi^2}{g^2} n, \quad n \in \mathbb{Z} \quad (7)$$

We shall now make the assumption that the space of gauge field configurations can be decomposed into sectors of different topological charge. (It should be pointed out that although this assumption is commonly made, it has never been rigorously justified. It certainly holds for sufficiently smooth configurations, but the "typical" configuration to be integrated over in the functional integral is known to be much more singular and presumably distribution-valued. For such configurations, $Q(A)$ ceases to be well defined.)

After these preliminary remarks, we can now introduce the mapping which linearizes the Yang-Mills theory. We define the new variables ξ_1^a by

$$\begin{aligned}\xi_i^a(A) &= \dot{A}_i^a - \frac{1}{2} \varepsilon_{ijk} F_{jk}^a(A) \quad \text{if } Q(A) \geq 0 \\ \xi_i^a(A) &= \dot{A}_i^a + \frac{1}{2} \varepsilon_{ijk} F_{jk}^a(A) \quad \text{if } Q(A) < 0\end{aligned}\tag{8}$$

where ε^{ijk} is the usual completely antisymmetric tensor in three (space) dimensions. The axial gauge choice (4) is convenient because the new variables should be in one-to-one correspondence with the original variables, at least locally. By a straightforward calculation one now proves the following relation

$$\begin{aligned}S(A) &= \frac{1}{4} \int F_{\mu\nu}^a F_{\mu\nu}^a d^4x = \\ &= \frac{1}{2} \int \xi_i^a(A) \xi_i^a(A) d^4x + |Q(A)|\end{aligned}\tag{9}$$

We have thus found a variable transformation which reduces the classical Yang-Mills theory to a free (Gaussian) theory. (A natural class of functions for which (8) and (9) are well defined consists of those gauge field configurations which have finite action. In terms of the new variables ξ_1^a this amounts to the requirement of square-integrability.) The mapping (8) is *not* one-to-one because

$$\xi(A) = 0 \implies A = \text{instanton solution}\tag{10}$$

and there exist instantons, i.e., solutions of the classical Euclidean Yang-Mills equations for any topological charge¹⁶⁾. Hence, the winding number of the mapping (8) is infinity.

It is instructive at this point to draw some analogy between the model considered here and supersymmetric quantum mechanics¹²⁾. The latter is described by one (complex) fermionic variable and one bosonic variable $q(t)$. The transformation which linearizes the bosonic part of the action is given by^{9),12)}

$$\xi(t) = \dot{q}(t) \pm V(q(t)) \quad (11)$$

where $V(q)$ is a certain function which determines the potential. The transformation (11) closely resembles (8), and one may easily verify that a formula analogous to (9) holds. It has been remarked^{10),13)} that (11) is a stochastic differential equation and that the reformulation of quantum field theories in terms of such differential equations may be advantageous because well-developed mathematical techniques are available¹⁷⁾. Of course, similar arguments apply here.

There is also a rather intriguing relation with Morse theory¹⁸⁾. One may regard the Yang-Mills action as a Morse function on the (infinite dimensional) manifold of gauge field configurations. Then (8) reduces the Morse function to a quadratic form at each stationary point^{*)} and, remarkably, Eq. (8) is valid on the whole manifold and not just in a neighbourhood of a stationary point. The possible relation between supersymmetry and Morse theory has been beautifully exploited in Ref. 20) and we refer the reader to that paper for a further discussion.

The change of variables (8) also affects the functional integration measure. The Jacobian associated with (8) reads

$$\begin{aligned} \det \frac{\delta \xi_i^a(x; A)}{\delta A_j^b(y)} &= \det \left[(\delta_{ij} \delta^{ab} \partial_0 + \varepsilon_{ijk} D_k^{ab}(A)) \delta(x-y) \right] = \\ &= \det \left[(\delta_{ij} \delta^{ab} \partial_0 - \varepsilon_{ijk} D_k^{ab}(A)) \delta(x-y) \right] \end{aligned} \quad (12)$$

The last step follows from the observation that $\det(H+A) = \det(H-A)$ for $H = H^\dagger$ and $A = -A^\dagger$. (This is possible, at least formally, since both ∂_0 and $D_i^{ab}(A)$ are antiHermitian operators.) We will now demonstrate that the Jacobi determinant (12) equals the product of the Faddeev-Popov determinant

*) This interpretation of the bosonic field transformation was first suggested in Ref. 19).

and the Matthews-Salam determinant^{6),7)} up to third order. For this purpose we rewrite (12) as

$$\det [\delta^{ab} \partial_0 \delta(x-y)]^3 \cdot \det [(\delta_{ij} \delta^{ab} \pm \varepsilon_{ijk} O_k^{ab}(A)) \delta(x-y)] \quad (13)$$

with the operator^{*)}

$$O_i^{ab}(A) = \partial_0^{-1} \circ D_i^{ab}(A) \quad (14)$$

The sign in (13) may be chosen arbitrarily since odd powers of O_i^{ab} do not contribute when one expands the determinant according to $\det = \exp \text{Tr} \log$.

The Faddeev-Popov determinant⁷⁾ associated with the gauge choice (4) is

$$\Delta_{FP} = \det [\delta^{ab} \partial_0 \delta(x-y)] \quad (15)$$

Being trivial, it is usually omitted, but here we keep it because it provides a necessary factor below. The Matthews-Salam determinant⁶⁾ is obtained by integrating out the fermions in the $N=1$ SYM action. Since we are dealing with Majorana spinors, it is actually the square root of a determinant and is equal to

$$\Delta_{MS}(A) = \left\{ \det [\gamma^\mu D_\mu^{ab}(A) \delta(x-y)] \right\}^{1/2} \quad (16)$$

In the axial gauge, one therefore gets the following expression for the product of these determinants

$$\Delta_{FP}(A) \Delta_{MS}(A) = \det [\delta^{ab} \partial_0 \delta(x-y)]^3 \cdot \left\{ \det [(\delta^{ab} \gamma^0 + \gamma^k O_k^{ab}(A)) \delta(x-y)] \right\}^{1/2} \quad (17)$$

^{*)} Since our arguments are formal anyway, we will not worry about mathematical subtleties such as invertibility of operators or the proper definition of infinite-dimensional determinants.

In order to bring this expression into a somewhat more tractable form and to eliminate the square root, it is most advantageous to choose the following representation of Euclidean γ matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \hat{\sigma}_\mu \\ \hat{\sigma}_\mu^+ & 0 \end{pmatrix}, \quad \hat{\sigma}_\mu = (1, i\sigma_k) \quad (18)$$

Then, for real but otherwise arbitrary matrices X_μ ,

$$\begin{aligned} (\det X_\mu \otimes \hat{\sigma}_\mu)^* &= \det X_\mu \otimes \hat{\sigma}_\mu^* = \\ &= \det X_\mu \otimes \sigma_2 \hat{\sigma}_\mu \sigma_2 = \det X_\mu \otimes \hat{\sigma}_\mu^+ \end{aligned} \quad (19)$$

and therefore

$$\det X_\mu \otimes \hat{\sigma}_\mu^+ = \det X_\mu^T \otimes \hat{\sigma}_\mu \quad (20)$$

This allows us to reexpress (17) as

$$\begin{aligned} \Delta_{FP}(A) \Delta_{MS}(A) &= \det [\delta^{ab} \partial_0 \delta(x-y)]^3 \cdot \\ &\cdot \det [(\delta_{\alpha\beta} \delta^{ab} \pm i\sigma_k \alpha\beta O_k^{ab}(A)) \delta(x-y)] \end{aligned} \quad (21)$$

which is to be compared with (13). As in (13), odd powers of O_i^{ab} do not contribute in the expansion of the determinant. It is now easy to check that (13) and (21) are equal to second order because

$$\epsilon_{k\ell i} \epsilon_{\ell k j} = \text{Tr} (i\sigma_i)(i\sigma_j) = -2\delta_{ij} \quad (22)$$

all other traces being the same. At fourth order, however, the two expressions no longer agree as

$$\epsilon_{mni} \epsilon_{npj} \epsilon_{pqk} \epsilon_{qme} = \delta_{ij} \delta_{ke} + \delta_{ie} \delta_{jk} \neq \quad (23)$$

$$\begin{aligned} & \neq \text{Tr} (i\sigma_i)(i\sigma_j)(i\sigma_k)(i\sigma_l) = \\ & = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \quad \begin{array}{l} (23) \\ \text{cont.} \end{array}$$

This disagreement is the main reason why $N=1$ SYM quantum field theory cannot be reduced to the problem of solving the comparatively simple differential equation (8) for A_i^a in terms of ξ_i^a , in contrast to the case of supersymmetric quantum mechanics where the corresponding expressions are equal to all orders.

Although the determinants cannot be computed explicitly for arbitrary gauge field configurations, a little more can be said. If

$$[O_i(A), O_j(A)] = 0 \quad (24)$$

the trace $\text{Tr}[O_i(A)O_j(A)\dots]$ is independent of the ordering of the operators $O_i(A)$, and (13) is exactly equal to (21). Another, more interesting, example is furnished by instanton configurations. These obey $\xi(A) = 0$, and one easily proves the relation

$$\frac{\delta^2 S(A_{\text{inst}})}{\delta A_i^a(x) \delta A_j^b(y)} = \int \frac{\delta \xi_k^c(z, A_{\text{inst}})}{\delta A_i^a(x)} \cdot \frac{\delta \xi_k^c(z, A_{\text{inst}})}{\delta A_j^b(y)} d^4 z \quad (25)$$

Hence,

$$\det \frac{\delta \xi}{\delta A} = \left\{ \det \frac{\delta^2 S}{\delta A^2} \right\}^{1/2} \quad (26)$$

It can now be shown that for (anti) selfdual gauge field configurations, the non-zero eigenvalues of the quadratic fluctuations operator (25) coincide with those of the Dirac operator up to zero modes^{(21), (22)}. Thus we have

$$\left\{ \det \frac{\delta \xi(A_{\text{inst}})}{\delta A} \right\}' = \Delta_{\text{FP}} \Delta'_{\text{MS}}(A_{\text{inst}}) \quad (27)$$

The primes in this equation indicate that zero eigenvalues have been removed.

The existence of zero modes partly explains our failure to construct a field transformation whose Jacobian exactly equals the expression (21). In general, the difference between the number of zero eigenmodes of the operator (25) and the

number of zero eigenmodes of the Dirac operator will be proportional to $|Q|$ and will be different in each topological sector. [See Ref. 22) which also contains further references, as well as Ref. 23) for a discussion of superindex theorems.] This means that equality cannot be achieved on all of the manifolds of gauge field configurations. Note that this complication neither arises in Wess-Zumino-type models because these are topologically trivial, nor in supersymmetric quantum mechanics because there the topological charge can only assume the values $0, \pm 1$ ¹²⁾. Thus, (27) appears to be the best one can hope for in the general case, but further clarification is obviously needed.

However, there is a well-known model which circumvents this difficulty: it is the $N = 4$ SYM theory²⁴⁾. The fermion content of this theory is just such that bosonic and fermionic zero modes cancel in a selfdual (anti-selfdual) background²²⁾. In this case, (27) remains valid without primes and there is some chance that this equality extends to the whole manifold. It has been suggested that the $N = 4$ SYM theory may be described as a gas of non-interacting monopoles (instantons?)²⁵⁾. Is it conceivable that this theory, which has so far proved intractable by conventional superspace methods²⁶⁾, can be effectively trivialized if only one can find the appropriate field transformation? Although this is a matter of speculation for the time being, the results described here are rather suggestive.

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