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SUPERGRAVITY WITH ONE AUXILIARY SPINOR

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A B S T R A C T

We present an "intermediate" off-shell version of $N = 1$ supergravity and its tensor calculus. The supergravity multiplet has $16 + 16$ field components. The formulation can be constrained to either of the minimal ones with $12 + 12$ components, or enlarged by matter couplings to several $20 + 20$ component versions. Self-coupled to its own axial gauge submultiplet it leads to spontaneous supersymmetry breaking in the form first discussed by Freedman and to a propagating gauge field.

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1. INTRODUCTION AND SUMMARY

Throughout the course of the development of supergravity, a major effort has been devoted to the search for off-shell formulations of the various models, i.e. for realizations of the algebra on sets of fields which are not subject to field equations. While this effort has been largely frustrated for theories with many supersymmetries (large N), quite a lot is now known about different off-shell formulations for $N = 1$ and $N = 2$.

Since every supermultiplet has equal numbers of fermionic and bosonic degrees of freedom, it follows that an $N = 1$ off-shell supergravity multiplet must contain the vierbein and gravitino fields and, additionally, bosonic auxiliary fields with $6 + 4p$ degrees of freedom if there are p auxiliary spinors with four degrees of freedom each ($p = 0, 1, \dots$). Clearly, a minimal formulation is one with no auxiliary fermion fields ($p = 0$) and just six bosonic auxiliary fields. There are two distinct minimal formulations which we call "old minimal" ¹⁾ and "new minimal" ²⁾. The auxiliary fields of the old minimal set are a scalar-pseudoscalar pair M, N and an axial-vector b_a , those of the new minimal set are an axial gauge-vector A_μ and a conserved axial-vector density b'^μ .

The simplest next possibility is to have one auxiliary spinor ($p = 1$). Such a multiplet has, however, not been studied, since by itself it cannot be used to construct a Lagrangian that describes pure $N = 1$ supergravity: the only known mechanism by which a spinor can be made to disappear on-shell is mutual cancellation of two spinors in a Lagrangian of the form $\mathcal{L} = \dots + \bar{\lambda}\chi + \dots$. Nevertheless, we here study the properties of such a multiplet with a total of $16 + 16$ field components for, as we shall see, its coupling to "matter" can describe interesting "physics".

A formulation of $N = 1$ supergravity with two auxiliary spinors ($p = 2$) and a total of $20 + 20$ field components was in fact the first known off-shell formulation of supergravity³⁾. Versions with more than two auxiliary spinors are unknown but not clearly ruled out.

Study of the tensor calculus for the old minimal^{4,5)} and the new minimal formulations⁶⁾ has given the couplings of these to matter. Such couplings have, for example, made possible a systematic study of spontaneous breaking of local supersymmetry⁷⁻⁹⁾. The different formulations have been found to couple to matter in different ways. The major difference is that in order to couple matter to the new minimal supergravity, and indeed to the non-minimal one presented in this paper, the theory must have a local chiral invariance. After elimination of the auxiliary fields (which usually involves gauge fixing), there should at least still be a global R-invariance. Hence, if an "on-shell" theory is not R-invariant, there is no hope of generalizing it off-shell with any of the "new" sets of auxiliary fields. On the other hand, we found that all theories with R-invariance can be formulated in the off-shell forms with local chiral invariance. This intimately relates supersymmetry-breaking with a Fayet-Iliopoulos term (which can be done for the new minimal set) to global R-invariance of the "on-shell" or the "old off-shell" form^{8,9)}.

These points, as well as the lessons that may be provided for larger N , make the study of others than the minimal formulations interesting^{*)}.

The present paper is devoted to a study of the $16 + 16$ component supergravity multiplet, its tensor calculus, and the associated matter couplings. Since the body of the paper is necessarily rather technical (like all work on tensor calculus), we proceed by summarizing our main results here.

The intimate relation¹¹⁾ between the multiplet of currents in flat-space supersymmetric theories and formulations of off-shell supergravity enables us to deduce one from the other. The continued interest in the multiplet structure of the anomalies of theories which in flat space have a superconformal invariance at the tree level [prompted by the finiteness question of the $N = 4$ Yang-Mills theory¹²⁾] has therefore revived the question: Which other off-shell $N = 1$ supergravity schemes can exist? This has, for instance, led to the discovery of the new minimal formulation.

*) Indeed, for $N = 2$ it is exactly the minimal Poincaré off-shell multiplets that have one "unbalanced" auxiliary spinor field¹⁰⁾.

The *supergravity multiplet* (Section 2) which we wish to consider is contragradient^{11,2)} to a multiplet of currents which contains the stress-tensor $\theta_{\mu\nu}$ and the supercurrent $J_{\mu\alpha}$, a conserved axial current, an axial-vector U_μ , a scalar-pseudoscalar pair H, K , a spinor χ , and a scalar C . The fields $(C; \chi; H, K, U_\mu; i\gamma \cdot J, 2\theta_\mu^\mu)$ form a submultiplet "of anomalies" with the structure of a general multiplet (or "vector multiplet"). Correspondingly, the fields of the supergravity multiplet are

$$\begin{aligned}
 e_\mu^a & \quad \text{the vierbein (dimension} = 0) \\
 \psi_{\mu\alpha} & \quad \text{the gravitino field (d} = \tfrac{1}{2}) \\
 A_\mu & \quad \text{the axial gauge vector (d} = 1) \\
 b_a & \quad \text{an axial-vector (d} = 1) \\
 M, N & \quad \text{a scalar-pseudoscalar pair (d} = 1) \\
 \lambda & \quad \text{a spinor (d} = 3/2) \\
 S & \quad \text{a scalar (d} = 2) ,
 \end{aligned} \tag{1.1}$$

with a total of $16 + 16$ components, after the gauge freedoms (four general coordinate transformations, local Lorentz rotations, one chiral gauge transformation, and four local supersymmetries) have been subtracted. The algebra of these gauge transformations closes on the fields.

This multiplet has been considered by de Wit and van Nieuwenhuizen¹³⁾ in a different context: in coupling the $(1, \tfrac{1}{2})$ -multiplet with a Fayet-Iliopoulos term to on-shell $N = 1$ supergravity, Freedman had observed¹⁴⁾ that the resulting Lagrangian had a local chiral invariance. The corresponding off-shell version, using the old minimal formulation of supergravity, leads to the above set of fields and their transformation laws¹³⁾.

The multiplet contains two *submultiplets* (Section 4): one is a chiral multiplet (or "scalar multiplet") with components $(M, N; \lambda; S, \mathcal{O}_a b^a)$ and the other is the gauge multiplet $(A_\mu; \lambda'; S')$. The λ' and S' in the gauge multiplet differ from the λ and S in the chiral multiplet by shifts of the form:

$$\begin{aligned} S' &= S + \frac{1}{4} R + \frac{4}{3} (M^2 + N^2 - \frac{1}{2} b_a b^a) \\ \lambda' &= \lambda - \frac{i}{4} \gamma^a \mathcal{R}_a \end{aligned} \quad (1.2)$$

Here R is the fully covariantized form of the Ricci scalar, and $\mathcal{R}_a$ is the fully covariantized form of the "Rarita-Schwinger field equation" [see Eqs. (2.9)-(2.18) for more detail].

If we set one of the two submultiplets to zero, we get the new and old *minimal formulations*, respectively. In the transition to the old minimal formulation, we set $A_\mu = 0$ and thus fix the chiral gauge. Hence that formulation has no local chiral invariance.

The *tensor calculus* (Sections 3, 5, 6) for matter fields can be carried through, and the results can again be constrained to either of the minimal tensor calculi. *Density formulae* exist both for general multiplets (D-densities) and chiral multiplets (F-densities), the latter however only if the chiral weight of the multiplet is such that its F-component is a chiral invariant. This is not the case for the chiral unit multiplet, and hence a term $[1]_F$ which gives a negative cosmological constant⁴⁾ is ruled out.

The D-density for the general unit multiplet, $[1]_D$, (the "volume of superspace") is a multiple of

$$e S + ie \bar{\Psi}_\mu \gamma^\mu \lambda - ie \bar{\Psi}_\mu \sigma^{\mu\nu} (M + \gamma_5 N) \Psi_\nu + 2e (M^2 + N^2) \quad (1.3)$$

and vanishes for $M = N = \lambda = S = 0$, i.e. for the new minimal case^{6,15)}. As it stands, expression (1.3) cannot be used as a Lagrangian, since it gives a field equation $e \equiv \det(e_\mu^a) = 0$. If, however, we set $S' = \lambda' = A_\mu = 0$, then $[1]_D$ becomes a multiple of the old *minimal supergravity Lagrangian* (Section 7).

The new minimal Lagrangian is obtained by observing that the action derived from the density

$$e S' + ie \bar{\Psi}_\mu \gamma^\mu \lambda' + \frac{4}{3} b'^\mu A_\mu \quad (1.4)$$

becomes invariant if $M = N = \lambda = S = \partial_\mu b'^\mu = 0$, i.e. for the new minimal set. This is due to an interesting phenomenon: the D-density for the product of a linear and a general multiplet will only involve the gauge-invariant fields of the general multiplet. In the new minimal case, and only there, the unit multiplet $\mathbb{1}$ is a linear multiplet, and hence the D-density of (unit multiplet) $\times$ (gauge multiplet) becomes well-defined and is a multiple of expression (1.4) which can be used as a Lagrangian⁶⁾.

Instead of constraining the $16 + 16$ fields "by hand" to obtain a minimal version, we can do so by *Lagrange multiplier* (Section 8). We use a linear multiplet $(C; \chi; U'^\mu)$ as a Lagrange multiplier for the chiral gauge multiplet and get a Lagrangian of the form

$$-2eS + eCS' + \text{many more.} \quad (1.5)$$

Variation with respect to C gives $S' = 0$, i.e. $S = -\frac{1}{4}R + \dots$ and similarly for the superpartners. The result will be the old minimal formulation. If we implement a different supersymmetric subset of all field equations which can be obtained from the Lagrangian, we get the new minimal version. This formulation involves $20 + 20$ fields (the $16 + 16$ plus the $4 + 4$ Lagrange multipliers) and consistently describes $N = 1$ supergravity. Its linearized limit corresponds to what Rivelles and Taylor have independently derived and called the "2nd new non-minimal set"¹⁶⁾.

Alternatively (Section 9), we can add the kinetic Lagrangian for a chiral multiplet to the action of expression (1.3). The result has the structure

$$eS + \frac{e}{3}(A^2 + B^2)S - \frac{w}{2}(A^2 + B^2)S' + \dots \quad (1.6)$$

and involves both S and S' , the latter in conjunction with the chiral weight w for the chiral multiplet. Variation with respect to S does not give $e = 0$ any more but rather a constraint

$$A^2 + B^2 = \text{constant.} \quad (1.7)$$

For most values of w this constraint can be implemented on the Lagrangian. Together with other equations of motion and the chiral gauge choice $B = 0$, this will leave us with the on-shell $N = 1$ Lagrangian $\mathcal{L}_E + \mathcal{L}_{RS}$. Thus we have *another* consistent $20 + 20$ version (16 + 16 plus the 4 + 4 of the chiral multiplet)*). The "1st new non-minimal set", found independently¹⁶⁾, is probably a shifted version of these $20 + 20$ fields.

Finally, and very interestingly, we can add to the invariant $-2eS + \dots$ the kinetic Lagrangian for the axial gauge multiplet which is actually part of the 16 + 16 fields (Section 10):

$$-2eS + \dots - \frac{e}{4} F_{ab} F^{ab} + \frac{ie}{2} \bar{\lambda} \not{D} \lambda + \frac{e}{2} S'^2 + \dots \quad (1.8)$$

The S -dependent terms give the equation of motion

$$S' = 2 \quad (1.9)$$

and hence give rise to a positive cosmological constant. This is the model of Freedman⁴⁾ and de Wit and van Nieuwenhuizen¹³⁾ which presents therefore a way (and the only one) to make the 16 + 16 formulation propagate without either constraining fields away to 12 + 12 or adding fields to 20 + 20. The price paid for this is *spontaneous breakdown of supersymmetry* and a positive *cosmological constant* of the order of (axial coupling constant)² $\times$ (Newton's constant)⁻¹. The prize won is a *propagating gauge field*. The "physical" spectrum now consists of the graviton, a massive gravitino (it has "eaten" the λ -field by a super-Higgs effect) and a massless axial gauge particle, all in a de Sitter universe. This model may be an $N = 1$ off-shell equivalent of the higher- N supergravities with gauged internal symmetries of de Wit and Nicolai¹⁸⁾.

In the appendices we give the *current multiplet* which is contragradient to our multiplet, and the superspace *torsion constraints* which can be derived from it and which define our theory.

*) A similar procedure has been used for $N = 2$ ¹⁷⁾.

2. THE SUPERGRAVITY MULTIPLY

We consider a multiplet of $N = 1$ supergravity with $16 + 16$ real off-shell field components. It contains the gauge fields for "local translations" (the vierbein e_μ^a), for supersymmetry transformations (the gravitino field $\psi_{\mu\alpha}$) and for chiral gauge transformations (the axial-vector A_μ). It also contains the following "covariant" fields which are scalars under general coordinate transformation, but transform under local frame rotations: an axial-vector b_a , a scalar-pseudoscalar pair M and N , a spinor^{*} λ , and another scalar S .

The multiplet is subject to various gauge transformations. These are first the *general coordinate transformations* $\delta_G(\xi^\mu)$ under which the gauge fields are vectors and the covariant fields are scalars, e.g.

$$\begin{aligned}\delta_G e_\mu^a &= \xi^\nu \partial_\nu e_\mu^a + (\partial_\mu \xi^\nu) e_\nu^a \\ \delta_G b_a &= \xi^\nu \partial_\nu b_a,\end{aligned}\tag{2.1}$$

second the *local Lorentz transformations* (frame rotations) $\delta_L(\lambda^{ab})$ under which e_μ^a and b_a are vectors, ψ_μ and λ are spinors and all others are scalars; e.g.

$$\begin{aligned}\delta_L e_\mu^a &= \lambda^{ab} e_{\mu b} \\ \delta_L \psi_\mu &= -\frac{i}{4} \lambda^{ab} \sigma_{ab} \psi_\mu \\ \delta_L A_\mu &= 0,\end{aligned}\tag{2.2}$$

third, the *chiral gauge transformations* $\delta_c(\alpha)$ under which e_μ^a , b_a and S are invariant and the other fields transform as

$$\begin{aligned}\delta_c \psi_\mu &= \alpha \gamma_5 \psi_\mu \quad ; \quad \delta_c \lambda = \alpha \gamma_5 \lambda \\ \delta_c A_\mu &= \partial_\mu \alpha \\ \delta_c M &= 2\alpha N \quad ; \quad \delta_c N = -2\alpha M,\end{aligned}\tag{2.3}$$

^{*}) Conventions: all spinors are Majorana spinors, $\lambda = C\bar{\lambda}^T$; $\eta_{00} = 1$;
 $\sigma_{ab} \equiv i/2[\gamma_a, \gamma_b]$; $\gamma_5 \equiv -1/4! \epsilon^{abcd} \gamma_a \gamma_b \gamma_c \gamma_d$, i.e. $\gamma_5^2 = -1$.

and fourth, the *local supersymmetry* transformations $\delta_S(\zeta)$ with anticommuting Majorana spinor parameters $\zeta_\alpha(x)$:

$$\begin{aligned}
 \delta_S e_\mu^a &= -2i \bar{\zeta} \gamma^a \psi_\mu \\
 \delta_S \psi_\mu &= (\mathcal{D}_\mu + \mathcal{T}_\mu) \zeta \\
 \delta_S A_\mu &= -i \bar{\zeta} \gamma_\mu \gamma_5 \lambda' \\
 \delta_S M &= \bar{\zeta} \lambda \\
 \delta_S N &= \bar{\zeta} \gamma_5 \lambda \\
 \delta_S b_a &= \frac{3}{4} \bar{\zeta} \gamma_5 \mathcal{Q}_a - i \bar{\zeta} \gamma_a \gamma_5 \lambda \\
 \delta_S \lambda &= - (S + \gamma_5 \mathcal{D}^a b_a + i \not{D} (M + \gamma_5 N)) \zeta \\
 \delta_S S &= i \bar{\zeta} (\not{D} - \frac{1}{3} \not{\gamma} \gamma_5) \lambda - 2 \bar{\zeta} (M + \gamma_5 N) (\lambda' + \frac{1}{3} \lambda) .
 \end{aligned} \tag{2.4}$$

A number of symbols and quantities are used in Eqs. (2.4) which require definition. The "gauge-covariantized" derivative $\mathcal{D}_\mu$ is

$$\mathcal{D}_\mu \equiv \partial_\mu - \delta_L(\omega_\mu^{ab}) - \delta_c(A_\mu) \tag{2.5}$$

and the formal expression $\mathcal{D}_\mu \zeta$ is to be seen as if ζ were a field with the gauge properties of ψ_μ . The *Lorentz connection* ω_μ^{ab} which appears in Eq. (2.5) is a function of e_μ^a and ψ_μ ("second-order formalism"):

$$\begin{aligned}
 \omega_\mu^{ab} &\equiv -\frac{1}{2} (e^{a\nu} \partial_\mu e_\nu^b - e^{b\nu} \partial_\mu e_\nu^a) + \frac{1}{2} (e^{a\sigma} e^{b\tau} - e^{b\sigma} e^{a\tau}) \partial_\sigma g_{\tau\mu} \\
 &\quad + i (\bar{\psi}_\mu \gamma^a \psi^b - \bar{\psi}_\mu \gamma^b \psi^a + \bar{\psi}^a \gamma_\mu \psi^b) .
 \end{aligned} \tag{2.6}$$

The e_a^μ in expression (2.6) is the *inverse vierbein*, and $g_{\mu\nu} \equiv e_\mu^a e_\nu^b \eta_{ab}$ is the *metric*. We generally make use of the vierbein and its inverse to convert vector indices on γ_a , σ_{ab} , ψ_μ and b_a , but *never* on any other quantities. The "super-covariantized" derivative $\mathcal{D}_a$ is:

$$\mathcal{D}_a \equiv e_a{}^\mu (\mathcal{D}_\mu - \delta_s(\psi_\mu)) , \quad (2.7)$$

and $\mathcal{D}$ always means $\gamma^a \mathcal{D}_a$ and never $\gamma^\mu \mathcal{D}_\mu$. The field-dependent matrix in spinor space, $\mathcal{T}_a$, which appears in $\delta_s \psi_\mu$, and the related T_a which appears in $\delta_s \bar{\psi}_\mu$ are

$$\mathcal{T}_a \equiv -\frac{i}{3} \gamma_a (M + \gamma_5 N - i \not{D} \gamma_5) + \gamma_5 b_a \quad (2.8)$$

$$T_a \equiv -\frac{i}{3} (M + \gamma_5 N - i \not{D} \gamma_5) \gamma_a - \gamma_5 b_a ,$$

and the vector-spinor $\mathcal{R}_\alpha^a$ is defined by the three equations

$$\mathcal{R}^a \equiv i \varepsilon^{abcd} \gamma_b \gamma_5 \tau_{cd} \quad (2.9)$$

$$\tau_{ab} \equiv e_a{}^\mu e_b{}^\nu \tau_{\mu\nu} + \mathcal{T}_a \psi_b - \mathcal{T}_b \psi_a \quad (2.10)$$

$$\tau_{\mu\nu} \equiv \mathcal{D}_\mu \psi_\nu - \mathcal{D}_\nu \psi_\mu . \quad (2.11)$$

Finally, the spinor λ' which appears in $\delta_s A_\mu$ and $\delta_s S$ is defined by

$$\lambda' \equiv \lambda - \frac{i}{4} \gamma_a \mathcal{R}^a . \quad (2.12)$$

The supersymmetry variation of λ' can be calculated to be

$$\delta_s \lambda' = - (S' + \frac{i}{2} \sigma^{ab} \gamma_5 F_{ab}) \lambda , \quad (2.13)$$

where

$$F_{ab} \equiv e_a{}^\mu e_b{}^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + i (\bar{\psi}_a \gamma_b - \bar{\psi}_b \gamma_a) \gamma_5 \lambda' \quad (2.14)$$

and

$$S' \equiv S + \frac{i}{4} R + \frac{4}{3} (M^2 + N^2 - \frac{1}{2} b^a b_a) \quad (2.15)$$

$$\delta_s S' = i \bar{\lambda} (\not{D} + \not{D} \gamma_5) \lambda' .$$

The supercovariant Ricci scalar R is defined by

$$R \equiv R_a{}^a ; \quad R_{ab} \equiv R_{acb}{}^c \quad (2.16)$$

$$R_{ab}{}^{cd} \equiv e_a{}^\mu e_b{}^\nu R_{\mu\nu}{}^{cd} + i \bar{\psi}_a (\gamma_b \tau^{cd} - \gamma^c \tau_b^d + \gamma^d \tau_b^c) - i \bar{\psi}_b (\gamma_a \tau^{cd} - \gamma^c \tau_a^d + \gamma^d \tau_a^c) - \frac{4i}{3} \bar{\psi}_a \sigma^{cd} (M + \gamma_5 N) \psi_b + \frac{4i}{3} \epsilon^{cdef} \bar{\psi}_a \gamma_e \psi_b b_f \quad (2.17)$$

$$R_{\mu\nu}{}^{ab} \equiv \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{ac} \omega_{\nu c}{}^b + \omega_\nu{}^{ac} \omega_{\mu c}{}^b \quad (2.18)$$

We can also define "local translations" $\delta_P(\xi^a)$ as combinations of field-dependent gauge transformations:

$$\delta_P(\xi^a) \equiv \delta_G(\xi^\mu) - \delta_S(\xi^\mu \psi_\mu) - \delta_C(\xi^\mu A_\mu) - \delta_L(\xi^\mu \omega_\mu{}^{ab}) \quad (2.19)$$

where $\xi^\mu = e_a{}^\mu \xi^a$. On the covariant fields we then have

$$\delta_P(\xi^a) = \xi^a \partial_a, \quad (2.20)$$

and on all fields the algebra of the gauge transformations closes. In particular, the commutator of two supersymmetry transformations is

$$[\delta_S(\zeta_1), \delta_S(\zeta_2)] = \delta_P(\xi^a) + \delta_L(\lambda^{ab}) \quad (2.21)$$

with

$$\xi^a = 2i \bar{\zeta}_1 \gamma^a \zeta_2; \quad \lambda^{ab} = -\frac{4i}{3} \bar{\zeta}_1 \sigma^{ab} (M + \gamma_5 N) \zeta_2 + \frac{2}{3} \epsilon^{abcd} \bar{\zeta}_c b_d. \quad (2.22)$$

3. TENSOR CALCULUS: MATTER MULTIPLETS

The algebra of gauge transformations which is realized on the supergravity multiplet of the previous section can also be represented on "matter" multiplets. The simplest one of these is the *chiral multiplet* $\mathcal{A}$ with 4 + 4 real field components

$$\mathcal{A} = (A, B; \psi; F, G). \quad (3.1)$$

Care has to be taken not to confuse the matter spinor field ψ with the gravitino field ψ_μ .

The multiplet is completely defined by the algebra (2.21) and the transformation laws of its lowest members A and B. Under supersymmetry and chiral transformations, $\delta = \delta_s + \delta_c$, the fields transform as follows:

$$\begin{aligned}
 \delta A &= \bar{\zeta} \psi + \alpha w B \\
 \delta B &= \bar{\zeta} \gamma_5 \psi - \alpha w A \\
 \delta \psi &= - (F + \gamma_5 G + i \not{D} (A + \gamma_5 B)) \zeta + \alpha (w-1) \gamma_5 \psi \\
 \delta F &= i \bar{\zeta} (\not{D} - \frac{1}{3} \not{D} \gamma_5) \psi - \frac{2}{3} \bar{\zeta} (M + \gamma_5 N) \psi - w \bar{\zeta} (A + \gamma_5 B) \lambda' - \alpha (w-2) G \\
 \delta G &= i \bar{\zeta} \gamma_5 (\not{D} - \frac{1}{3} \not{D} \gamma_5) \psi + \frac{2}{3} \bar{\zeta} \gamma_5 (M + \gamma_5 N) \psi - w \bar{\zeta} \gamma_5 (A + \gamma_5 B) \lambda' + \alpha (w-2) F.
 \end{aligned} \tag{3.2}$$

The real number w is called the *chiral weight* of the multiplet. These transformations are derived in the following way: δA and δB define the chiral multiplet. For $\delta \psi$ we make an ansatz

$$\delta \psi = - (F + \gamma_5 G + i \gamma^a X_a + i \gamma^a \gamma_5 Y_a + i \sigma^{ab} Z_{ab}) \zeta + \alpha \gamma_5 \psi',$$

and if we enforce the algebra, namely Eq. (2.21) and $[\delta_s, \delta_c] = \delta_s (\alpha \gamma_5 \zeta)$, on A and B, we find conditions

$$X_a = \partial_a A \quad ; \quad Y_a = \partial_a B \quad , \quad Z_{ab} = 0 \quad ; \quad \psi' = (w-1) \psi,$$

and hence $\delta \psi$ as given above. Similarly, we make the ansatz

$$\delta F = \bar{\zeta} \psi'' + \alpha F' \quad ; \quad \delta G = i \bar{\zeta} \gamma_5 \psi''' + \alpha G'$$

and enforcing the algebra on ψ gives

$$\frac{1}{2} (\psi'' + \psi''') = i (\not{D} - \frac{1}{3} \not{B} \gamma_5) \psi - w (A + \gamma_5 B) \lambda'$$

$$\frac{1}{2} (\psi'' - \psi''') = -\frac{2}{3} (M + \gamma_5 N) \psi$$

$$F' = -(w-2) G \quad ; \quad G' = (w-2) F$$

We can then check that the algebra also closes on F and G.

Note that the supersymmetry transformations as well as the chiral ones depend on the weight w.

Another matter multiplet is the (scalar) *general multiplet* $\mathbb{C}$, defined by the property of its lowest lying component field C to be a chirally invariant real scalar. The multiplet contains 8 + 8 field components

$$\mathbb{C} = (C; \chi; H, K, U_a; \varphi; \mathbb{D}) \quad (3.3)$$

and their transformation laws can be calculated in a similar fashion as described above for the chiral multiplet. They are

$$\begin{aligned} \delta C &= \bar{\zeta} \chi \\ \delta \chi &= (H + \gamma_5 K - i \not{D} C - i \not{U} \gamma_5) \zeta - \alpha \gamma_5 \chi \\ \delta H &= \bar{\zeta} (\varphi - i \not{D} \chi + \frac{i}{3} \not{B} \gamma_5 \chi + \frac{2}{3} (M + \gamma_5 N) \chi) + 2 \alpha K \\ \delta K &= \bar{\zeta} \gamma_5 (\varphi - i \not{D} \chi + \frac{i}{3} \not{B} \gamma_5 \chi - \frac{2}{3} (M + \gamma_5 N) \chi) - 2 \alpha H \\ \delta U_a &= -i \bar{\zeta} \gamma_a \gamma_5 \varphi + \bar{\zeta} (\not{D}_a + T_a) \gamma_5 \chi \\ \delta \varphi &= -(\mathbb{D} + \frac{i}{2} \sigma^{ab} \gamma_5 u_{ab}) \zeta + \alpha \gamma_5 \varphi \\ \delta \mathbb{D} &= i \bar{\zeta} (\not{D} + \not{U} \gamma_5) \varphi \end{aligned} \quad (3.4)$$

with

$$u_{ab} \equiv \not{D}_a u_b - \not{D}_b u_a + \bar{\chi} \gamma_5 \tau_{ab} \chi \quad (3.5)$$

We find that $\mathcal{C}$ contains a submultiplet, the *curl multiplet* $d\mathcal{C}$ with components

$$d\mathcal{C} = (\varphi; u_{ab}, \mathcal{D}). \quad (3.6)$$

This is a submultiplet because δu_{ab} is linear in φ :

$$\begin{aligned} \delta u_{ab} = & -i \bar{\zeta} (\partial_a \gamma_b - \partial_b \gamma_a) \gamma_5 \varphi - \frac{i}{3} \bar{\zeta} (\gamma_a b_b - \gamma_b b_a) \varphi \\ & - \frac{2i}{3} \varepsilon_{abcd} \bar{\zeta} \gamma^c \gamma_5 \varphi b^d + \frac{2i}{3} \bar{\zeta} \sigma_{ab} (M + \gamma_5 N) \gamma_5 \varphi. \end{aligned} \quad (3.7)$$

The constraint $u_{ab} = 0$ has the solution

$$u_a = \partial_a \mathcal{U} \quad \text{with} \quad \delta \mathcal{U} = \bar{\zeta} \gamma_5 \chi, \quad (3.8)$$

and if we set $d\mathcal{C} = 0$, the general multiplet is reduced to the form

$$\mathcal{C} = (C; \chi; H, K, \partial_a \mathcal{U}; 0; 0) \equiv d\mathcal{H} \quad (3.9)$$

where $\mathcal{H}$ is a chiral multiplet with components

$$\mathcal{H} = (C, \mathcal{U}; \chi; -H, -K) \quad (3.10)$$

and weight $w = 0$. Note that d and later ∂ mean different things on different multiplets.

It can be very useful to replace U_a by a covariant "world" axial-vector

$$U_\mu \equiv e_\mu^a U_a + \bar{\psi}_\mu \gamma_5 \chi \quad (3.11)$$

with transformations

$$\delta U_\mu = -i \bar{\zeta} \gamma_\mu \gamma_5 \varphi + \partial_\mu (\bar{\zeta} \gamma_5 \chi). \quad (3.12)$$

The curl of U_μ is related to u_{ab} through

$$\partial_\mu U_\nu - \partial_\nu U_\mu = e_\mu^a e_\nu^b u_{ab} - i (\bar{\psi}_\mu \gamma_\nu - \bar{\psi}_\nu \gamma_\mu) \gamma_5 \varphi, \quad (3.13)$$

and the solution (3.8) to the constraint $d\mathbb{C} = 0$ now reads $U_\mu = \partial_\mu U$. In a Wess-Zumino gauge $0 = C = \chi = H = K$, the transformation law for U_μ becomes

$$\delta U_\mu = -i \bar{\psi} \gamma_\mu \gamma_5 \varphi, \quad (3.14)$$

and the algebra (2.21) closes on the remaining *gauge multiplet* $\mathcal{Y}$,

$$\mathcal{Y} = (U_\mu; \varphi; \mathcal{D}), \quad (3.15)$$

only modulo an Abelian gauge transformation $U_\mu \rightarrow U_\mu - \partial_\mu \beta$, as follows:

$$[\delta_1, \delta_2] U_\mu = \dots - \partial_\mu \beta \quad \text{with} \quad \beta = \int U_\mu. \quad (3.16)$$

Since $d\mathbb{C}$ involves only gauge-invariant quantities, we can define $d\mathcal{Y}$ to be

$$d\mathcal{Y} \equiv d\mathbb{C}. \quad (3.17)$$

Another way of writing the general multiplet is in terms of *shifted fields*

$$\mathbb{C}_{sh} = (C'; \chi'; H', K', U'_a; \varphi'; \mathcal{D}') \quad (3.18)$$

with

$$\begin{aligned} C' &= C & \chi' &= \chi \\ H' &= H - \frac{4}{3} MC \\ K' &= K - \frac{4}{3} NC \\ U'_a &= U_a - \frac{4}{3} b_a C \\ \varphi' &= \varphi - i \left(\not{D} - \frac{1}{3} \not{b} \gamma_5 \right) \chi - \frac{2}{3} (M - \gamma_5 N) \chi - \frac{4}{3} \lambda C \\ \mathcal{D}' &= \mathcal{D} + \not{D}^a \not{D}_a C - \frac{2}{3} (MH + NK - 2b^a U_a) - \bar{\chi} \left(\lambda' + \frac{1}{3} \lambda \right) - \frac{4}{3} SC \end{aligned} \quad (3.19)$$

The transformations for the shifted field are

$$\begin{aligned}
 \delta C' &= \bar{\zeta} \chi' \\
 \delta \chi' &= (H' + \gamma_5 K' - i \not{D} C - i \not{U} \gamma_5 + \frac{4}{3} (M + \gamma_5 N - i \not{V} \gamma_5) C) \zeta - \alpha \gamma_5 \chi' \\
 \delta H' &= \bar{\zeta} \varphi' + 2\alpha K' \\
 \delta K' &= \bar{\zeta} \gamma_5 \varphi' - 2\alpha H' \\
 \delta U'_a &= -i \bar{\zeta} \gamma_a \gamma_5 \varphi' + i \bar{\zeta} (\not{D}^b + T^b) \sigma_{ab} \gamma_5 \chi - \bar{\zeta} \gamma_5 \not{D}_a C \\
 \delta \varphi' &= -(\not{D}' + \gamma_5 \not{P}' + i \not{D} (H' + \gamma_5 K')) \zeta + \alpha \gamma_5 \varphi' \\
 \delta D' &= i \bar{\zeta} (\not{D} - \frac{1}{3} \not{V} \gamma_5) \varphi' - \frac{2}{3} \bar{\zeta} (M + \gamma_5 N) \varphi' - 2 \bar{\zeta} (H' + \gamma_5 K') \lambda',
 \end{aligned} \tag{3.20}$$

where P' is an abbreviation for

$$P' = \not{D}^a U'_a + 2(MK' - NH') + \frac{i}{4} \bar{\chi} \gamma_5 \gamma_a \not{D}^a. \tag{3.21}$$

We now see that the general multiplet contains another submultiplet, a chiral multiplet ∂C , which has weight $w = 2$ and components

$$\partial C = (H', K'; \varphi', D', P'). \tag{3.22}$$

If we impose the constraint $\partial C = 0$, we set $H' = K' = \varphi' = D' = 0$, i.e. express H, K, ϕ and D through other fields, and constrain U'_a by

$$\not{D}^a U'_a = -\frac{i}{4} \bar{\chi} \gamma_5 \gamma_a \not{D}^a \tag{3.23}$$

or equivalently

$$\not{D}^a U_a = \frac{4}{3} \not{D}^a (b_a C) - \frac{i}{4} \bar{\chi} \gamma_5 \gamma_a \not{D}^a.$$

The multiplet which is obtained from C by setting $\partial C = 0$ is called the *linear multiplet* $\mathbb{L}$,

$$\mathbb{L} = (C; \chi; U'_a). \tag{3.24}$$

In this context, it is useful to replace U'_a by a contravariant "world" axial-vector density^{*)}

$$U'^\mu \equiv e e^{\mu\rho} (U'_\rho - i \bar{\chi} \sigma_{\rho\sigma} \gamma_5 \psi^\sigma) + i \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\nu \gamma_\rho \psi_\sigma C, \quad (3.25)$$

which transforms as

$$\begin{aligned} \delta U'^\mu = & -ie \bar{\chi} \gamma^\mu \gamma_5 \varphi' - 2ie \bar{\chi} (H' + \gamma_5 K') \sigma^{\mu\nu} \gamma_5 \psi_\nu \\ & + \varepsilon^{\mu\nu\rho\sigma} \partial_\nu \left(\frac{i}{2} \bar{\chi} \sigma_{\rho\sigma} \chi + 2i \bar{\chi} \gamma_\rho \psi_\sigma C \right) \end{aligned} \quad (3.26)$$

and whose divergence is

$$\partial_\mu U'^\mu = e P' - 2e (MK' - NH') - ie \bar{\psi}_\mu \gamma^\mu \gamma_5 \varphi' - ie \bar{\psi}_\mu \sigma^{\mu\nu} \gamma_5 (H' + \gamma_5 K') \psi_\nu. \quad (3.27)$$

The linear multiplet constraint then takes the form $\partial_\mu U'^\mu = 0$, which has the solution

$$U'^\mu = \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\nu b_{\rho\sigma}. \quad (3.28)$$

The transformation law for the antisymmetric tensor $b_{\mu\nu}$ is

$$\delta b_{\mu\nu} = i \bar{\chi} \sigma_{\mu\nu} \chi + 2i \bar{\chi} (\gamma_\mu \psi_\nu - \gamma_\nu \psi_\mu) C, \quad (3.29)$$

but the algebra (2.21) closes on $b_{\mu\nu}$ only modulo a "vector gauge transformation"

$b_{\mu\nu} \rightarrow b_{\mu\nu} + \partial_\mu \beta_\nu - \partial_\nu \beta_\mu$, as follows

$$[\delta_1, \delta_2] b_{\mu\nu} = \dots + \partial_\mu \beta_\nu - \partial_\nu \beta_\mu \quad \text{with} \quad \beta_\mu = b_{\mu\nu} \xi^\nu + \xi_\mu C. \quad (3.30)$$

We can introduce a tensor gauge multiplet b ,

$$b = (b_{\mu\nu}, C; \chi), \quad (3.31)$$

^{*)} $e \equiv \det e_\mu^a = -1/4! \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} e_\mu^a e_\nu^b e_\rho^c e_\sigma^d$.

and write

$$L = \partial b . \quad (3.32)$$

4. MINIMAL SUPERGRAVITY MULTIPLETS

The large supersymmetry multiplet of Section 2 contains two invariant sub-multiplets: first, a chiral multiplet M of weight $w = 2$ with components

$$M = (M, N; \lambda; S, \partial^a b_a), \quad (4.1)$$

second, a gauge multiplet Υ_c with chiral transformations as gauge transformations and components

$$\Upsilon_c = (A_\mu; \lambda'; S'). \quad (4.2)$$

If we constrain our supergravity multiplet by setting

$$\Upsilon_c = 0, \quad (4.3)$$

we eliminate A_μ , λ , and S as independent fields. The requirement $A_\mu = 0$ moreover fixes the chiral gauge in a way consistent with supersymmetry, since the chiral transformation in the commutator of two supersymmetry transformations has parameter $\xi^\mu A_\mu$ and thus vanishes for $\Upsilon_c = 0$. What remains of the multiplet is the "old" minimal set of 12 + 12 field components with supersymmetry transformations

$$\begin{aligned} \delta e_\mu^a &= -2i \bar{\zeta} \gamma^a \psi_\mu \\ \delta \psi_\mu &= (\partial_\mu + \mathcal{T}_\mu) \zeta \\ \delta M &= \frac{i}{4} \bar{\zeta} \gamma^a R_a \\ \delta N &= \frac{i}{4} \bar{\zeta} \gamma_5 \gamma^a R_a \\ \delta b_a &= \frac{3}{4} \bar{\zeta} \gamma_5 (R_a - \frac{1}{3} \gamma_a \gamma_b R^b). \end{aligned} \quad (4.4)$$

Owing to $\lambda' = 0$, the w -dependent parts in the transformation laws of the chiral multiplet disappear.

If, on the other hand, we constrain

$$M = 0, \quad (4.5)$$

we eliminate M , N , λ , and S , and require $\mathcal{D}^a b_a = 0$ while retaining the chiral transformations. This results in the "new" minimal set of fields with transformations

$$\begin{aligned} \delta e_\mu^a &= -2i \bar{\zeta} \gamma^a \psi_\mu \\ \delta \psi_\mu &= (\mathcal{D}_\mu + \mathcal{T}_\mu) \zeta + \alpha \gamma_5 \psi_\mu \\ \delta A_\mu &= \frac{1}{4} \bar{\zeta} \gamma_5 \gamma_\mu \gamma_a \mathcal{R}^a + \partial_\mu \alpha \\ \delta b_a &= \frac{3}{4} \bar{\zeta} \gamma_5 \mathcal{R}_a. \end{aligned} \quad (4.6)$$

It is interesting to note that a constant *chiral* multiplet

$$\mathbf{1} = (1, 0; 0; 0, 0) \quad (4.7)$$

is in general consistent with the "old" minimal geometry, or otherwise must have $w = 0$, while a constant *general* multiplet

$$\mathbf{1} = (1; 0; 0, 0, 0; 0, 0) \quad (4.8)$$

is always consistent and has (cf. Ref. 5)

$$\partial \mathbf{1} = -\frac{4}{3} M. \quad (4.9)$$

Thus a constant *linear* multiplet is consistent only for the "new" minimal geometry. With $U_a = 0$ for the constant multiplet, we have $U'_a = -\frac{4}{3} b_a$ and $U'^\mu = -\frac{4}{3} b'^\mu$ with

$$b'^\mu = e b^\mu - \frac{3i}{4} \varepsilon^{\mu\nu\sigma} \bar{\psi}_\nu \gamma_5 \psi_\sigma. \quad (4.10)$$

The "new minimal" constraint $\mathcal{D}^a b_a = 0$ thus reads

$$\partial_\mu b'^\mu = 0 \quad (4.11)$$

and has the solution

$$b'^\mu = -\frac{3}{8} \varepsilon^{\mu\nu\sigma\tau} \partial_\nu a_{\sigma\tau} \quad (4.12)$$

with

$$\delta a_{\mu\nu} = 2i \bar{\zeta} (\gamma_\mu \psi_\nu - \gamma_\nu \psi_\mu), \quad (4.13)$$

and an algebra which closes on $a_{\mu\nu}$ only modulo the vector gauge transformation

$$[\delta_1, \delta_2] a_{\mu\nu} = \dots + \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu \quad \text{with} \quad \alpha_\mu = a_{\mu\nu} \zeta^\nu + \zeta_\mu. \quad (4.14)$$

5. TENSOR CALCULUS: DENSITIES

Since an invariant action must, in particular, be chirally invariant, and since the chiral transformation of the F-component in a chiral multiplet is proportional to $w-2$, we expect *F-densities* only for chiral multiplets with weight $w = 2$. Indeed, for $w = 2$, we find that

$$[\mathbb{A}]_F = eF + ie\bar{\psi}\gamma^\mu\psi + 2e(MA + NB) - ie\bar{\psi}\sigma^{\mu\nu}(A + \gamma_5 B)\psi_\nu \quad (5.1)$$

transforms like a density:

$$\delta[\mathbb{A}]_F = \partial_\mu \left[ie\bar{\zeta}\gamma^\mu\psi - 2ie\bar{\zeta}\sigma^{\mu\nu}(A + \gamma_5 B)\psi_\nu \right]. \quad (5.2)$$

A *D'-density* can be constructed from a general multiplet $\mathbb{C}$ by using expression (5.1) on the chiral submultiplet $\partial\mathbb{C}$ which has indeed weight $w = 2$:

$$[\mathbb{C}]_{\mathcal{D}'} = e\mathcal{D}' + ie\bar{\psi}\gamma^\mu\psi' + 2e(MH' + NK') - ie\bar{\psi}\sigma^{\mu\nu}(H' + \gamma_5 K')\psi_\nu. \quad (5.3)$$

This is equal to the following *D-density*:

$$\begin{aligned}
 [C]_{\mathcal{D}} = & e \mathcal{D} - \frac{4e}{3} (CS + \bar{\chi} \lambda - HM - KN + 2C(M^2 + N^2)) + \frac{4}{3} b'^{\mu} u_{\mu} \\
 & + ie \bar{\psi}_{\mu} \gamma^{\mu} (\varphi - \frac{4}{3} (M - \gamma_5 N) \chi - \frac{4}{3} \lambda C) \\
 & + \frac{4i}{3} e \bar{\psi}_{\mu} \sigma^{\mu\nu} (M + \gamma_5 N) \psi_{\nu} C,
 \end{aligned} \tag{5.4}$$

up to a total divergence of the form $\partial_{\mu} (e \partial^{\mu} C + \text{many more terms})$. $[C]_{\mathcal{D}}$ transforms as

$$\begin{aligned}
 \delta[C]_{\mathcal{D}} = & \partial_{\mu} \left[ie \bar{\chi} \gamma^{\mu} (\varphi - \frac{4}{3} (M - \gamma_5 N) \chi - \frac{4}{3} \lambda C) + \frac{4e}{3} \bar{\chi} \gamma_5 \chi b'^{\mu} \right. \\
 & \left. + \frac{4i}{3} e \bar{\chi} \sigma^{\mu\nu} (M + \gamma_5 N) \psi_{\nu} C + 2i \varepsilon^{\mu\nu\sigma\tau} \bar{\chi} \gamma_{\nu} \psi_{\sigma} u_{\tau} \right].
 \end{aligned} \tag{5.5}$$

By construction, we see that for a linear multiplet we have

$$0 = [\mathcal{L}]_{\mathcal{D}'} = \int d^4x [\mathcal{L}]_{\mathcal{D}}. \tag{5.6}$$

The simplest density which we can construct is $[\mathbb{1}]_{\mathcal{D}}$, the "volume of superspace":

$$-\frac{3}{4} [\mathbb{1}]_{\mathcal{D}} = [M]_{\mathcal{F}} = eS + ie \bar{\psi}_{\mu} \gamma^{\mu} \lambda - ie \bar{\psi}_{\mu} \sigma^{\mu\nu} (M + \gamma_5 N) \psi_{\nu} + 2e(M^2 + N^2) \tag{5.7}$$

This density vanishes for the "new" minimal geometry, since there $\mathbb{1}$ is a linear multiplet.

6. TENSOR CALCULUS: PARITY-FLIPPED AND PRODUCT MULTIPLETS

The *parity flipped* versions of chiral, general, curl, and gauge multiplets are, respectively:

$$\begin{aligned}
 \mathbb{H}^P = & (-B, A; -\gamma_5 \chi; G, -F) \\
 \mathbb{C}^P = & (C; \gamma_5 \chi; -K, H, -u_a; -\gamma_5 \varphi; \mathcal{D}) \\
 (d\mathbb{C})^P = & (-\gamma_5 \varphi; -u_{ab}, \mathcal{D}) \\
 \Psi^P = & (-u_{\mu}; -\gamma_5 \varphi; \mathcal{D}).
 \end{aligned} \tag{6.1}$$

The weight is the same for $\mathbb{R}$ and $\mathbb{R}^P$, and we have:

$$(d\mathbb{C})^P = d\mathbb{C}^P \quad ; \quad (\partial\mathbb{C})^P = \partial\mathbb{C}^P . \quad (6.2)$$

The transformation laws for the flipped multiplets are obtained by inserting the components given in Eqs. (6.1) into the old transformation laws.

There are three *products* of two chiral multiplets: $\mathbb{R}_1 \circ \mathbb{R}_2$, $\mathbb{R}_1 \times \mathbb{R}_2$ and $\mathbb{R}_1 \wedge \mathbb{R}_2$.

The (symmetric) product

$$\mathbb{R}_3 = \mathbb{R}_1 \circ \mathbb{R}_2 \quad (6.3)$$

gives a chiral multiplet of weight $w_3 = w_1 + w_2$ and has components

$$\begin{aligned} A_3 &= A_1 A_2 - B_1 B_2 \\ B_3 &= A_1 B_2 + A_2 B_1 \\ \psi_3 &= (A_1 - \gamma_5 B_1) \psi_2 + (A_2 - \gamma_5 B_2) \psi_1 \\ F_3 &= A_1 F_2 + B_1 G_2 + F_1 A_2 + G_1 B_2 + \bar{\psi}_1 \psi_2 \\ G_3 &= A_1 G_2 - B_1 F_2 + G_1 A_2 - F_1 B_2 - \bar{\psi}_1 \gamma_5 \psi_2 . \end{aligned} \quad (6.4)$$

The (symmetric) product

$$\mathbb{C} = \mathbb{R}_1 \times \mathbb{R}_2 \quad (6.5)$$

is defined only for $w_1 = w_2$, and gives a general multiplet with components:

$$\begin{aligned} C &= A_1 A_2 + B_1 B_2 \\ \chi &= (A_1 + \gamma_5 B_1) \psi_2 + (A_2 + \gamma_5 B_2) \psi_1 \\ H &= -A_1 F_2 - A_2 F_1 + B_1 G_2 + B_2 G_1 \end{aligned} \quad (6.6)$$

$$K = -A_1 G_2 - A_2 G_1 - B_1 F_2 - B_2 F_1$$

(contd.)

(contd.)

$$u_a = A_1 \vec{\partial}_a B_2 + A_2 \vec{\partial}_a B_1 - i \bar{\psi}_1 \gamma_a \gamma_5 \psi_2 \quad (6.7)$$

$$\varphi = -(F_1 - \gamma_5 G_1 - i \not{D}(A_1 + \gamma_5 B_1)) \psi_2 - (F_2 - \gamma_5 G_2 - i \not{D}(A_2 + \gamma_5 B_2)) \psi_1 + 2w\lambda' C$$

$$D = -2 \left[F_1 F_2 + G_1 G_2 + \partial_a A_1 \partial^a A_2 + \partial_a B_1 \partial^a B_2 + \frac{i}{2} \bar{\psi}_1 \not{D} \psi_2 - i \bar{\psi}_1 \not{D} \gamma_5 \psi_2 - w(S'C + \bar{\lambda}'\chi) \right].$$

The (antisymmetric) product $\mathbb{A}_1 \wedge \mathbb{A}_2$ is defined by

$$\mathbb{A}_1 \wedge \mathbb{A}_2 \equiv (\mathbb{A}_1^p \times \mathbb{A}_2)^p \quad (6.8)$$

and has lowest component $C = A_1 B_2 - A_2 B_1$.

The product of two general multiplets $\mathbb{C}_1 \circ \mathbb{C}_2$ gives a third general multiplet,

$$\mathbb{C}_3 = \mathbb{C}_1 \circ \mathbb{C}_2$$

with components:

$$C_3 = C_1 C_2$$

$$\chi_3 = \chi_1 C_2 + \chi_2 C_1$$

$$H_3 = C_1 H_2 + C_2 H_1 - \frac{i}{2} \bar{\chi}_1 \chi_2$$

$$K_3 = C_1 K_2 + C_2 K_1 + \frac{i}{2} \bar{\chi}_1 \gamma_5 \chi_2$$

$$u_{a3} = u_{a1} C_2 + u_{a2} C_1 - \frac{i}{2} \bar{\chi}_1 \gamma_a \gamma_5 \chi_2 \quad (6.9)$$

$$\begin{aligned} \varphi_3 = & \varphi_1 C_2 + \frac{i}{2} (H_1 - \gamma_5 K_1 + i \not{D} C_1 + i \psi_1 \gamma_5) \chi_2 \\ & + \varphi_2 C_1 + \frac{i}{2} (H_2 - \gamma_5 K_2 + i \not{D} C_2 + i \psi_2 \gamma_5) \chi_1 \end{aligned}$$

$$\begin{aligned} D_3 = & D_1 C_2 + D_2 C_1 + \bar{\chi}_1 \varphi_2 + \bar{\chi}_2 \varphi_1 - \frac{i}{2} \bar{\chi}_1 \not{D} \chi_2 \\ & - H_1 H_2 - K_1 K_2 - u_1^a u_{a2} - \partial_a C_1 \partial^a C_2 + i \bar{\chi}_1 \not{D} \gamma_5 \chi_2 . \end{aligned}$$

The densities for product multiplets can be derived straightforwardly; a particularly interesting case, however, is $[\mathbb{C}' \circ \mathbb{C}]_D$ which becomes reasonably simple if the components of $\mathbb{C}'$ are given in the "shifted" form of Eqs. (3.19). Then:

$$\begin{aligned}
 [\mathcal{C}' \cdot \mathcal{C}]_{\mathfrak{D}} = & e \left(\mathcal{D}' \mathcal{C} + \mathcal{D} \mathcal{C}' + \bar{\chi}' \not{\varphi} + \bar{\chi} \not{\varphi}' - H' H - \kappa' \kappa + 2 (M H' + N \kappa') \mathcal{C} \right) \\
 & - \mathcal{U}'^\mu \mathcal{U}_\mu + i e \bar{\psi}_\mu \gamma^\mu (\not{\varphi}' \mathcal{C} + \not{\varphi} \mathcal{C}') + (H' - \gamma_5 \kappa') \chi - i e \bar{\psi}_\mu \sigma^{\mu\nu} (H' + \gamma_5 \kappa') \psi_\nu \mathcal{C}.
 \end{aligned} \tag{6.10}$$

This is remarkable in so far as, if $\mathcal{C}'$ is a linear multiplet, the D-density

$$[\mathcal{L}' \cdot \mathcal{C}]_{\mathfrak{D}} = e (\mathcal{D} \mathcal{C}' + \bar{\chi} \not{\varphi}' + i \bar{\psi}_\mu \gamma^\mu \not{\varphi} \mathcal{C}') - \mathcal{U}'^\mu \mathcal{U}_\mu = [\mathcal{L}' \cdot \Psi]_{\mathfrak{D}} \tag{6.11}$$

involves only the fields of the gauge submultiplet Ψ of $\mathcal{C}$ and transforms as a total divergence under gauge transformations $\mathcal{C} \rightarrow \mathcal{C} + d\mathcal{H}$:

$$[\mathcal{L}' \cdot \mathcal{C}]_{\mathfrak{D}} = [\mathcal{L}' \cdot (\mathcal{C} + d\mathcal{H})]_{\mathfrak{D}} + \text{div.} \tag{6.12}$$

Since for the "new" minimal formulation $\mathbb{1}$ is a linear multiplet with $U'^\mu = -\frac{4}{3} b'^\mu$, we get the result

$$[\mathcal{C}]_{\mathfrak{D}} = [\Psi_{\mathfrak{D}}] \quad \text{and} \quad [\mathcal{C} + d\mathcal{H}]_{\mathfrak{D}} = [\mathcal{C}]_{\mathfrak{D}} + \text{div.} \tag{6.13}$$

i.e. the D-density is gauge "invariant".

7. MINIMAL SUPERGRAVITY LAGRANGIANS

The density (5.7) cannot straightforwardly be used as a Lagrangian for supergravity, since variation with respect to the independent auxiliary field S gives rise to the inconsistent field equation $e = 0$. However, once we restrict ourselves to the *old minimal* set, where $S' = \lambda' = 0$ and thus $S = -\frac{1}{4} R - \frac{4}{3} [M^2 + N^2 - \frac{1}{2} b_a b^a]$ and $\lambda = \frac{i}{4} \gamma_a \mathcal{Q}^a$, we get the familiar result⁵⁾:

$$\begin{aligned}
 \mathcal{L}_{SG}^{\text{old}} = & \frac{3}{2} [\mathbb{1}]_{\mathfrak{D}} = \frac{e}{2} R + \frac{e}{2} \bar{\psi}_\mu \gamma^\mu \not{\gamma}^a \mathcal{Q}_a + 2 i e \bar{\psi}_\mu \sigma^{\mu\nu} (M + \gamma_5 N) \psi_\nu \\
 & - \frac{4e}{3} (M^2 + N^2 + b_a b^a).
 \end{aligned} \tag{7.1}$$

This can be "decovariantized" by use of Eqs. (2.10) and (2.17) into the well-known form

$$\mathcal{L}_{SG}^{\text{old}} = \frac{e}{2} e_a^\mu e_b^\nu R_{\mu\nu}^{ab} + 2i \varepsilon^{\mu\nu\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 \partial_\sigma \psi_\sigma - \frac{4e}{3} (M^2 + N^2 + b_a b^a). \quad (7.2)$$

In the case of the *new minimal* set, we know from the closing remarks of the previous section that we can use $[\Psi_c]_D$ as a Lagrangian, and we find, with

$$S' = \frac{1}{4} R - \frac{2}{3} b_a b^a \text{ and } \lambda' = -\frac{i}{4} \gamma_a \mathcal{R}^a, \text{ the result}^{(6)}:$$

$$\begin{aligned} [\Psi_c]_D &= e S' + ie \bar{\psi}_\mu \gamma^\mu \lambda' + \frac{4}{3} b'^\mu A_\mu \\ &= \frac{e}{4} R - \frac{2e}{3} b_a b^a + \frac{e}{4} \bar{\psi}_\mu \gamma^\mu \gamma^a \mathcal{R}_a + \frac{4}{3} b'^\mu A_\mu \end{aligned} \quad (7.3)$$

i.e. after decovariantization

$$\begin{aligned} \mathcal{L}_{SG}^{\text{new}} &= 2 [\Psi_c]_D = \frac{e}{2} e_a^\mu e_b^\nu R_{\mu\nu}^{ab} + 2i \varepsilon^{\mu\nu\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 \partial_\sigma \psi_\sigma \\ &\quad - \frac{4e}{3} b_a b^a - \varepsilon^{\mu\nu\sigma} A_\mu \partial_\nu a_{\sigma} . \end{aligned} \quad (7.4)$$

8. A "GRANDFATHER" FORMULATION

One way to write down a consistent Lagrangian without having to impose constraints which lead us back to the well-known minimal sets is to impose the constraints by Lagrange-multiplier methods. Thus, if we consider a linear multiplet of Lagrange multipliers and augment our invariant $[\mathbb{1}]_D$ by $[\mathbb{L} \cdot \Psi_c]_D$,

$$\begin{aligned} \mathcal{L}_{SG} &= \frac{3}{2} [\mathbb{1}]_D + 2 [\mathbb{L} \cdot \Psi_c]_D \\ &= -2eS - 2ie \bar{\psi}_\mu \gamma^\mu \lambda + 2ie \bar{\psi}_\mu \sigma^{\mu\nu} (M + \gamma_5 N) \psi_\nu - 4e (M^2 + N^2) \\ &\quad + 2e S' C + 2e \bar{\chi} \lambda' + 2ie \bar{\psi}_\mu \gamma^\mu \lambda' C - 2 U'^\mu A_\mu , \end{aligned} \quad (8.1)$$

then variation with respect to the fields of $\mathbb{L} = (C; \chi; U_a)$ gives $d\Psi_c = 0$, or after fixing the chiral gauge to $A_\mu = 0$:

$$0 = A_\mu = \lambda' = S' , \quad (8.2)$$

and upon implementation of this *supercovariant* set of equations of motion we get:

$$\mathcal{L}_{SG} = \mathcal{L}_{SG}^{\text{old}} . \quad (8.3)$$

From $\mathcal{L}^{\text{old}}$ we get, of course, the usual equations of motion

$$0 = M = N = b_a = Q_a = R_{ab} - \frac{1}{2} \eta_{ab} R . \quad (8.4)$$

Had we, instead, first varied Eq. (8.1) with respect to the fields S' , λ' , and A_μ , we would have got equations of motion

$$C = 1 \quad ; \quad \chi = 0 \quad ; \quad U'_a = 0 , \quad (8.5)$$

which when implemented into $\mathcal{L}_{SG}$ also give $\mathcal{L}_{SG} = \mathcal{L}_{SG}^{\text{old}}$.

If however, we implement the following *supercovariant* set of field equations (all of which can be derived from $\mathcal{L}_{SG}$):

$$0 = M = N = \lambda = S = \partial^a b_a \quad (8.6)$$

$$C = 1 \quad ; \quad \chi = 0 \quad ; \quad U'_a = -\frac{4}{3} b_a ,$$

then we get

$$\mathcal{L}_{SG} = \mathcal{L}_{SG}^{\text{new}} . \quad (8.7)$$

We note that Eq. (8.1) is a consistent formulation for off-shell $N = 1$ supergravity with $20 + 20$ field components: there are the $16 + 16$ of our supergravity multiplet and the $4 + 4$ components of the linear multiplet L , the Lagrange multipliers.

We list the auxiliary fields:

C	a scalar
χ	a spinor
U'^μ	a transverse axial vector density ($\partial_\mu U'^\mu = 0$)
A_μ	the axial gauge vector
b_a	an axial-vector
M, N	a scalar-pseudoscalar pair
λ	a spinor
S	a scalar .

This spectrum corresponds to that which Rivelles and Taylor found independently and called the "2nd new non-minimal set"¹⁶⁾. There are no field redefinitions which could eliminate the chiral gauge freedom or the constraint on U'^μ , since there are no pseudoscalars of the correct dimensions to provide either a gauge-compensating field for A_μ or a divergence for U'^μ .

9. COUPLING MATTER: THE CHIRAL MULTIPLIET

The kinetic Lagrangian for the chiral multiplet is given by the tensor calculus expression

$$\mathcal{L}_{kin} = -\frac{1}{4} [A \times A]_{\mathcal{D}} \quad (9.1)$$

and can be evaluated to be

$$\begin{aligned} \mathcal{L}_{kin} = & \frac{e}{2} (\partial_a A \partial^a A + \partial_a B \partial^a B + i \bar{\psi} \not{\partial} \psi + F^2 + G^2) \\ & - \frac{ie}{2} \bar{\psi} \not{\partial} \gamma_5 \psi - \frac{2}{3} b'^a (A \overset{\leftrightarrow}{\partial}_a B - \frac{i}{2} \bar{\psi} \gamma_a \gamma_5 \psi + \bar{\psi}_a \gamma_5 (A + \gamma_5 B) \psi) \\ & + \frac{ie}{2} \bar{\psi}_\mu \gamma'^\mu (F - \gamma_5 G - i \not{\partial} (A + \gamma_5 B)) \psi \\ & + \frac{2e}{3} (MAF + NAG + NBF - MBG + \bar{\lambda} (A + \gamma_5 B) \psi + i \bar{\psi}_\mu \gamma'^\mu (M - \gamma_5 N) (A + \gamma_5 B) \psi) \\ & + \frac{1}{3} (A^2 + B^2) (eS + ie \bar{\psi}_\mu \gamma'^\mu \lambda - ie \bar{\psi}_\mu \sigma^{\mu\nu} (M + \gamma_5 N) \psi_\nu + 2e (M^2 + N^2)) \\ & - \frac{w}{2} (A^2 + B^2) (eS' + ie \bar{\psi}_\mu \gamma'^\mu \lambda') - we \bar{\psi} (A + \gamma_5 B) \lambda'. \end{aligned} \quad (9.2)$$

This is, after suitable rescaling to get rid of the $(A^2 + B^2)R$ terms, the kinetic Lagrangian for the chiral multiplet in both minimal cases.

If we do not restrict our gravitational multiplet to either of the minimal cases, but keep the 16 + 16 field components, we encounter a more interesting phenomenon: if $0 \neq w \neq 2/3$, the Lagrangian

$$\mathcal{L} = -\frac{w'}{w} [\mathbb{1}]_{\mathcal{D}} - \frac{4}{w} \mathcal{L}_{kin} \quad ; \quad w' = 1 - \frac{3w}{2} \quad (9.3)$$

gives the following equations of motion when varied with respect to S , λ , F , and G :

$$0 = 1 - (A^2 + B^2) \quad (9.4)$$

$$0 = \psi = F = G. \quad (9.5)$$

If we now fix the chiral gauge to $B = 0$, then

$$B = 0; \quad A = 1; \quad \partial_\mu B = w A_\mu. \quad (9.6)$$

We can insert this into $\mathcal{L}$ and eliminate the remaining auxiliary fields M , N , b_a , A_μ from the resulting Lagrangian (the equations of motion will give $A_\mu = 0$ rather than $\partial_\mu A_\nu - \partial_\nu A_\mu = 0$, since the chiral gauge has been fixed). The result is the Lagrangian of $N = 1$ on-shell supersymmetry, $\mathcal{L} = \mathcal{L}_E + \mathcal{L}_{RS}$.

We see that the kinetic Lagrangian for a chiral "matter" multiplet allows the construction of a Lagrangian which consistently describes the $N = 1$ supergravity theory. The total number of field components is now $20 + 20$. We list the auxiliary fields:

A, B	a scalar-pseudoscalar pair
ψ	a spinor
A_μ	the axial gauge vector
b_a	an axial-vector
$M, N;$ F, G	two scalar-pseudoscalar pairs
λ	a spinor
S	a scalar

We expect that a set of field-redefinitions can be found which eliminates the chiral gauge transformations from the algebra and combines A_μ and B into a non-gauge four-component axial vector. The result of these redefinitions should be the $20 + 20$ fields of the "1st new non-minimal set" ¹⁶⁾. If we coupled more than one chiral multiplet, we would get a σ -model-like constraint instead of Eq. (9.4), and the additional chiral multiplets would propagate. Also, rather than $A_\mu = 0$, we would get something like:

$$A_\mu = - \sum_{i=2} w_i A_i \overleftrightarrow{\partial}_\mu B_i$$

for the chiral gauge field, cf. Ref. 17.

By simply counting fields, we should expect Breitenlohner's fields³⁾ to come out of the coupling to our fields of a linear multiplet with the opposite parity to that of Section 8. Field redefinitions would then combine A_μ and C to a non-gauge axial-vector, and S would provide a divergence for U'^μ . We have, however, not been able to do this, in agreement with Breitenlohner's observation¹⁸⁾ that one needs $24 + 24$ fields for a version which contains both his and the old minimal set as subsets. We should probably couple the $8 + 8$ components of a general multiplet to our multiplet to then, after suitable manipulations, obtain Breitenlohner's fields.

10. COUPLING MATTER: THE GAUGE MULTIPLY

The kinetic Lagrangian for the gauge-multiplet can be obtained from the F-density for the following chiral multiplet of weight 2, which is built from a gauge multiplet $\mathcal{V} = (U_\mu; \varphi; D)$:

$$A = - \frac{1}{4} \bar{\varphi} \varphi$$

$$B = - \frac{1}{4} \bar{\varphi} \gamma_5 \varphi$$

$$\mathcal{V} = \frac{1}{2} \varphi \mathcal{D} - \frac{i}{4} \sigma^{ab} \gamma_5 \varphi u_{ab} \quad (10.1)$$

$$F = \frac{1}{2} \mathcal{D}^2 - \frac{1}{4} u_{ab} u^{ab} + \frac{i}{2} \bar{\varphi} (\not{\mathcal{D}} + \not{u} \gamma_5) \varphi + \frac{1}{2} \bar{\varphi} (M + \gamma_5 N) \varphi$$

$$G = \frac{1}{8} \epsilon^{abcd} u_{ab} u_{cd} + \frac{i}{4} \partial^a (\bar{\varphi} \gamma_a \gamma_5 \varphi) - \frac{1}{2} \bar{\varphi} \gamma_5 (M + \gamma_5 N) \varphi.$$

The F-density for this multiplet is

$$\begin{aligned} \mathcal{L}_g(\Psi) = & e \left[-\frac{1}{4} u_{ab} u^{ab} + \frac{i}{2} \bar{\varphi} (\not{D} + \not{u} \gamma_5) \varphi + \frac{1}{2} D^2 \right] \\ & + \frac{ie}{2} \bar{\psi}_\mu \gamma^\mu (\not{D} - \frac{i}{2} \sigma^{ab} \gamma_5 \varphi u_{ab}) \\ & + \frac{ie}{4} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu \bar{\varphi} \varphi + \frac{ie}{4} \bar{\psi}_\mu \sigma^{\mu\nu} \gamma_5 \psi_\nu \bar{\varphi} \gamma_5 \varphi . \end{aligned} \quad (10.2)$$

For either of the minimal sets, this is the supersymmetric Maxwell Lagrangian for a (chiral) U(1)-gauge group. In contrast to chiral matter, this Lagrangian cannot be used to enlarge the 16 + 16 set consistently to 20 + 20 fields.

We can, however, use the $\mathcal{L}_g$ for the axial gauge multiplet

$$Y_c = (A_\mu ; \lambda' ; S') ,$$

which is a submultiplet of our gravitational multiplet:

$$\mathcal{L} = \frac{3}{2} [\mathbb{1}]_D + \mathcal{L}_g(Y_c) . \quad (10.3)$$

The bosonic part of $\mathcal{L}$ is now

$$\begin{aligned} \mathcal{L} = & -2eS - 4e(M^2 + N^2) - \frac{e}{4} F_{ab} F^{ab} + \frac{e}{2} S'^2 + \dots \\ = & \frac{e}{2} R - \frac{4e}{3} (M^2 + N^2 + b_a b^a) - \frac{e}{4} F_{ab} F^{ab} + e \left(\frac{1}{2} S'^2 - 2S' \right) + \dots \end{aligned}$$

Elimination of S' gives now not the inconsistent condition $e = 0$, but rather

$$S' = 2 ,$$

i.e. a positive cosmological constant

$$\mathcal{L} = \frac{e}{2} R - 2e + \dots$$

and spontaneous breaking of supersymmetry, since $\langle S' \rangle \neq 0$ and $\delta_S S' \neq 0$.

This is the result which Freedman¹⁴⁾ obtained in an on-shell formulation, and which de Wit and van Nieuwenhuizen¹³⁾ have previously discussed.

We see that, contrary to common belief, an off-shell formulation with an odd number of auxiliary spinor fields can give rise to a consistent propagating theory, which then, however, must break supersymmetry spontaneously: the superfluous spinor becomes the Goldstino and is swallowed by the gravitino in a super-Higgs effect.

The considerable difficulties encountered for higher N in finding the auxiliary field formulations of supergravity with even numbers of auxiliary fermions may be less severe in the formulations with odd numbers of auxiliary fermions, like the one considered here. Indeed, these difficulties have been highlighted in a recent counting argument¹⁹⁾ which, however, does not apply to formulations with an odd number of fermions.

A LARGE SUPERCURRENT MULTIPLY

The most general supersymmetry variation of an axial-vector C_μ involves a vector-spinor $J_{\mu\alpha}$:

$$\delta C_\mu = \bar{\zeta} \gamma_5 J_\mu.$$

If we constrain it to be γ -traceless,

$$0 = \gamma^\mu J_\mu \quad (A.1)$$

then it follows from the supersymmetry algebra^{*)} that the multiplet contains only C_μ , J_μ , and a symmetric tensor $\theta_{\mu\nu}$, that the transformation laws are

$$\begin{aligned} \delta C_\mu &= \bar{\zeta} \gamma_5 J_\mu \\ \delta J_\mu &= i \gamma^\nu (2 \theta_{\mu\nu} - \gamma_5 \partial_\nu C_\mu - \frac{i}{2} \epsilon_{\mu\nu\lambda\kappa} \partial^\kappa C^\lambda) \zeta \\ \delta \theta_{\mu\nu} &= -\frac{i}{4} \bar{\zeta} (\sigma_{\mu\lambda} \partial^\lambda J_\nu + \sigma_{\nu\lambda} \partial^\lambda J_\mu), \end{aligned} \quad (A.2)$$

that all fields are conserved,

$$0 = \partial^\mu C_\mu = \partial^\mu J_\mu = \partial^\mu \theta_{\mu\nu}, \quad (A.3)$$

and that $\theta_{\mu\nu}$ is traceless:

$$0 = \theta_\mu{}^\mu \quad (A.4)$$

For this *conformal current multiplet*, we now wish to relax the conformal constraints $\gamma \cdot J = \theta_\mu{}^\mu = 0$, while retaining conservation laws for at least some vector-spinor and some symmetric tensor. We therefore replace Eq. (A.1) by the weaker constraint

$$i \gamma^\mu J_\mu = \varphi, \quad (A.5)$$

where φ is a member of the flat-space general multiplet, cf. Eqs. (3.4). The constraint (A.5) still implies conservation laws for C_μ , J_μ , and $\theta_{\mu\nu}$, but now $\gamma \cdot J \neq 0 \neq \theta_\mu{}^\mu$. Apart from the currents, the multiplet contains the fields C , χ , H , K , and U_μ from the general multiplet. φ and D have been replaced by $i\gamma \cdot J$ and $2\theta_\mu{}^\mu$. The transformations are:

*) In this Appendix we assume the space to be flat.

$$\begin{aligned}
 \delta C &= \bar{\zeta} \chi \\
 \delta \chi &= (H + \gamma_5 K - i \not{\partial} C - i \not{U} \gamma_5) \zeta \\
 \delta H &= i \bar{\zeta} \gamma^\mu (\not{\partial}_\mu - \partial_\mu C) \\
 \delta K &= i \bar{\zeta} \gamma_5 \gamma^\mu (\not{\partial}_\mu - \partial_\mu C) \\
 \delta U_\mu &= -\bar{\zeta} \gamma_5 \gamma_\mu \gamma \cdot \not{\partial} + \bar{\zeta} \gamma_5 \partial_\mu \chi \\
 \delta C_\mu &= \bar{\zeta} \gamma_5 \not{\partial}_\mu \\
 \delta \not{\partial}_\mu &= i \gamma^\nu (2 \theta_{\mu\nu} - \gamma_5 \partial_\nu C_\mu - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} \partial^\alpha C^\beta - \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} \partial^\alpha U^\beta) \zeta \\
 \delta \theta_{\mu\nu} &= -\frac{i}{4} \bar{\zeta} (\sigma_{\mu\lambda} \partial^\lambda \not{\partial}_\nu + \sigma_{\nu\lambda} \partial^\lambda \not{\partial}_\mu)
 \end{aligned} \tag{A.6}$$

and the "anomalies submultiplet" is the general multiplet

$$\mathcal{C} = (C; \chi; H, K, U_\mu; i \gamma \cdot \not{\partial}; 2 \theta_{\mu\nu}). \tag{A.7}$$

LINEARIZED SUPERGRAVITY AND THE TORSION CONSTRAINTS

We now try to construct an invariant from the current multiplet of Appendix A and its contragradient multiplet. For the following ansatz,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} h^{\mu\nu} \partial_{\mu\nu} + \bar{\psi}^\mu \mathcal{J}_\mu + (A^\mu - b^\mu) C_\mu \\ & - \frac{1}{3} (b^\mu u_\mu + MH + NK - \bar{\lambda}\chi - SC), \end{aligned} \quad (\text{B.1})$$

we demand that $\mathcal{L}$ transform as a divergence and find transformation laws for the linearized supergravity multiplet:

$$\begin{aligned} \delta h_{\mu\nu} &= -2i \bar{\zeta} (\gamma_\mu \psi_\nu + \gamma_\nu \psi_\mu) + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu \\ \delta \psi_\mu &= \frac{i}{4} \sigma^{\kappa\lambda} \zeta \partial_\kappa h_{\lambda\mu} - \gamma_5 \zeta A_\mu + \gamma_5 \zeta b_\mu - \frac{i}{3} \gamma_\mu (M + \gamma_5 N - i \not{D} \gamma_5) \zeta + \partial_\mu \varepsilon \\ \delta A_\mu &= -i \bar{\zeta} \gamma_\mu \gamma_5 (\lambda + i \sigma_{\kappa\lambda} \partial^\kappa \psi^\lambda) + \partial_\mu \alpha \\ \delta b_\mu &= \frac{3i}{2} \varepsilon_{\mu\nu\kappa\lambda} \bar{\zeta} \gamma^\nu \partial^\kappa \psi^\lambda - i \bar{\zeta} \gamma_\mu \gamma_5 \lambda \\ \delta M &= \bar{\zeta} \lambda \\ \delta N &= \bar{\zeta} \gamma_5 \lambda \\ \delta \lambda &= -(S + \gamma_5 \not{D} b_\mu + i \not{D} (M + \gamma_5 N)) \zeta \\ \delta S &= i \bar{\zeta} \not{D} \lambda, \end{aligned} \quad (\text{B.2})$$

i.e. the linearized version of the transformation laws (2.4). For a fuller discussion see Refs. 2 and 6.

Knowing that in general the transformation law for the supervielbein $E_\mu^A = (e_\mu^a, \bar{\psi}_\mu^\alpha)$ is given by

$$\delta E_\mu^A = \partial_\mu \zeta^A + E_\mu^C \zeta^B T_{BC}^A, \quad (\text{B.3})$$

we can read off the following superspace torsion constraints:

$$\begin{aligned} T_{\alpha a}^b &= T_{\alpha\beta}^\gamma = 0 \\ T_{\alpha\beta}^a &= 2i (\gamma^a C)_{\alpha\beta} \\ T_{a\alpha}^\beta &= - \left(\gamma_5 b_a + \frac{i}{3} (M + \gamma_5 N - i \not{D} \gamma_5) \gamma_a \right)_\alpha^\beta . \end{aligned} \quad (B.4)$$

We assume these to hold also beyond the linearized level and together with a conventional constraint

$$T_{ab}^c = 0 , \quad (B.5)$$

which can be solved for the Lorentz connection, they give the full non-linear theory when evaluated by means of Bianchi identities.

Obviously, all our results can also be written in a superspace notation, and we hint at their form ($E \equiv \text{sdet } E$) without working out details:

The chiral action formula, Eq. (5.1), is (for $\bar{D}\phi = 0$):

$$[R]_F \sim \int d^2\theta E \phi + h.c. \quad (B.6)$$

and the D-action formula, Eq. (5.4), is [for a real superfield $C(x, \theta, \bar{\theta})$]:

$$[C]_D \sim \int d^4\theta E C . \quad (B.7)$$

A linear superfield has $\bar{D}^2 L = D^2 L = 0$, and

$$0 = \int d^4x d^4\theta E L , \quad (B.8)$$

cf. Eq. (5.6).

Generally, $\bar{D}^2 1$ is not zero but a chiral multiplet: $\bar{D} \bar{D}^2 1 = 0$. The relation

$$M \sim \bar{D}^2 1 \quad (B.9)$$

gives us the equivalent of Eq. (5.7):

$$\int d^4\theta E \sim \int d^2\theta E \bar{D}^2 1 + h.c. \quad (B.10)$$

and we see that exactly for the new minimal geometry we have

$$0 = D^2 1 = \bar{D}^2 1 , \quad \text{and hence} \quad 0 = \int d^4x d^4\theta E .$$

Equation (6.12) has the superfield equivalent:

$$\int d^4x d^4\theta E L C = \int d^4x d^4\theta E L (C + \phi + \bar{\phi}), \quad (B.11)$$

i.e. the gauge invariance of the density obtained from multiplying a linear superfield L and a general one, C . For the new minimal geometry this results in the gauge invariance of the Fayet-Iliopoulos term

$$L_{FI} \sim \int d^4\theta E C. \quad (B.12)$$

The old minimal Lagrangian is, of course, $\mathcal{L} \sim \int d^4\theta E$, while the new minimal one can be written as

$$\mathcal{L}_{SG}^{new} \sim \int d^4\theta E Y_c \quad (B.13)$$

where Y_c is the gauge superfield for the axial gauge-invariance. The "grandfather Lagrangian", Eq. (8.1), is (neglecting constants):

$$\mathcal{L} \sim \int d^4\theta E (1 + L Y_c), \quad (B.14)$$

and the matter coupling Lagrangians (9.1) and (10.2) are

$$\mathcal{L}_{kin} \sim \int d^4\theta E \phi \bar{\phi} \quad (B.15)$$

$$\mathcal{L}_{kin} \sim \int d^2\theta E (\bar{D}^2 D C)^2 + h.c. \equiv \int d^2\theta E W^\alpha W_\alpha + h.c. \quad (B.16)$$

The "self-coupling" Lagrangian (10.3) which leads to spontaneous supersymmetry breaking is

$$\mathcal{L} \sim \int d^4\theta E + \int d^2\theta E (\bar{D}^2 D Y_c)^2 + h.c. \quad (B.17)$$

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