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LIGHT-CONE SUPERSPACE AND THE FINITENESS
OF THE $N = 4$ MODEL

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ABSTRACT

Superspace in the light-cone frame takes a simple form, and application to extended supersymmetries is straightforward. If we use a certain form of the light-cone gauge, the $N = 4$ model has no divergences in any order of perturbation theory. It follows that the β function has no perturbative divergences in any gauge.

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I should like to discuss superspace in light-cone co-ordinates, since extended supersymmetries may then be treated on the same footing as ordinary supersymmetries. We shall be able to show, in any order of perturbation theory, that the $N = 4$ model has no ultra-violet divergences in a certain form of the light-cone gauge, and therefore that the β -function vanishes in any gauge. The Wess-Zumino algebra for $N = 1$ supersymmetry has been quoted by Siegel and Gates¹⁾, and a treatment of $N = 4$ symmetry has been given independently of us by Brink, Lindgren and Nilsson²⁾, but neither of these groups considered the ultra-violet problem.

The difficulty in obtaining covariant extended superspaces is connected with the lack of correspondence between the number of fields and the number of particles. In any supersymmetric model, the particle multiplets will form a representation of the supersymmetry algebra. In particular, the number of bosons and fermions will be equal. When going from fields to particles this correspondence is lost, and one restores the balance by adding auxiliary fields. No such set of auxiliary fields has been found for the $N = 4$ model. With light-cone co-ordinates the number of fields and particles is the same, and no problem arises.

In light-cone co-ordinates the number of spinor field components, and therefore the number of supersymmetry operators, is half the number in the co-variant formalism. Invariance under these operators, together with Lorentz invariance, guarantees full supersymmetry invariance. The Wess-Zumino algebra is simple:

$$\{Q_i, Q_j\} = 4i(\partial/\partial x^-)\delta_{ij} = 4p^+\delta_{ij} \quad (1)$$

For $N = 4$, the algebra is as follows:

$$\{Q_{\alpha i}, Q_{\beta j}\} = 4p^+\delta_{\alpha\beta}\delta_{ij} \quad (ij = 1, 2; \alpha, \beta = 1, 2, 3, 4). \quad (2)$$

To construct a superspace for the Wess-Zumino model, we note that the first quantized model has operators x and p , together with the two fermionic operators Q_1 and Q_2 . After second quantization, the wave functions become fields which are functions of co-ordinates corresponding to x and the Q 's. We require one co-ordinate corresponding to each pair of conjugate variables, so that we have four x^μ 's and one anticommuting θ . Our approach thus differs from the usual approach, which would have introduced a θ and a $\bar{\theta}$. For the $N = 4$ model we require four θ_α 's. A treatment closer to the conventional one is given in Refs. 1) and 2).

Corresponding to $Q_{1\alpha}$ and $Q_{2\alpha}$, we define the two operators

$$Q_{2\alpha}: \quad \mathcal{Q} = i\left\{\frac{\partial}{\partial\theta^\alpha} - 2P^+\theta^\alpha\right\}, \quad (3a)$$

$$Q_{1\alpha}: \quad \tilde{\mathcal{Q}} = \frac{\partial}{\partial\theta^\alpha} + 2P^+\theta^\alpha. \quad (3b)$$

In writing down the superfield for the $N = 4$ model, we start from the highest helicity field and go downwards. Thus

$$\phi = i(2p^+)^{-1} V + (2p^+)^{-1} \theta^\alpha \psi_\alpha + \frac{i}{4} \theta^\alpha \theta^\beta \rho^\mu_{\alpha\beta} A_\mu + \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} \theta^\alpha \theta^\beta \theta^\gamma \psi^{\delta\tau} + 2i p^+ \theta_1 \theta_2 \theta_3 \theta_4 V^\dagger. \quad (4)$$

Each field carries a gauge index which has been suppressed. The positive and negative helicity vector and spinor fields, and the six spinless fields, have been represented by the symbols V , $V^\dagger$, ψ_α , $\psi^{\dagger\alpha}$ and A respectively. The matrix ρ is that of Gliozzi, Olive and Scherk³⁾.

The supermultiplet for the $N = 4$ theory is self-conjugate. The fields ϕ and $\phi^\dagger$ are therefore not independent, but are related by the equation

$$\phi^\dagger = (2p^+)^{-2} \tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2 \tilde{\mathcal{D}}_3 \tilde{\mathcal{D}}_4 \phi. \quad (5)$$

We shall omit the derivation of the formula for the Lagrangian; the result is:

$$\begin{aligned} \mathcal{L} = & (\partial^\mu \phi)(\partial_\mu \phi) - \frac{\sqrt{2}ig}{3} \phi \cdot \{(\partial_1 \phi) \times (2p^+ \phi)\} \\ & - \frac{\sqrt{2}ig}{3} (2p^+)^{-2} \phi \cdot \prod_\alpha (2i\partial_{x^-}, \theta^\alpha) \{[(2p^+)^{-2} \partial_r \phi] \times (2p^+)^{-1} \phi\} \\ & - \frac{g^2}{64} \sum_\alpha (2i\partial_{x^-}, \theta^\alpha)(\phi \times \phi) \cdot (2p^+)^{-2} \prod_{B \neq \alpha} (2i\partial_{x^-}, \theta^B) \{(2p^+)^{-1} \phi \times (2p^+)^{-1} \phi\}. \end{aligned} \quad (6)$$

Dot and cross products refer to the gauge degrees of freedom. The derivatives $\partial_{r,l}$ are $\partial_1 \pm i\partial_2$, while

$$\partial_{x^-}, \theta^\alpha (\phi_a, \phi_b) = \frac{\partial \phi_a}{\partial x^-} \frac{\partial \phi_b}{\partial \theta^\alpha} - \frac{\partial \phi_a}{\partial \theta^\alpha} \frac{\partial \phi_b}{\partial x^-}. \quad (7)$$

We observe that $\mathcal{L}$ involves only $\partial/\partial\theta^\alpha$ and not θ^α , so that it commutes with the operator θ^α . Furthermore, it only involves $\partial/\partial\theta^\alpha$ in the combination (7), so that it commutes with $P^\dagger \theta^\alpha$. The Lagrangian thus commutes with the supersymmetry operators (3). It is understood that the θ derivatives in the third and fourth terms of (7) are to be written in cyclic order. The derivatives thus all occur combined with the symbol $\epsilon^{\alpha\beta\gamma\delta}$ and, since this is an $SU(4)$ invariant combination, the Lagrangian is manifestly $SU(4)$ invariant.

The third term of $\mathcal{L}$ can be obtained from the second by replacing ϕ by $\phi^\dagger$, which is related to ϕ by Eq. (5). The terms are thus the Hermitian conjugates of one another.

To prove the finiteness of the model, we examine a general vertex diagram. Let us consider an external three-point vertex of the form of the second term of (6). By using the identity

$$\phi_A \cdot \{(\partial_1 \phi_B \times 2p^+ \phi_C) - (2p^+ \phi_B \times \partial_1 \phi_C)\} = 2p^+ \phi_A \cdot (\phi_B \times \partial_1 \phi_C) - \partial_1 \phi_A \cdot (\phi_B \times 2p^+ \phi_C), \quad (8)$$

we note that we can choose one pair of lines meeting at the vertex, including the external line, and apply the factors ∂_l and $2p^+$ to them. Thus the external line has at least one factor ∂_l or $2p^+$. We can treat the other terms of (6) similarly; each external line has at least one factor of ∂_l , ∂_r , $2p^+$ or $\partial/\partial\theta$.

We thus find that there are more powers of p on the external lines of the vertex corrections than on the external lines of the bare vertex. (A factor $\partial/\partial\theta$ is dimensionally equivalent to a factor $p^{1/2}$.) It might therefore be expected that the number of powers of p on internal lines is insufficient to give a divergence. The power counting is easily performed and confirms this result: the vertex corrections are finite, provided it is legitimate to employ naïve power counting with all components of p treated equally.

In the usual light-cone gauge, such power counting is not in fact permissible. The reason is that ultra-violet divergences appear from the region where p^+ and $p^+p^- - p^2$ are finite, while p^- and p^2 are large. The poles in the factors $(p^+)^{-1}$ prevent us from continuing to imaginary p^0 and thus avoiding these dangerous regions.

The condition $A^+ = 0$ does not define the light-cone gauge uniquely, since it remains true under a gauge transformation which depends on x^1 and x^+ but not on x^- . The ambiguity is reflected in the $i\epsilon$ prescription in the factors $(p^+)^{-1}$. Usually, one takes a principal-value prescription. If we could use the prescription $(p^+)^{-1} \rightarrow (p^+ + i\epsilon p^-)^{-1}$, there would be no difficulty in continuing the p^0 integration to imaginary p^0 . After such continuation it is easy to see that naïve power counting is valid.

It is not difficult to show that one can define a light-cone gauge with the above $i\epsilon$ prescription. Such a "modified light-cone gauge" is inconvenient for most purposes, since it is only invariant under Lorentz transformations which leave both p^+ and p^- unchanged. For our purposes this is the best gauge to use, since the vertex functions are finite. It is also easy to prove that all n -point functions, with $n \geq 3$, are finite.

For the two-point functions, the above reasoning would still allow a divergent term of the form $A p^2 \delta_{ij} + B p_i p_j$. To show that such a term does not in fact occur, we use the Ward identity in the form $\Lambda^i(p, p, 0) = -\partial/\partial p_i \Pi(p, p)$. This version of the Ward identity is valid only if proper Green's functions involving gluons of all four polarizations are free of singularities when any of the p^+ 's become zero; a condition which is true in the modified light-cone gauge (though not in the usual light-cone gauge). From the differential form of the Ward identity and the finiteness of Λ , we can conclude that divergent terms proportional to $p^2 \delta_{ij}$ or to $p_i p_j$ cannot occur in Π . The two-point function, and the complete model, are thus finite in any order of perturbation theory.

In other gauges the wave function renormalization will generally not be finite. The divergence is a pure gauge artifact. The β function will always vanish, however, since its vanishing is a gauge-invariant condition.

REFERENCES

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