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SPONTANEOUS COMPACTIFICATION OF 11-DIMENSIONAL
SUPERGRAVITY⁺

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ABSTRACT

Three distinct geometries on S_7 arise as solutions of the classical equations of motion in eleven dimensions. In addition to the conventional Riemannian geometry, one can also obtain the two exceptional Cartan-Schouten compact flat geometries with torsion. Possible implications of these results are mentioned.

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Simple supergravity in eleven dimensions¹⁾ may turn out to be of more fundamental significance than the ungauged version of the maximal (N = 8) extension of four-dimensional supergravity²⁾: in the trivial dimensional reduction process leading from the former theory to the latter, relevant information may well have been lost. Following this line of thought, it is of interest to look for possible non-trivial classical solutions in eleven dimensions which would describe a compactification of seven dimensions into a sphere S_7 and which would keep four space-time dimensions non-compact. Indeed a Kaluza-Klein dimensional reduction³⁾ around $S_7 \equiv SO(8)/SO(7)$ would lead to an $SO(8)$ gauge theory which could be compared and perhaps identified with the gauged version of the N = 8 theory⁴⁾: in this way one would learn whether or not this gauged version is also contained in the eleven-dimensional theory^{*)}

In this paper we shall derive the surprising result that there exists not only one, but three geometries on S_7 which emerge as solutions of the equations of motion in eleven dimensions. In addition to the "trivial" Riemannian geometry on S_7 , one also obtains the two "non-trivial" geometries for which the linear connexion in S_7 contains just the torsion terms required to define an absolute parallelism on the seven sphere. This property of the eleven-dimensional action is particularly remarkable because, apart from group spaces, S_7 is the only parallelizable compact manifold, a feature linked to the fact that it is the space of unit octonions⁵⁾.

We now present and briefly discuss these solutions. We defer to a separate publication the detailed analysis of the supersymmetry and gauge symmetries they would induce in the four-dimensional space-time background.

The bosonic action of the theory is

$$S = \int \sqrt{g^{(11)}} d^{(11)}x \left\{ -\frac{1}{2} R - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right. \\ \left. + \frac{\sqrt{2}}{6 \cdot (4!)^2} \frac{1}{\sqrt{g^{(11)}}} \epsilon^{M_1 M_2 \dots M_{11}} F_{M_1 M_2 M_3 M_4} F_{M_5 M_6 M_7 M_8} A_{M_9 M_{10} M_{11}} \right\}^{(1)}$$

We use the metric $(+, ---)$, $\epsilon^{0123\dots} = +1$ and $F_{MNPQ} = 4! \partial_{[M} A_{NPQ]}$. The field equations are

$$R_{MN} - \frac{1}{2} g_{MN} R = -\frac{1}{48} \left[8 F_{MPQR} F_N{}^{PQR} - g_{MN} F_{SPQR} F^{SPQR} \right] \quad (2)$$

*) This possibility was already suggested by M.J. Duff and C.N. Pope⁷⁾ [see M.J. Duff, CERN preprint TH-3232 (1982); M.J. Duff and D.J. Toms, CERN preprint TH-3259 (1982)].

$$F^{MNPQ}{}_{;M} = -\frac{\sqrt{2}}{2 \cdot (4!)^2} \frac{1}{\sqrt{g^{(4)}}} \epsilon^{M_1 \dots M_8 N P Q} F_{M_1 M_2 M_3 M_4} F_{M_5 M_6 M_7 M_8}. \quad (3)$$

We look for solutions of (2) and (3) which describe a direct product of S_7 with a four-dimensional Einstein space to cope with an eventual cosmological constant. Hence we must have

$$R_{mn} = \gamma g_{mn} \quad (m, n = 4, 5, 6, 7, 8, 9, 10) \quad (4)$$

$$R_{\mu\nu} = \gamma' g_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3) \quad (5)$$

$$R_{m\mu} = 0 \quad (6)$$

where γ and γ' are constants and $\gamma < 0$.

Equation (3) is consistent with the existence of a non-vanishing expectation value for the auxiliary field $F^{\mu\nu\sigma\lambda}$, namely⁶⁾

$$F^{\mu\nu\sigma\lambda} = \frac{1}{\sqrt{-g^{(4)}}} \epsilon^{\mu\nu\sigma\lambda} \frac{f}{(4!)^{1/2}}; \quad f = \text{constant}. \quad (7)$$

If we assume that no other components of F^{MNPQ} acquire a non-vanishing expectation value, we see that (4), (5) and (6) solve Eq. (2) provided

$$f^2 = -F_{\mu\nu\sigma\lambda} F^{\mu\nu\sigma\lambda} = -72 \gamma \quad (8)$$

$$\gamma' = -2\gamma \quad (9)$$

Thus the "trivial" S_7 is indeed a solution for arbitrary γ , or equivalently arbitrary radius.

Let us now admit non-vanishing expectation values for the 4-scalars F^{mnpq} . The solution (7) remains valid, but (3) now yields an additional equation:

$$F^{mnpq}{}_{;m} = \frac{\sqrt{2}}{(4!)^{3/2}} f \frac{1}{\sqrt{-g^{(7)}}} \epsilon^{npqrst} F_{rstu}. \quad (10)$$

It will turn out that the solutions of Eq. (10) are related to the torsion tensors required to parallelize S_7 . We shall therefore first rederive the Cartan-Schouten equations which these torsion tensors must satisfy.

A Riemann space is parallelizable if, by adding to the Riemannian symmetric connection Γ_{np}^m a suitable totally antisymmetric torsion tensor S_{np}^m ($S_{mnp} = S_{[mnp]}$), one can "flatten" the space in the sense that

$$R^m_{npq} (\{\Gamma^a_{tu} + S^a_{tu}\}) = 0. \quad (11)$$

Using the cyclic identities for $R^m_{npq}(\{\Gamma^s_{tu}\})$ one easily obtains the necessary and sufficient conditions for such a torsion, and hence for a parallelism, to exist:

$$\overset{\circ}{R}_{mnpq} = S_{t mn} S^t_{pq} - S_{t[mn} S^t_{p]q} \quad (12)$$

$$S_{mnp;q} = S_{t[mn} S^t_{p]q} = S_{[mn p, q]} \quad (13)$$

Here and in what follows $\overset{\circ}{R}_{mnpq} \equiv R_{mnpq}(\{\Gamma^s_{tu}\})$ and the covariant derivatives are taken with respect to the Riemann connexion Γ^s_{tu} ; the last equality in (13) follows from the antisymmetry of $S_{t[mn} S^t_{p]q}$ in the four indices m, n, p and q . Contracting any two indices in (12) with g^{rs} and S^{rs}_t and using the explicit form of $\overset{\circ}{R}_{mnpq}$ for S_7 , namely

$$\overset{\circ}{R}_{mnpq} = \frac{\gamma}{6} (g_{mp} g_{nq} - g_{mq} g_{np}), \quad (14)$$

one gets

$$S^{tr}_m S_{trn} = \gamma g_{mn} \quad (15)$$

$$S_{tm}{}^r S_{rn}{}^a S_{ap}{}^t = \frac{\gamma}{2} S_{mnp} \quad (16)$$

Equations (15) and (16) were first obtained by Cartan and Schouten⁵⁾ who showed that their solutions define two types of parallelisms on S_7 .

We shall show that each parallelism is described by a torsion S_{+mnp} or S_{-mnp} which satisfy different linear partial differential equations. To this effect, consider the algebraic equations

$$X^{mnp} \equiv X^{[mnp]} = \pm \sqrt{\frac{6}{-\gamma}} \frac{1}{4!} \frac{1}{\sqrt{-g^{(7)}}} \epsilon^{mnpqrst} X_{qrst} X^u \eta_u \quad (17)$$

Performing the ϵ algebra, one may verify that both Eqs. (17) admit solutions satisfying (12) and which of course also satisfy (15) and (16). These solutions determine the torsions S_{+} and S_{-} up to a sign which can be fixed by (13). Using the latter equation in the right-hand side of (17) one gets

$$S_{+}^{mnp} = + \sqrt{\frac{6}{-\gamma}} \frac{1}{4!} \frac{1}{\sqrt{-g^{(7)}}} \epsilon^{mnpqrst} S_{+}[rst, q] \quad (18)$$

$$S_{-}^{mnp} = - \sqrt{\frac{6}{-\gamma}} \frac{1}{4!} \frac{1}{\sqrt{-g^{(7)}}} \epsilon^{mnpqrst} S_{-}[rst, q] \quad (19)$$

These "fields" satisfy $S_{\pm}^{mnp}; m = 0$; they may be interpreted as "potentials" which, in this "Landau gauge", are dual (or anti-dual) to their field strength $S_{+}[mnp, q]$ ($S_{-}[mnp, q]$). These potentials are eigenfunctions of the Laplace operator on S_7 :

$$S_{\pm}^{mnp}; q = -\frac{2}{3} \gamma S_{\pm}^{mnp} \quad (20)$$

Equation (20) follows from (13) using (15) and (16); it may also be obtained directly from (18) or (19) using $S_{\pm}[rst, q] = S_{\pm}rst; q$.

We now solve Eq. (10) by postulating

$$F_{mnpq} = \lambda S_{+}[npq, m] \quad , \quad (21)$$

or

$$F_{mnpq} = \lambda S_{-}[npq, m] \quad , \quad (22)$$

where λ is a constant. Indeed, using (18), (19) and (20) we can satisfy (10) provided

$$\pm f = \sqrt{-\gamma} e^{5/2} \quad (23)$$

As the right-hand side of (23) is positive, one must choose (21) or (22) according to the sign of f .

The expressions (7), (21) [or (22)] and (23) define solutions of Eq. (3) if g_{mn} is the metric tensor of S_7 . One must still verify (4), (5) and (6). From (2) we see that these equations can be satisfied only if $F_{mpqr} F_n{}^{pqr}$ is proportional to g_{mn} . This is indeed the case as follows from (21) [or (22)] and (18) [or (19)]: inverting (18) [or (19)] and performing the ϵ algebra, we get, using (15)

$$F_{mpqr} F_n{}^{pqr} = \frac{2}{3} \lambda^2 \gamma^2 g_{mn} . \quad (24)$$

Equation (2) now yields (4), (5) and (6) and determines λ^2 :

$$\lambda^2 = - \frac{12}{\gamma} \quad (25)$$

Thus we have obtained, in addition to the "trivial" S_7 solution characterized by $F_{mnpq} = 0$, new solutions where the 4-scalars take the values (21) or (22) with λ^2 determined by (25). The results corresponding to (8) and (9) are, for these new solutions,

$$f^2 = - F_{\mu\nu\sigma\lambda} F^{\mu\nu\sigma\lambda} = - 32 \gamma \quad (26)$$

$$F^2 = F_{mpqr} F^{mpqr} = - 56 \gamma \quad (27)$$

$$\gamma' = - \frac{5}{3} \gamma \quad , \quad (28)$$

and γ is, of course, still arbitrary. The geometrical significance of (21) and (22) is clear from Eqs. (18) and (19): the admissible expectation values of the 4-scalars F_{mnpq} are just those for which the potentials A_{npq} in the "Landau gauge" are, up to a scale factor whose absolute value is determined by (25), equal to one of the torsion tensors $S_{\mp}^{npq}$ which "flattens" the seven-sphere.

A few concluding remarks are in order:

a) the non-trivial solutions [(26), (27), (28)], as well as the trivial one [(8), (9)], will necessarily lead, after a Kaluza-Klein reduction around $S_7 \equiv SO(8)/SO(7)$, to four-dimensional actions containing $SO(8)$ gauge fields with a coupling constant proportional to $|\gamma|$ in Planck units. What about supersymmetry? It can be shown⁷⁾ that, provided four-dimensional space-time is taken to be the anti-de Sitter space satisfying Eq. (9), the trivial solution on S_7 preserves the full supersymmetry of the original action (1) with fermions included, but that supersymmetry is broken by the non-trivial solution. Hence the four-dimensional action deducible from the reduction around the trivial S_7 geometry will be fully

supersymmetric and this suggests of course that it might describe the gauged version of $N = 8$ supergravity⁴⁾. On the other hand, the reduction around the non-trivial S_7 geometries will reveal a spontaneous breakdown of supersymmetry^{*)}.

b) The appealing mathematical properties of the non-trivial S_7 solutions are marred by the reappearance of a cosmological constant [Eq. (28)] which was already there in the trivial S_7 solution: thus the "flattening" of S_7 is not accompanied by a vanishing of the cosmological constant. However, the value of the measurable cosmological constant can anyhow not be predicted without understanding the effect of the quantum fluctuations which are not considered here.

c) Finally, we have not at all discussed the relative stability of the different classical solutions. There may be no sensible way of attacking this problem as long as the physical implications of the "wrong" sign induced by the conformal part of gravity are not mastered. This may require the link between field theory and cosmology provided by the conjecture that the path integral over metrics is stabilized by the entropy stored in a "foam of universes"⁸⁾. In this respect, it is perhaps encouraging to notice that if eleven-dimensional supergravity is physically relevant, and if spontaneous compactification does take place, a multiplicity of "universes" will already be unavoidably generated at the level of "multi-soliton" solutions.

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