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HIDDEN SYMMETRIES AS CANONICAL TRANSFORMATIONS
FOR THE CHIRAL MODEL

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ABSTRACT

The hidden symmetries of the principal chiral models are examined within the Hamiltonian formalism. Problems which arise when they are rewritten as canonical transformations are discussed. It is shown that the simple form of the symmetry algebra found by Dolan is modified.

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It is known¹⁾ that the equations of motion of two-dimensional chiral models admit an infinite number of non-local conserved currents. More recently, it has been claimed^{2),3)} that, for the principal chiral model, these conserved currents are the Noether currents of certain symmetry transformations on the chiral fields, and that the symmetry algebra has the form of a subalgebra of a Kac-Moody algebra^{*)}. Further discussion of these transformations has been given in Refs. 4) and 5). That the transformations involved are symmetries, in the sense that they preserve the equations of motion, is clear. However, the claim that we examine in this paper is that the transformations are "symmetries of the entire space of fields"²⁾. We discuss difficulties that arise if a naïve approach is taken and we show that, when the problem is formulated carefully, the symmetry algebra no longer has the simple form of Ref. 3).

The transformations of Ref. 3) are defined on the field configurations $g(x,t)$ of functions on $\mathbb{R}^2$ valued in a Lie group G by

$$\begin{aligned} g &\longrightarrow g + \Delta_T^{(n)} g, \\ \Delta_T^{(n)} g &= -g \Lambda_T^{(n)}, \\ \Lambda_T^{(n)}(x,t) &= \int_{-\infty}^x dy (\partial_0 \Lambda_T^{(n-1)}(y,t) + [A_0(y,t), \Lambda_T^{(n-1)}(y,t)]), \end{aligned} \quad (1)$$

where

$$A_\mu = g^{-1} \partial_\mu g,$$

n is a non-negative integer and

$$\Lambda_T^{(0)} = T,$$

is a constant element of the Lie algebra $\mathfrak{G}$ of G . Under (1) the equations of motion of the principal chiral model

$$\partial_\mu A_\mu = 0,$$

are preserved and the Lagrangian density

$$\mathcal{L}(x,t) = \frac{1}{2} \text{tr} (\partial_\mu g \partial_\mu g^{-1}),$$

changes by a total divergence

$$\Delta^{(n)} \mathcal{L} = \partial_\mu K_\mu^{(n)},$$

*) Actually, just $\mathbb{R}[t] \times \mathfrak{G}$, where $\mathbb{R}[t]$ is the algebra of formal power series in the variable t , and $\mathfrak{G}$ is the Lie algebra of the chiral group.

where $K_{\mu}^{(n)}$ is such that the Noether currents

$$J_{\mu}^{(n)} = -K_{\mu}^{(n)} + \text{tr} \left(\left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} g)} \right)^T \Delta_T^{(n)} g \right), \quad (5)$$

correspond, on the solution set of (2), to those found in Ref. 1).

By way of clarification and motivation for taking the Hamiltonian approach, let us first note here that the fact that under a transformation $g \rightarrow g + \Delta g$ $\mathcal{L}$ changes by total divergence

$$\Delta \mathcal{L} = \partial_{\mu} K_{\mu},$$

is not sufficient to warrant the conclusion that the transformation is a symmetry in a stronger sense than just preserving the equations of motion. Indeed, in itself, this condition does not imply that we have any symmetry at all (even for the equations of motion) since $\Delta \mathcal{L}$ may always be written in this way with the choice, for example,

$$K_0 = 0, \quad K_1 = -\int_{-\infty}^{\infty} dy \Delta \mathcal{L}(y, t).$$

What is important is the effect this change in $\mathcal{L}$ has on the action

$$S(t_1, t_2) = \int_{t_1}^{t_2} L(t) dt,$$

where the Lagrangian function is

$$L(t) = \int_{-\infty}^{\infty} dx \mathcal{L}(x, t).$$

Usually one requires for symmetry that the change $\Delta S(t_1, t_2)$ in the action depend only on the field configuration (and not its time derivatives) on the time slices t_1, t_2 . Then, when we make a variation in the fields, as prescribed by Hamilton's principle, the variation of ΔS is zero. Thus the equations of motion derived from the variational principle are unchanged in form.

The transformations $\Delta^{(n)}$, $n \geq 1$ do not satisfy the above condition for symmetry. If, in particular, we consider the transformation $\Delta^{(1)}$

$$\Delta^{(1)} \mathcal{L} = \partial_{\mu} \epsilon_{\mu\nu} \text{tr} \left\{ \left(A_{\nu} - \frac{1}{2} [\partial_{\nu} \chi, \chi] \right)^T \right\}, \quad (6)$$

where

$$\chi = \int_{-\infty}^{\infty} A_0 dy,$$

it gives the following expression for $\Delta^{(1)} S$

$$\Delta^{(1)} S(t_1, t_2) = - \left[\int_{-\infty}^{\infty} dx \operatorname{tr} \left\{ (A_1 - \frac{1}{2} [A_0, \chi]) T \right\} \right]_{t_1}^{t_2} + \left[\int_{t_1}^{t_2} dt \operatorname{tr} \left\{ (A_0 - \frac{1}{2} [\partial_0 \chi, \chi]) T \right\} \right]_{x=-\infty}^{x=\infty}. \quad (7)$$

The second term in Eq. (7) depends on the field g at all times t , $t_1 < t < t_2$. We could try to get rid of this dependence on fields at intermediate times by demanding that $A_0, \partial_0 \chi \rightarrow 0$ as $|x| \rightarrow \infty$, but [except on the solution set of (2)] an infinity of further conditions will result from the requirement that these constraints be preserved under the transformation $\Delta^{(1)}$. A similar problem arises when we consider the conservation of the charge:

$$\begin{aligned} \frac{d}{dt} Q_T^{(1)} &= \frac{d}{dt} \int dx \operatorname{tr} \left((A_1 + \frac{1}{2} [A_0, \chi]) T \right) \\ &= \int dx \operatorname{tr} \left(\partial_1 (A_0 + \frac{1}{2} [A_1, \chi]) T \right) - \int dx \operatorname{tr} \left(\frac{1}{2} [A_0, A_1] T \right) \quad (8) \\ &= \operatorname{tr} \left((A_0 + \frac{1}{2} [A_1, \chi]) T \right) \Big|_{x=-\infty}^{x=\infty} - \int dx \operatorname{tr} \left(\frac{1}{2} [A_0, A_1] T \right). \end{aligned}$$

(8) vanishes if $A_\mu \rightarrow 0$ as $|x| \rightarrow +\infty$, but this condition is not preserved by $\Delta^{(1)}$ and so cannot consistently be imposed on our configuration space if we are to consider such transformations.

The first term in (7) depends on the velocity $\partial_0 g$ at $t = t_1$ and t_2 , and arises because $\Delta^{(1)}$ is a velocity dependent transformation. In the context of such velocity dependent transformations, the attempt to find a sensible generalization of Noether's theorem and its inverse has a long history [see Ref. 6) for a recent review]. The conclusion to be drawn is that we can deal with these transformations in a consistent fashion if they can be written as canonical transformations within the Hamiltonian framework. There, by symmetry, we mean that the Hamiltonian is invariant under the transformation, i.e., that the generator of the transformation is a constant of the motion. We might hope that, modulo anomalies, such symmetries will carry over to the quantized theory, via canonical quantization.

We now therefore consider the Hamiltonian formulation of the principal chiral model. If we regard G as a subgroup of a real general linear group, so that $g(x, t)$ is a matrix, then the momentum is given by

$$\pi = \partial_0 (g^{-1})^T = -g^{-1T} A_0^T \quad (9)$$

and the Hamiltonian is then

$$\begin{aligned} H &= \frac{1}{2} \int_{-\infty}^{\infty} dx \operatorname{tr} (-g^T \pi g^T \pi + \partial_1 g \partial_1 g^{-1}) \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} dx \operatorname{tr} (A_0^2 + A_1^2). \end{aligned}$$

An alternative set of variables to g, π is A_0, A_1 and with respect to these the time evolution is

$$\begin{aligned} \partial_0 A_0 &= \{A_0, H\} = \partial_1 A_1, \\ \partial_0 A_1 &= \{A_1, H\} = \partial_1 A_0 + [A_1, A_0], \end{aligned} \quad (10)$$

where $\{.,.\}$ denotes the Poisson bracket. From Eq. (10) it follows that all properties derived for $g(x,t)$ valued in a general linear group will also be true for a subgroup G^* . This fact makes all ensuing computations tractable for arbitrary G .

We consider the first Noether charge $Q_T^{(1)} = \int dx J_0^{(1)}(x,t)$ defined by (4), (5) and (6)

$$Q_T^{(1)} = \int_{-\infty}^{\infty} dx \operatorname{tr} \left((A_1 + \frac{1}{2} [A_0, \chi]) \tau \right). \quad (11)$$

Since $Q_T^{(1)}$ depends only on the field g and its first time derivative, we may write it as a function on phase space, to define the generator of a canonical transformation, denoted by $\delta_T^{(1)}$.

Before proceeding further, we note that we require H and $Q_T^{(1)}$ to be finite for our manipulations to be other than purely formal. It is sufficient for this purpose to impose the following conditions^{**}): a) g and π are C^∞ functions on $\mathbb{R}$ such that their limits and the limits of all their spatial derivatives exist as $|x| \rightarrow \infty$; b) $|x|^{1+\delta} A_\mu \rightarrow 0$, some $\delta > 0$, as $|x| \rightarrow \infty$. In particular, these conditions imply

$$\int dx |A_\mu|^m < \infty, \quad m=1, 2, \dots \quad (12)$$

*) The proof of this is given by noting that if A_0, A_1 are initially valued in the corresponding subalgebra $\mathcal{G}$ then from Eq. (10) they will be valued in $\mathcal{G}$ for all time.

**) Note that the time evolution (10) preserves conditions a) and b).

Under the canonical transformation $\delta_T^{(1)}$ we have

$$\delta_T^{(1)} g = \{g, Q_T^{(1)}\} = -g[\eta, T], \quad (13a)$$

$$\delta_T^{(1)} \pi = ([A_1, T]g^{-1} + [\eta, T]\pi^T)^T. \quad (13b)$$

In terms of the variables A_μ , these become

$$\delta_T^{(1)} A_\mu = -[A_\mu, [\eta, T]] - \epsilon_{\mu\nu} [A_\nu, T], \quad (14)$$

where

$$\begin{aligned} \eta(x) &= \frac{1}{2} \int_{-\infty}^{\infty} \epsilon(x-y) A_0(y) dy, \\ &= \chi(x) - \frac{1}{2} \int_{-\infty}^{\infty} A_0(y) dy, \end{aligned} \quad (15)$$

and where $\epsilon(x) = 1$ for $x > 0$, and $= -1$ for $x < 0$. Equation (14) implies immediately

$$\delta_T^{(1)} H = \{H, Q_T^{(1)}\} = 0,$$

so that $\delta_T^{(1)}$ is a symmetry. We note two points about this derivation. First, $\delta_T^{(1)}$ is closed, in the sense that the conditions a) and b) are preserved by the transformation, and second we see that from Eqs. (1) and (13a)

$$\delta_T^{(1)} g \neq \Delta^{(1)} g,$$

even though $Q_T^{(1)}$ was originally defined using $\Delta^{(1)}$. This may be easily understood when it is realized that $\Delta^{(1)}g$, even when it is written as a transformation on phase space, is not a canonical transformation since

$$\{g(x) + \Delta^{(1)}g(x), g(y) + \Delta^{(1)}g(y)\} \neq 0, \quad (16)$$

and hence $\Delta^{(1)}$ does not preserve the canonical Poisson bracket relation

$$\{g(x), g(y)\} = 0.$$

We further note that the new transformation $\delta_T^{(1)}$ can be considered to solve some of the problems we encountered in the Lagrangian formulation of the theory, at the expense of a naive notion of the action: we have

$$\delta_T^{(1)} \mathcal{L} = 2 \partial_\mu \text{tr} \epsilon_{\mu\nu} A_\nu T, \quad (17)$$

without using the equations of motion, and so

$$\delta_T^{(1)} S(t_1, t_2) = -2 \int_{-\infty}^{\infty} dx \operatorname{tr} (A_1 T) \Big|_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} dt \operatorname{tr} (A_0 T) \Big|_{x=-\infty}^{x=\infty}, \quad (18)$$

but because of the result (14) the condition $A_\mu \rightarrow 0, |x| \rightarrow \infty$ [which is implied by b)] is preserved by $\delta_T^{(1)}$, and so we can consistently impose it; the second term in (18) then vanishes and $\delta_T^{(1)} S(t_1, t_2)$ depends only on the field configuration at t_1, t_2 and not on its time derivatives there. Similarly, we find that the boundary condition on A_μ means that $Q_T^{(1)}$ is conserved as expected.

The difficulty we encounter now, however, is that

$$\delta_T^{(1)} (\partial_0 g) = g [A_1, T] + \partial_0 g [\eta, T] \neq \partial_0 (\delta_T^{(1)} g) = g [\partial_0 \eta, T] + \partial_0 g [\eta, T],$$

except for solutions of (2), and this prevents us from thinking of the action as a functional on paths $g(t)$ (we must see it rather as a functional of fields and field time derivatives treated as independent variables): this is the price we have to pay for introducing velocity dependent transformations in a Lagrangian theory.

We now consider the commutator algebra $Q_T^{(1)}, \{Q_T^{(1)}, Q_T^{(1)}\}$, etc. via the Poisson bracket, and obtain an infinite set of conserved generators (and hence an infinity of canonical symmetry transformations) equivalent to the solution set to the conserved charges of Ref. 1). We would now like to check whether this commutator algebra remains that found in Ref. 3), i.e., an algebra $R(t) \times \mathcal{G}$ generated by $Q_T^{(0)}, Q_T^{(1)}$ ($T \in \mathcal{G}$). Here

$$Q_T^{(0)} = \int_{-\infty}^{\infty} dx \operatorname{tr} (A_0 T),$$

is the generator of the constant symmetry transformation. What is required is that a set of functions $\{R_T^{(n)}; n \geq 1\}$ can be defined on phase space with

$$R_T^{(1)} = Q_T^{(1)},$$

such that

$$\begin{aligned} \{Q_T^{(0)}, R_{T'}^{(n)}\} &= R_{[T, T']}^{(n)}, \\ \{R_T^{(1)}, R_{T'}^{(n)}\} &= R_{[T, T']}^{(n+1)}. \end{aligned} \quad (19)$$

Once (19) is consistent for $n = k$, we may use the equation recursively to define $R_T^{(k+1)}$. The remainder of the algebra is assured using (19) and the fact that the Poisson bracket is itself a Lie product. We find, however, that

$$\{Q_T^{(1)}, Q_{T'}^{(1)}\} = \int dx \operatorname{tr} (A_0 - [\eta, A_1] + \frac{1}{2} [\eta, [\eta, A_0]]) [T, T'] + \frac{2}{3} \operatorname{tr} \eta(\omega) [[\eta(\omega), T], [\eta(\omega), T']]. \quad (20)$$

Because of the last term in Eq. (20) it follows that (19) is violated for all simple non-Abelian Lie algebras $\mathfrak{g}$ except that of $SO(3)$, in which case we may write the last term of (20) as

$$-\frac{1}{6} \operatorname{tr} Q^{(0)2} \operatorname{tr} Q^{(0)} [T, T'],$$

since $Q^{(0)} = 1/2 \eta(\omega)$. For $SO(3)$, we must go to the second step. There we find that $\{R_T^{(1)}, R_{T'}^{(2)}\}$ is not antisymmetric in T, T' and so cannot be written in terms of $[T, T']$.

Hence (19) is violated for all $\mathfrak{g}$, and the algebra we construct is not isomorphic to $\mathbb{R}[t] \times \mathfrak{g}$. It is not possible to reinstate the algebra by the simple expedient of changing $Q_T^{(1)}$ to $\tilde{Q}_T^{(1)}$ where

$$\tilde{Q}_T^{(1)} = Q_T^{(1)} + f_T(Q^{(0)}),$$

if we are to retain the property

$$\{Q_T^{(0)}, \tilde{Q}_{T'}^{(1)}\} = \tilde{Q}_{[T, T']}^{(1)}. \quad (21)$$

With the form of $f_T(Q^{(0)})$ allowed by Eq. (21) it is not possible either to eliminate the last term in Eq. (20) in the case of Lie algebras other than that of $SO(3)$ or to eliminate the symmetric part of $\{R_T^{(1)}, R_{T'}^{(2)}\}$ in the case of $SO(3)$, as may be checked explicitly.

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