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IS THERE A CONTINUUM LIMIT IN FOUR-DIMENSIONAL

Z(2) GAUGE THEORY ?

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ABSTRACT

A Monte Carlo simulation of Wilson loops and order-disorder loops in Z(2) gauge theory in four dimensions gives indications of long-range correlations at the self-dual temperature. This may have important implications on our understanding of non-Abelian Yang-Mills theory.

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Some time ago, Creutz, Jacobs and Rebbi¹⁾ presented evidence that the $d = 4$ $Z(2)$ gauge theory has a first order phase transition at the self-dual temperature. In this paper, we try to argue that this theory may have infinite range correlations at the self-dual point in spite of exhibiting a first order transition at precisely the same temperature. This would re-establish the general analogy between the critical behaviour of $d = 2$ spin systems and $d = 4$ gauge systems as advocated by Migdal²⁾ and Kadanoff³⁾. In conjunction with recent work⁴⁾ on a reformulation of this theory in terms of random surfaces, such findings would suggest that some kind of continuum string theory could describe the critical behaviour of the theory.

Furthermore, as will be explained at the end of this paper, it would then be quite likely that $U(N)$ Yang-Mills theory is also described by a similar kind of string theory.

In both the $d = 2$ Ising model and the $d = 4$ $Z(2)$ gauge theory, products of order and disorder variables play a special role because of their duality properties and because they linearize the theory^{4)*)}. However, such variables are non-local and quite unlike customary observables in statistical systems analyzed by computer simulations. We therefore decided to check whether standard Monte-Carlo methods are adequate for measuring correlations of order-disorder pairs in the $d = 2$ Ising model where they are known analytically.

The $d = 2$ Ising model is defined by

$$Z = \sum_{\{\sigma\}} \exp \left\{ \beta \sum_{x,\mu} \sigma(x) \sigma(x+\mu) \right\} \quad (1)$$

where $\sigma = \pm 1$ and x, μ are the site and direction labels of a two-dimensional lattice. In our simulation we used a 30×30 lattice with periodic boundary conditions.

Order-disorder variables in this theory were introduced by Onsager⁵⁾ and discussed further by Kadanoff and Ceva⁶⁾. Denoting by $\psi_i(x,y)$ (or $\bar{\psi}_i(x,y)$) with $i = 1,2,3,4$ a spin variable at (x,y) multiplied by a disorder variable at $(x + 1/2, y + 1/2)$, $(x - 1/2, y + 1/2)$, $(x - 1/2, y - 1/2)$ and $(x + 1/2, y - 1/2)$, respectively, it can be shown that the correlations between order-disorder pairs are given by⁷⁾

*) They also linearize the $d = 3$ Ising model, but in $d = 3$ they are not self-dual.

$$\langle \bar{\Psi}_i(x,y) \Psi_j(x',y') \rangle = \int \frac{d^2 p}{(2\pi)^2} M_{ij}(p) e^{i[p_x(x-x') + p_y(y-y')]} \quad (2)$$

with

$$M(p) = \frac{1}{\text{Det}(p)} \begin{pmatrix} 1 - cs(\tau_x + \tau_y) & c - s\tau_x & c^2 - s^2\tau_x\tau_y & c - s\tau_y \\ -c + s\tau_x & 1 - cs(\tau_y + \tau_x) & -c + s\tau_y & s^2\tau_y\tau_x - c^2 \\ s^2\tau_x\tau_y - c^2 & c - s\tau_y & 1 - cs(\tau_x + \tau_y) & -c + s\tau_x \\ -c + s\tau_y & c^2 - s^2\tau_x\tau_y & c - s\tau_x & 1 - cs(\tau_x + \tau_y) \end{pmatrix} \quad (3a)$$

where

$$\tau_{\pm x} = e^{\pm i p_x}, \quad \tau_{\pm y} = e^{\pm i p_y} \quad (3b)$$

and

$$\begin{aligned} \text{Det}(p) &= 2(1 - cs(\cos p_x + \cos p_y)) \\ c &= \frac{1}{\cosh(2\beta)}, \quad s = \tanh(2\beta) \end{aligned} \quad (3c)$$

Note that c, s are interchanged under a duality transformation. The notation $\psi, \bar{\psi}$ is just a reminder that these correlations were computed using complex Grassmann variables. The correlation function in Eq. (2) can be written directly in terms of the original spin variables in Eq. (1) as

$$\langle \bar{\Psi}_i(x,y) \Psi_j(x',y') \rangle = \langle \sigma(x,y) \sigma(x',y') \exp(-2\beta \sum_{(\ell,\mu)} \sigma_\ell \sigma_{\ell+\mu}) \rangle \quad (4)$$

where $\sigma_\ell, \sigma_{\ell+\mu}$ are spins at the ends of the links intersected by a path drawn from the location of the disorder variable near (x,y) to the disorder variable near (x',y') . A typical path for $i, j = 1$ is shown in Fig. 1. It is in the form of Eq. (4) that we compute this correlation function in our Monte Carlo simulations.

Figure 3 shows Monte Carlo data for $\langle \psi_1(o,d)\psi_1(o,o) \rangle$ for $d = 1,2,3,4$ and a range of temperatures β . We also plot (solid lines) the analytic evaluation [by direct numerical integration of Eq. (2)] of this correlation. From the excellent agreement between Monte Carlo data and the analytic formula, we conclude that measurements of order-disorder variable correlations require no modification of standard Monte Carlo techniques.

The $Z(2)$ gauge theory in four dimensions is defined by the partition function

$$Z_\beta = \sum_{\{\sigma\}} \exp \left\{ \beta \sum_{\square} \sigma_{\square} \right\} \quad (5)$$

where $\sigma = \pm 1$ are variables living on the links of a four-dimensional hypercubic lattice and $\sigma_{\square}$ is the product of link variables bordering an elementary square. This theory is self-dual. Thus

$$Z_\beta = C(\beta) Z_{\beta^*} \quad (6)$$

with

$$Z_{\beta^*} = \sum_{\{\mu\}} \exp \left\{ \beta^* \sum_{\square^*} \mu_{\square^*} \right\} \quad (7)$$

where the variables $\mu = \pm 1$ now sit on the dual lattice links and

$$\sinh(2\beta) \sinh(2\beta^*) = 1 \quad (8)$$

The $Z(2)$ gauge theory has a dramatic first order phase transition with a large latent heat in four dimensions and exhibits two distinct, co-existent phases at the critical point $\beta = \beta^*$ ¹⁾. For spin systems, we generally do not expect long-range correlations at a first order transition point. However, in gauge theories, we have no such intuition and a direct determination of correlations is necessary.

We choose to measure expectation values of products of nearby order and disorder loops in the tilted $\hat{x} + \hat{t}$, $\hat{y} + \hat{z}$ planes as suggested in Ref. 7). To understand what these are, consider Fig. 2 where we show the contour that goes into the construction of a 3×3 order-disorder loop.

A disorder loop is defined by flipping the sign of the coupling β for all squares that intersect a surface on the dual lattice that closes the disorder loop (also on the dual lattice). In the geometry we have chosen, the squares

where $\beta \rightarrow -\beta$ form a closed surface that fills the order loop. Therefore, order-disorder loops in a tilted plane such as the $\hat{x} + \hat{t}$, $\hat{y} + \hat{t}$ plane are given by

$$L_F = \left\langle \prod_i [\sigma_{\square}^i \exp(-2\beta \sigma_{\square}^i)] \right\rangle \quad (9)$$

where $\{\sigma_{\square}^i\}$ is the set of plaquettes that fill the surface of the loop (see Fig. 3). We shall measure rectangular $I \times J$ loops that are denoted by $L_F(I, J)$.

In addition, we also measure "rough" Wilson loops L_R of size $I \times J$ in the tilted plane defined by

$$L_R(I, J) = \left\langle \prod_i \sigma_{\square}^i \right\rangle \quad (10)$$

as well as normal, planar Wilson loop expectation values $L_W(I, J)$ in the principal lattice planes.

Following Creutz⁸⁾, we define a quantity $\chi(I)$ as

$$\chi(I) = \ln \left| \frac{L(I, I-1) L(I-1, I)}{L(I, I) L(I-1, I-1)} \right| \quad (11)$$

which has the property that as $I \rightarrow \infty$, it becomes the string tension. We take the modulus because $L_F(I, J)$ is negative for $\beta < \beta_c$ if IJ is odd. This is a purely kinematical effect deriving from the fact that $L_F(I, J)$ is antiself-dual for IJ odd

$$L_F(I, J) = (-1)^{IJ} L_F^*(I, J) \quad (12)$$

where $*$ means that one must use the temperature β^* instead of β .

In Fig. 4, we show our data for $\chi_F(I)$ and $\chi_R(I)$ for $I = 1, 2^{*)}$. The Monte Carlo runs were made on a $7 \times 7 \times 7 \times 7$ lattice with periodic boundary conditions. The lattice was initialized as a random configuration of links at $\beta = 0$ and slowly cooled (increasing β) in steps of $\Delta\beta = 0.02$ up to $\beta = 0.24$ and then in steps of $\Delta\beta = 0.01$ up to the critical temperature $\beta_c = \frac{1}{2} \ln(1+\sqrt{2})$. At each step, we performed 200 Monte Carlo sweeps of the lattice with the heat-bath algorithm, discarding the first 100 sweeps and averaging over the rest. The data points represent about five hours of CDC-7600 computer time.

^{*)} $\chi(1) = -\ln L(1, 1)$.

The solid curve plotted in Fig. 4 is the strong coupling result to lowest order for χ_F, χ_R, χ_W which reads

$$\chi(I) = -\ln(\tanh(\beta)) \quad (13)$$

The points for $\beta > \beta_c$ in the figure were obtained from those for $\beta < \beta_c$ by using duality.

To check our simulation, we verified duality by measuring correlations on both sides of the critical point for a few mutually dual temperatures. In Fig. 5, we show our data for $\chi_W(I)$, $I = 1, 2, 3$ which were obtained using the same Monte Carlo averaging procedure as for χ_F for $\beta < \beta_c$. The $\beta > \beta_c$ points were again obtained by duality and represent the corresponding χ_W for the disorder loop (which has an area law behaviour in weak coupling). The solid line is Münster's strong coupling expansion for the string tension to 12th order in β ⁹⁾).

From Figs. 4 and 5, it is apparent that χ_F, χ_R and χ_W all show a considerable dip in the critical region. The data for $\chi_W(3)$ dips down considerably below what one expects from a 14th order strong coupling series¹⁰⁾. One may define a correlation length ξ that is related to the string tension $\chi(\infty)$

$$\xi^2 = \frac{1}{\chi(\infty)} \quad (14)$$

The decrease in $\chi_F(2)$, $\chi_R(2)$ and $\chi_W(3)$ near the critical point is an indication that the correlation length is increasing (string tension decreasing) there. We are unable to measure $\chi_{F,R}(I)$ for $I > 2$ because of a low signal to noise ratio and limits on available computer time. Our expectation is that $\chi(I) \rightarrow 0$ as $I \rightarrow \infty$ and as the computer cost diverges. At this point we notice that we have considerably deflected from our original goal of measuring mainly order-disorder variables.

It should be noted that our order-disorder and rough loops lie in the tilted $\hat{t} + \hat{x}, \hat{y} + \hat{z}$ plane. If we take the midpoints between order and disorder loops [the superloop of Ref. 4)] as defining the geometry, we notice that an $n \times n$ tilted superloop has an area $n^2/2$ in lattice units. This means that in standard lattice units, the value we measure for $\chi_W(3)$ must be multiplied by 1/2 before being compared with χ_F or χ_R . When this is done, one sees that χ_W becomes consistent with χ_F and χ_R (they all equal ~ 0.25 for $I = 2, 3$) as one

^{*}) Münster's high temperature expansion is very unstable at the self-dual point, i.e., including successive terms in his series changes the string tension without any indication of convergence. This is also seen directly by using the ratio test on the coefficients of the series. There exists a fourteenth order series for the string tension due to Drouffe and Zuber¹⁰⁾. The effect of the fourteenth order term at the self-dual point is to reduce the twelfth order number by 0.019. This is an invisible effect in Figure 5.

approaches the critical point. This indicates that the theory might be approaching a scale invariant limit and has forgotten that some of the χ 's are measured with rough loops and some with planar loops. It is somewhat surprising, at first sight, that precisely near $\beta = \beta^*$ where one could hope for the correlation length to be large enough for 3×3 loops to be measurable, these quantities are actually very difficult to measure and show very strong fluctuations. We might interpret this as a "critical slowing down" of the Monte Carlo method.

In conclusion, subject to the limitations of the size of the lattice and available computer time, we find "long" range correlations of 1-2 lattice units and a significant dip in the string tension of order-disorder loops, rough loops and planar loops. Note that a correlation length of 1-2 lattice units is already within the scaling region for the $SU(2)$ Wilson action and might be considered sufficiently large⁸⁾.

Our data might resurrect the general correspondence between critical behaviour in spin systems in two dimensions and gauge systems in four dimensions. The nature of the phase transition may be different for these systems. For example, it is of infinite order in the 2-d XY model¹¹⁾ and second order for the 4-d $U(1)$ gauge theory¹²⁾; it is second and first order respectively for the Ising model⁵⁾ and 4-d $Z(2)$ gauge theory¹⁾. However, when one has long range correlations in 2-d spin models, one may also expect long range correlations in the corresponding gauge model.

If we accept the arguments of Ref. 4) that connect the $d = 4$ $Z(2)$ gauge theory to a theory of fermionic random surfaces, the existence of a continuum limit in $Z(2)$ gauge theory would indicate the existence of some continuum fermionic string theory in $d = 4$ dimensions. For an attempt to define such models, see Ref. 13).

We shall now explain why long-range correlations in $d = 4$ $Z(2)$ gauge theory may have important implications for $U(N)$ Yang-Mills theory. Above, we had already mentioned that the $d = 2$ Ising model has a reformulation in terms of complex Grassmann variables, i.e., in terms of a Grassmann action having a global $U(1)$ symmetry. Appropriately, the $d = 3$ $Z(2)$ gauge theory leads to a Grassmann action having a local $U(1)$ symmetry^{*)}. For $d = 4$ $Z(2)$ gauge theory, a quartic fermionic Hamiltonian without local $U(1)$ symmetry has been derived by Srednicki¹⁴⁾ but from the work of Refs. 4) and 7) we expect a quartic action with local $U(1)$ symmetry also to exist in $d = 4$. If this turns out to be

*) The details will be described elsewhere.

correct, then it is easy to replace the local $U(1)$ symmetry by a local $U(N)$ symmetry just by replacing the complex link variables $\psi, \bar{\psi}$ by $N \times N$ matrices ψ_{ij} and $\bar{\psi}_{ij}$. The resulting $U(N)$ gauge theory may then be regarded as an alternative to Wilson's formulation, provided this $U(N)$ Grassmann gauge theory has a continuum limit. From the point of view of random surfaces, the $U(N)$ local symmetry leads to the usual suppression of handles via a weight $N = N^{2-2h}$, h = number of handles. One would expect that if the theory has a continuum limit before suppressing the handles [i.e., $d = 4$ $Z(2)$ gauge theory] it then will also have a continuum limit with the handles suppressed [i.e., $d = 4$ $U(N)$ gauge theory]. So it appears that the existence of a continuum limit in $d = 4$ $Z(2)$ gauge theory might imply the existence of a new formulation of $U(N)$ gauge theory (different from the Wilson one) that is manifestly equivalent to some theory of strings. So the implications of long-range correlations in $d = 4$ $Z(2)$ gauge theory might indeed be far reaching and a more careful investigation of this problem seems amply justified.

Finally, we would like to mention that a first order transition in the free energy per plaquette coincident with an infinite correlation length is not unprecedented. This indeed happens in the 4-d $SU(N)$ lattice gauge theory for even N . This theory is expected to have an infinite correlation length at $g_0^2 = 0$. Also, in this theory, there exists an exact change of variables on the links whose effect is to change the sign of every plaquette¹⁵⁾. Thus, every configuration at positive g_0^2 is mapped into one at negative g_0^2 and the order parameter $1/N \langle \text{tr Re } U_{\square} \rangle$ has a first order jump at $g_0^2 = 0$ with latent heat two.

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FIGURE CAPTIONS

- Fig. 1 : Spin-frustration correlation function in 2-d Ising model [see Eq. (4)]. $\sigma_\ell, \sigma_{\ell+\mu}$ are the spins at the ends of the links marked with a double bar. The single line denotes one possible path connecting the frustrations.
- Fig. 2 : 3×3 order-disorder loop in the $\hat{t} + \hat{x}, \hat{y} + \hat{z}$ plane.
- Fig. 3 : $\langle \psi_1(o,d) \psi_1(o,o) \rangle$ for the 2-d Ising model; $d = 1, 2, 3, 4$. The data was taken on a 30×30 lattice with periodic boundary conditions. The solid curve is the exact result [see Eq. (2)]. The crosses (dots) represent an average over 40 (1000) iterations after discarding 20 (500) iterations starting the lattice in an ordered (disordered) configuration.
- Fig. 4 : $X_F(1), X_F(2), X_R(2)$ from Monte Carlo simulation on a $7 \times 7 \times 7 \times 7$ lattice for the $Z(2)$ gauge theory. The solid line is the leading strong (weak) coupling expansion result.
- Fig. 5 : $X_W(1), X_W(2), X_W(3)$ also from simulations on a $7 \times 7 \times 7 \times 7$ lattice. The solid line is Münster's 12th order strong (weak) coupling expansion of Ref. 9).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5