Ref.TH.3400-CERN

DOES THE QUENCHED REDUCED $U(\infty)$ CHIRAL MODEL BREAK SPONTANEOUSLY
IN WEAK COUPLING ?

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A B S T R A C T

The $U(N)$ chiral model, when quenched using Parisi's rule, has a $[U(1) \times U(1)]^N / U(1)$ global invariance. To determine whether this symmetry breaks spontaneously in weak coupling for $N = \infty$, a one-loop calculation of the distribution of eigenvalues of the single $U(N)$ matrix of the model is performed. This distribution is shown to be uniform on the unit circle and hence, no symmetry breaking occurs. Further, the order parameter $|\text{tr} U|^2 / N^2$, which should be zero at $N = \infty$ in the absence of spontaneous symmetry breaking, is evaluated in the weak coupling phase for one, two and three dimensions for N varying from 2 to 50 by Monte Carlo simulation of the quenched model. The data indicate that this parameter indeed goes to zero as $N \rightarrow \infty$ implying that the symmetry does not break.

Ref.TH.3400-CERN

3 August 1982

Recently, there has been renewed interest in large N matrix theories. It was initiated by the observation of Eguchi and Kawai ¹⁾ that the loop equations of the $N = \infty$, d dimensional lattice gauge theory are identical to those of a reduced model with d links and periodic boundary conditions. However, this model was shown by Bhanot, Heller and Neuberger ²⁾ to be an incomplete description of $N = \infty$ QCD because of spontaneous symmetry breaking in the weak coupling (continuum) limit of the theory. The recipe suggested to cure this problem and obtain the correct weak coupling limit was to "quench" the eigenvalues of the d link matrices ²⁾. This means that the eigenvalues are not to be treated as dynamical variables to be integrated over in the partition function. Instead, one must average over them after computing Green's functions in a fixed distribution of these eigenvalues.

To illustrate using a rather obvious though partly symbolic notation, consider the free energy

$$F = \ln Z = \ln \int d\lambda da e^{S(\lambda, a)} \quad (1)$$

where the λ 's are the eigenvalues and "a" stands symbolically for the "rest" of the matrix.

Under quenching, this is replaced by

$$F_Q = \int d\lambda \ln Z(\lambda) = \int d\lambda \ln \left[\int da e^{S(\lambda, a)} \right] \quad (2)$$

Similarly, a Green's function

$$\langle \phi \rangle = \frac{1}{Z} \int d\lambda da \phi e^{S(\lambda, a)} \quad (3)$$

is replaced under quenching by

$$\langle \phi \rangle_Q = \int d\lambda \frac{1}{Z(\lambda)} \int da \phi e^{S(\lambda, a)} \quad (4)$$

Parisi ³⁾ suggested and developed ⁴⁾ another approach to quenching valid for all $N = \infty$ matrix theories. Using Parisi's approach, Gross and Kitazawa ⁵⁾ and Das and Wadia ⁶⁾ showed that the reduced, quenched gauge model reproduces the planar graphs of weak coupling perturbation theory for $N = \infty$ QCD.

It is now of interest to apply these ideas to $N = \infty$ chiral models. Using the quenching prescription of Ref. 3), Bars, Gunaydin and Yankielowicz ⁷⁾ have succeeded in performing the group integrals in the $N = \infty$ continuum quenched chiral model. This is potentially a very important development. However, it has been pointed out by Heller and Neuberger ⁸⁾ that there is an additional $[U(1) \times U(1)]^N / U(1)$ symmetry left over in the chiral model after quenching. In Monte Carlo studies, this symmetry seems to break spontaneously in weak coupling. This would suggest that the naively quenched chiral model ends up with the wrong vacuum state in the continuum limit. The authors of Ref. 8) suggest an alternate model which is quartic and therefore difficult to integrate in the spirit of Ref. 7).

In the present paper, I study the question of the spontaneous symmetry breaking of the quenched lattice chiral model in weak coupling within a one-loop approximation. To lowest order, one finds that there is no spontaneous symmetry breaking because the distribution of eigenvalues of the single matrix U of the model is uniform on the unit circle. In the one-loop approximation also, this picture persists as the distribution remains uniform. This strongly indicates that there is no spontaneous symmetry breaking.

To confirm this issue, a Monte Carlo simulation was performed with the order parameter $L = 1/N^2 \langle |\text{tr} U|^2 \rangle$ for $d = 1, 2, 3$ dimensions and N ranging from 2 to 50. By factorization, L should vanish as $N \rightarrow \infty$. This indeed happens, both for weak and strong coupling in the quenched model. The conclusion is that there is no spontaneous symmetry breaking in this model and the vacuum state in the continuum limit is the correct one.

The lattice chiral model is defined by the action

$$S_L = \beta N \sum_{x, \mu} \text{Re} [\text{tr} (U(x) U^\dagger(x+\mu))] \quad (5)$$

where x, μ label the sites and directions of a hypercubic lattice and the U 's are elements of the group $U(N)$.

The reduced model à la Parisi ^{3), 7)} equivalent to this one at $N = \infty$ is given by the action

$$S = \beta N \sum_{\mu} \text{Re} [\text{tr} (U D_{\mu} U^\dagger D_{\mu}^\dagger)] \quad (6)$$

where D_{μ} is a diagonal matrix of momenta

$$D_\mu = \text{Diag} \left\{ D_\mu^i, D_\mu^i = e^{ip_\mu^i}, i=1,2,\dots,N, -\pi \leq p_\mu^i \leq \pi \right\} \quad (7)$$

and the p_μ^i 's are distributed uniformly between $-\pi$ and π .

The action of Eq. (6) obviously has a $[U(1) \times U(1)]^N / U(1)$ invariance.

The partition function is given by

$$Z = \int dU e^S \quad (8)$$

where dU is the Haar measure on $U(N)$. The free energy, according to the rules of quenching ^{2),3),8)} is defined by

$$F = \int \left(\prod_\mu dp_\mu^i \right) \ln Z \quad (9)$$

Z will now be computed to one-loop approximation in weak coupling (large β).

First, rewrite the action as

$$S = \beta N^2 d - \frac{\beta N}{2} \sum_\mu \text{tr}(F_\mu F_\mu^\dagger) \quad (10)$$

with

$$F_\mu = [U, D_\mu] \quad (11)$$

Maximum action corresponds to $F_\mu = 0$ or to U 's which are diagonal. Hence, a convenient parametrization to construct perturbation theory is ^{2),8),9)}

$$U = e^{ia} \lambda e^{-ia} \quad (12)$$

where λ is a diagonal matrix of the eigenvalues of U ,

$$\lambda = \text{Diag} \left\{ \lambda^i, i=1,2,\dots,N, \lambda_i = e^{i\theta_i}, -\pi \leq \theta_i \leq \pi \right\} \quad (13)$$

and "a" is a Hermitean $N \times N$ matrix.

The Haar measure becomes ^{8),9)}

$$dU = \left(\prod_i d\lambda^i \right) \left(\prod_{i>j} |\lambda_i - \lambda_j|^2 \right) \left(\prod_{i,j} d^2 h_{ij} \right) \quad (14a)$$

where

$$d^2 h_{ij} = d^2 a_{ij} f(a) \quad (14b)$$

$$\delta h = \sum_{n=0}^{\infty} \frac{i^{n+1}}{(n+1)!} [a, [a, \dots [a, \delta a] \dots]] \quad (14c)$$

and the last equation is to be thought of as a matrix equation.

To compute Z to one-loop in weak coupling, one must expand the non-Gaussian parts of e^S in a power series in "a" up to $O(a^4)$ and the density $\rho(a)$ to $O(a^2)$. This is tedious but straightforward. The result is

$$\begin{aligned} Z &= e^{d\beta N^2} \int \left(\prod_i d\lambda^i \right) \left\{ \prod_{i,j} |\lambda^i - \lambda^j|^2 \tilde{Z}(\lambda) \right\} \quad (15) \\ \tilde{Z}(\lambda) &= \int \prod_{i,j} d^2 a_{ij} \exp \left[-\beta N \sum_{i,j} |a_{ij}|^2 |\lambda_i - \lambda_j|^2 D_{ij} \right] \\ &\left\{ 1 - \frac{N}{3} \sum_{i \neq j} |a_{ij}|^2 - \frac{\beta N}{8} \sum_{\substack{i \neq k \\ k \neq j \\ j \neq l \\ l \neq i}} [D_{ij} a_{ik} a_{kj} a_{jl} a_{li} (\lambda_j - 2\lambda_k + \lambda_i) \right. \\ &(\lambda_j^* - 2\lambda_k^* + \lambda_i^*)] + \frac{\beta N}{12} \sum_{\substack{i \neq l \\ l \neq k \\ k \neq j \\ j \neq i}} [D_{ij} a_{il} a_{lk} a_{kj} a_{ji} ((\lambda_j - \lambda_i)(\lambda_j^* - \\ &\lambda_i^* + 3(\lambda_k^* - \lambda_k)) + c.c.)] \left. \right\} + \text{higher orders in } 1/\beta \end{aligned} \quad (16)$$

where

$$D_{ij} = \sum_{\mu} |D_{\mu}^i - D_{\mu}^j|^2 \quad (17)$$

As discussed in Ref. 2), the issue of spontaneous symmetry breaking depends on the effective action governing the dynamics of the λ 's. If the distribution of these λ 's [which is determined by the curly bracket in Eq. (15)] is such that the λ 's are uniformly distributed on the unit circle, there is no spontaneous symmetry breaking. If the λ 's can clump together on the unit circle, the symmetry is broken.

To decide the issue, one must compute $\tilde{Z}(\lambda)$. To zeroth order (no loops), $\tilde{Z}(\lambda)$ is given solely by integrating over the exponential term in Eq. (16). These integrals are easily done and the result is

$$Z \sim e^{d\beta N^2} \left(\frac{1}{\beta}\right)^{\frac{N^2-N}{2}} \int \prod_i d\lambda^i \quad (18)$$

It is therefore apparent that to this order, the distribution of the λ 's is uniform on the unit circle and there is no spontaneous symmetry breaking.

To compute the one-loop correction to Eq. (18), one introduces in the usual way the generating function

$$G(\gamma) = \int \prod_{ij} d^2 a_{ij} \exp \left[-\beta N \sum_{ij} |a_{ij}|^2 Q_{ij} + \sum_{ij} a_{ij} \gamma_{ij} \right] \quad (19a)$$

$$G(\gamma) \cong \exp \left[\frac{1}{\beta N} \sum_{ij} \gamma_{ij} \gamma_{ji} \frac{1}{Q_{ij}} \right], \quad Q_{ij} > 0, \quad \forall i, j, i \neq j \quad (19b)$$

From this, it is easy to perform the integrations over the a_{ij} 's in Eq. (16) by differentiating with respect to the γ 's. After some algebra, this gives

$$Z \sim e^{d\beta N^2} \left(\frac{1}{\beta}\right)^{\frac{N^2-N}{2}} \int \prod_i d\lambda^i \left[e^{-\frac{1}{\beta} f(D, \lambda)} + O\left(\frac{1}{\beta^2}\right) \right] \quad (20)$$

where

$$\begin{aligned} f(D, \lambda) = & \frac{1}{6} \sum_{i \neq j} \frac{1}{|\lambda_i - \lambda_j|^2} \frac{1}{D_{ij}} \\ & + \frac{1}{32N} \sum_{\substack{i \neq k \\ j \neq k}} \frac{D_{ij}}{D_{ik} D_{jk}} \frac{|\lambda_i - 2\lambda_k + \lambda_j|^2}{|\lambda_i - \lambda_k|^2 |\lambda_j - \lambda_k|^2} \\ & - \frac{1}{24N} \sum_{\substack{i \neq k \\ j \neq k}} \frac{(\lambda_k^* - \lambda_i^*)(4\lambda_k - \lambda_i - 3\lambda_j) + c.c.}{D_{kj} |\lambda_i - \lambda_k|^2 |\lambda_j - \lambda_k|^2} \end{aligned} \quad (21)$$

$f(D, \lambda)$ is the effective action for the λ integration. The λ 's like to distribute themselves so as to minimize f . If the uniform distribution of λ 's remains an absolute minimum of f , the symmetry remains unbroken to one-loop order. This issue is most readily decided numerically. This is done as follows.

Choose $D_{\mu}^i = e^{ip_{\mu}^i}$ with the p_{μ}^i 's uniformly spaced in the interval $[-\pi, \pi]$ 2), 5), 7). Choose $\lambda^i = e^{i\theta_i}$ with the θ 's uniformly spaced in the interval $(-\phi, \phi)$. Compute

$$\xi(\phi) = \frac{1}{N^2} [f(D, \lambda(\phi)) - f(D, \lambda(\pi))] \quad (22)$$

as $N \rightarrow \infty$. When $\phi < \pi$, the λ 's are forced closer together on the unit circle. If the uniform distribution of λ 's were an unstable saddle point, $\xi(\phi)$ would be negative for $\phi < \pi$. In Fig. 1, $\xi(\phi)$ is plotted vs. ϕ for N varying from 2 to 50 for $d = 2$ dimensions. It is clear that ξ is always positive for large N . Thus, pushing the λ 's together increases the action. Hence the uniform distribution for the λ 's remains a stable saddle in the one-loop approximation. Note that for small N ($N \sim 5$ or less), the uniform distribution seems unstable. This may explain the results of Ref. 8) where it was concluded that the λ 's like to clump together. In one dimension, $\xi(\phi)$ behaves just as in two dimensions and the conclusion is that the symmetry there also does not break as is well known from the exact solution.

To obtain a more concrete proof for the absence of spontaneous symmetry breaking, one must resort to Monte Carlo simulation.

Consider the order parameter

$$L = \frac{1}{N^2} \langle |\text{tr } U|^2 \rangle \quad (23)$$

where the average is in the sense of Eqs. (1)-(4).

If the eigenvalues of U like to cluster together, it is easy to see that L does not vanish, even at $N = \infty$. Indeed, if the distribution of θ^i 's ($\lambda^i = e^{i\theta^i}$) is Gaussian with width σ , then for $\sigma \ll \pi$, $L \approx e^{-\sigma^2/2}$ at $N = \infty$.

If, on the other hand, the distribution of the θ^i 's is uniform on the unit circle, L tends to vanish when $N \rightarrow \infty$ in general like $1/N^2$. However, in the present case, it is sufficient that L vanishes like $1/N$ because the quenched model is expected to be correct only to $O(1/N)$ (2), (3), (5), (7) and not $O(1/N^2)$.

A standard Monte Carlo simulation was done to determine L both in the weak and strong coupling region for N varying from 2 to 50. The quenched momenta p_μ^i were chosen uniformly in the interval $[-\pi, \pi]$ but care was taken to compute L by averaging it over many different random permutations of these momenta.

In Fig. 2, L is plotted versus $1/N$ for $d = 1, 2, 3$ dimensions. In strong coupling, L was evaluated for $d = 1, 2$ for $\beta = 0.4$ and it is clear that as $N \rightarrow \infty$, L vanishes faster than $1/N$ in this regime. The choice of which β value to use to determine the weak coupling behaviour of L was made as follows.

Consider the order parameter

$$E = \frac{1}{2Nd} \left\langle \sum_{\mu} \text{tr}(U D_{\mu} U^{\dagger} D_{\mu}^{\dagger} + \text{h.c.}) \right\rangle = \frac{1}{N^2 d} \frac{\partial F}{\partial \beta} \quad (24)$$

E has the weak coupling expansion

$$E = 1 - \frac{1}{2d\beta} \left(1 - \frac{1}{N}\right) + O\left(\frac{1}{\beta^2}\right) \quad (25)$$

It was determined by numerical simulations that at $\beta = 2.0$, $\beta = 1.2$, $\beta = 0.8$, respectively for $d = 1, 2, 3$ dimensions, E had a value somewhere between 0.75-0.8 and was well within the weak coupling region described by Eq. (25).

From Fig. 2, it is also clear that in weak coupling, L vanishes as $N \rightarrow \infty$ at least as fast as $1/N$ for two and three dimensions and like $1/N^2$ for $d = 1$. As stated above, this is sufficient in the quenched model because it is expected to be correct only to $O(1/N^{2,3,5,7})$. Figure 2 also shows that if one does simulations in the quenched model with $N \sim 10$, one will see a rather large signal for L . This should not be interpreted to mean that the symmetry breaks spontaneously because it is an artifact of the finiteness of N . This observation probably explains why the authors of Ref. 8) concluded that this model suffers spontaneous symmetry breaking.

In summary, it has been shown in this paper both by a one-loop weak coupling computation of the distribution of eigenvalues of the matrix U and by Monte Carlo simulation on the complete quenched model that the $[U(1) \times U(1)]^{\infty}/U(1)$ invariance of the reduced quenched chiral $U(N)$ model at $N = \infty$ does not break spontaneously.

I would like to thank Itzhak Bars, Dietrich Foerster, Murat Günaydin, Giorgio Parisi, Spenta Wadia and Shimon Yankielowicz for very useful discussions.

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FIGURE CAPTIONS

Figure 1 The effective action $\xi(\phi)$ for the λ 's when they are restricted to the interval $(-\phi, \phi)$ for $d = 2$ dimensions and various values of N .

Figure 2 The order parameter L vs. $1/N$ for N varying from 4 to 50 in $d = 1, 2, 3$ dimensions both for weak coupling (dots) and strong coupling (crosses).

- Figure 1 -

- Figure 2 -