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SUPERSYMMETRIC MODELS ON THE LATTICE

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ABSTRACT

We construct the $d = 2$ Wess-Zumino model and the $d = 4$ $N = 2$ supersymmetric gauge theory on the lattice, in the Hamiltonian formalism. The models have the correct tree level continuum limit. Using the Kähler-Dirac formalism, we show that on the lattice a full cubic symmetry is preserved. A strong coupling calculation shows the possibility of spontaneous breaking of supersymmetry for the $d = 2$ model.

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Theories formulated on the lattice allow us to use calculational schemes which go beyond ordinary perturbation expansion¹⁾. It would, therefore, be extremely useful to have a lattice formulation of supersymmetric models.

It is rather easy to invent a lattice model which is invariant under a transformation of a supersymmetric type, i.e., mixing bosons with fermions. The non-trivial part of the problem is finding a lattice formulation whose continuum limit will turn into a Lorentz invariant supersymmetric model with the right supersymmetry algebra²⁾⁻⁴⁾. One faces two basic difficulties.

a) The continuum supersymmetry algebra

$$\{Q_F, Q_G\} = \delta_{FG} H + \alpha_{FG}^i P_i \quad (1)$$

where $\alpha^i = \gamma^0 \gamma^i$, contains the momenta P_i which, being the continuous translation generators, do not exist on the lattice. We solve this problem by working in the Hamiltonian formulation⁴⁾ (continuous time, discrete space) and using a subalgebra of (1) which involves only the Hamiltonian^{*}). The minimal subalgebra which still implies the basic features of supersymmetry is:

$$H = Q^2 \quad (2)$$

where Q is a Hermitean fermionic generator. As a consequence of (2) unbroken supersymmetry causes a zero vacuum energy and fermion-boson degeneracy.

Finding the subalgebra (2) inside the algebras (1) will limit the models which can be formulated on the lattice.

b) The problem of fermion doubling: as is well known⁵⁾, the naive continuum limit cannot be taken for Hamiltonians containing fermions since spurious soft modes appear. For space dimension d in a Susskind formulation⁵⁾, the minimal number of real fermionic components in the continuum describable by a lattice Hamiltonian is 2^d .

We will choose supersymmetric models which contain exactly the above number of fermions. This limitation on the models coincides with the one mentioned under point a). Models having this minimal doubling admit a natural description in the Kähler-Dirac formalism⁶⁾. This formalism allows us to clearly formulate the lattice symmetries which hopefully lead to the emergence of the full supersymmetry algebra in the continuum.

^{*}) A more radical solution of the problem would require a supersymmetry algebra which involves the finite translations.

Let us start with the $N = 1$ two-dimensional Wess-Zumino model. The algebra (1) in a Majorana basis in which $\gamma_0 = \sigma_2$, $\gamma_1 = i\sigma_1$, $\alpha_1 = \sigma_3$ takes the form

$$Q_1^2 = H + P \quad (3)$$

$$Q_2^2 = H - P$$

$$\{Q_1, Q_2\} = 0 \quad (4)$$

Define $Q_{\pm} = 1/\sqrt{2}(Q_1 \pm Q_2)$. This is a change of the basis of the α matrices of (1) into one in which $\alpha^1 = \sigma_1$. The algebra becomes

$$Q_+^2 = H \quad (5a)$$

$$Q_-^2 = H$$

$$\{Q_+, Q_-\} = P \quad (5c)$$

On the lattice, we shall hold (5a) say, and expect (5b) and (5c) to be recovered in the continuum. We cannot have both (5a) and (5b) on the lattice since if Q_+ and Q_- are good symmetries then by (5c) so is P which is a continuous translation.

In the two-dimensional Wess-Zumino model there is a two-component real Majorana fermion ψ and a real boson field ϕ . The charges in the continuum model are:

$$Q_{\pm} = \int dx \left[\pi(x) \psi_{1,2}(x) + \frac{\partial \psi}{\partial x} \psi_{2,1}(x) \pm V(\psi(x)) \psi_{2,1}(x) \right] \quad (6)$$

with H given by (5).

To formulate the theory on the lattice we shall discretize in a straightforward way, say the operator Q_+ . Let

$$Q = \sum_x \pi(x) \psi_1(x) + \Delta^+ \psi(x) \psi_2(x) + V(\psi(x)) \psi_2(x) \quad (7)$$

where $\psi(x)$ is a two-component real Fermi field at the one-dimensional lattice point x . We use the notation

$$(\Delta^+ \psi)(x) = \psi(x+1) - \psi(x) \quad (\Delta^- \psi)(x) = \psi(x) - \psi(x-1)$$

Canonical commutation rules are assumed, namely

$$[\pi(x), \psi(y)] = i\delta_{x,y} \quad \{\psi_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha,\beta} \delta_{x,y} \quad (8)$$

Now, let us examine how the above choice is consistent with the doubling picture. The form of the operator Q in (8) becomes more natural if we view the two components of $\psi(x)$ not as sitting both at the site x , but rather let $\psi_1(x)$ sit at the site x and associate $\psi_2(x)$ with the link connecting x and $x+1$. It is then natural for $\psi_2(x)$ to couple under supersymmetry to the gradient of ϕ across its associated link (Fig. 1). This is of course equivalent to the Susskind picture, a lattice with twice the number of points and $\psi_1(x) = \chi(2x)$ and $\psi_2(x) = \chi(2x+1)$ with $\chi(x)$ a single component real Fermi field. The scalar field ϕ is defined only on even sites.

The lattice Hamiltonian is then defined by

$$H = Q^2 = \sum_x \left[\frac{1}{2} \pi^2(x) + \frac{1}{2} (\Delta^+ \phi)^2 + \frac{1}{2} V^2(\phi(x)) - i\psi_1(x) \Delta^- \psi_2(x) + iV'(\phi(x)) \psi_1(x) \psi_2(x) + V(\phi(x)) \Delta^+ \phi(x) \right] \quad (9)$$

Consider the kinetic fermionic part of H . In the single component Susskind picture it takes the form

$$-i \sum_x \psi_1(x) \Delta^- \psi_2(x) = -i \sum_x \chi(x) \chi(x+1) \quad (10)$$

In momentum space it becomes

$$\int_{-\pi}^{\pi} dk \sin k \tilde{\chi}(k) \tilde{\chi}(-k)$$

The single component field undergoes doubling since there are two soft regions in momentum space around 0 and π . Its dispersion law is $E(k) = \sin k$. In terms of the momentum in the original lattice of the two component field, it is $\sin k/2$ which is identical to the bosonic dispersion as required by supersymmetry.

In terms of the two-component field ψ the fermionic kinetic term gives, after integration by parts,

$$\begin{aligned} -i \sum_x \psi_1 \Delta^- \psi_2 &= i \sum_x (\Delta^+ \psi_1) \psi_2 = -i \sum_x \psi_2 \Delta^+ \psi_1 \\ &= -\frac{i}{2} \sum_x [\psi_1 \Delta^a \psi_2 + \psi_2 \Delta^a \psi_1 - (\psi_1 \Delta^s \psi_2 - \psi_2 \Delta^s \psi_1)] = -\frac{i}{2} \sum_x [\psi^\dagger \Delta^a \psi - \psi^\dagger \Delta^s \psi] \end{aligned} \quad (11)$$

where $(\Delta^a) \Delta^s$ is the (anti) symmetric part of Δ^+ , i.e.,

$$(\Delta^a \psi)(x) = \frac{1}{2} (\Delta^+ - \Delta^-) \psi(x) = \frac{1}{2} (\psi(x+1) - \psi(x-1))$$

$$(\Delta^s \psi)(x) = \frac{1}{2} (\Delta^+ + \Delta^-) \psi(x) = \frac{1}{2} (\psi(x+1) + \psi(x-1) - 2\psi(x)) = \frac{1}{2} \Delta^+ \Delta^- \psi(x)$$

The term $i \psi \Delta^a \psi$ in (11) is linear in momentum space near $p = 0$. It gives there the right continuum behaviour and dominates the second term which is quadratic in p . The second term is a Wilson term⁷⁾ which lifts the energy at $p = \pi$ which would be zero for the first term alone. The two terms combine together to give the $\sin p/2$ dispersion law mentioned above.

Models in which two components Majorana particles do not double have previously been constructed⁸⁾. Note, however, that we did not need to put this Wilson term in by hand, rather it was formed as a result of supersymmetry which requires identical dispersion for bosons and fermions. Thus all the terms in H of (10) tend to the right limit in the continuum. The last term then becomes

$$V(\psi) \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} w(\psi) \quad \text{with} \quad V(\psi) = \frac{dw}{d\psi}$$

As a total derivative it drops out of H . It may still be included in the algebra as a central topological charge⁹⁾. On the lattice it is not a total derivative and does not commute with Q . Therefore it should be kept in H to maintain supersymmetry.

Let us study the question of supersymmetry breaking by the strong coupling method. Since in two dimensions the field ϕ is dimensionless, the potential $V(\phi)$ is arbitrary and the number of couplings in the system is infinite. The term strong coupling then requires further specification. Define

$$V_\lambda(\phi) = \lambda V(\lambda \phi) \quad (12)$$

and take the limit of (9) when $\lambda \rightarrow \infty$. In that limit the leading term in H , the order λ^2 term, gives indeed many decoupled single site theories. Furthermore, the single site theory is nothing but Witten's supersymmetric quantum mechanics¹⁰⁾. According to Witten, for V a polynomial supersymmetry is broken in the large λ limit if and only if $V(\phi)$ is of an even degree. Then one can perform an expansion for the Hamiltonian (9) in $1/\lambda$ about the large λ limit. Note that if we want to approach (9) from the strong coupling limit by an expansion with a finite radius of convergence, then we better keep at least the π^2 term and the leading term of the potential polynomial. But, the question of whether supersymmetry is broken in the strong coupling limit depends just on the leading term of V . Of course, as in every other strong coupling treatment, the supersymmetric behaviour of the strong coupling phase may not extend throughout the entire parameter space if there are phase transitions for finite coupling. Such transitions may depend on the detailed form of $V(\phi)$; see Ref. 11).

We have seen that the naive continuum limit of (10) is indeed the right covariant supersymmetric Hamiltonian. One might worry that the last term of (10) which breaks parity or, in two dimensions, spatial rotation symmetry, may spoil in the course of renormalization the Lorentz covariance of the continuum limit. Since, however, at least for polynomial $V(\phi)$, the theory is super-renormalizable with dimension one coupling, the only terms induced by renormalization have dimensions zero or one. The only such terms that can break parity are of the form $F(\phi)\partial\phi/\partial X$ with arbitrary F . Such terms are harmless being total derivatives.

Let us now turn to four dimensions. We seek again a real combination Q of the four Hermitean operators Q_ρ which satisfy the algebra (1) such that $Q^2 = H$. If $Q = \sum_\rho v_\rho Q_\rho$ we need by (1) four real numbers v_ρ such that

$$\sum_{\rho, \sigma} \alpha_{\rho\sigma}^i v^\rho v^\sigma = 0$$

for $i = 1, 2, 3$. It turns out that unlike in two dimensions there is no solution to this constraint in four dimensions. To execute our programme we are forced to enlarge the algebra (1) to the case of $N = 2$ supersymmetry. Then we have four complex operators Q_α obeying the algebra.

$$\{Q_\alpha^{\dagger i}, Q_\beta^{\dagger j}\} = \delta_{ij} \Gamma_{\alpha\beta}^r P_r \quad (13)$$

under which, as we will show explicitly, we can separate many combinations, each of which obeys (2). Here appears the connection with doubling. In (13) there

is an $SU(2)$ internal symmetry, related to the minimal doubling in three space dimensions⁸⁾. Then one can conveniently combine the space and internal symmetries by representing Q as a matrix.

$$Q_{\alpha}^{(i)} = (\sigma_2)_{\alpha j} S + (\vec{\sigma} \sigma_2)_{\alpha j} \vec{V} \quad (14)$$

S and $\vec{V}$ behaving as a scalar and a vector under the direct sum of the two groups. The algebra (13) becomes

$$\begin{aligned} \{S^{\dagger}, S\} &= H & \{S^{\dagger}, \vec{V}\} &= \vec{P} \\ \{V_j^{\dagger}, V_k\} &= \delta_{jk} H + \varepsilon_{jkl} P_l \end{aligned} \quad (15)$$

the other anticommutators being zero.

The algebra (15) has an additional discrete symmetry ("parity"): $S \rightarrow S^{\dagger}$, $\vec{V} \rightarrow -\vec{V}^{\dagger}$, $\vec{P} \rightarrow -\vec{P}$, $H \rightarrow H$ such that the real and imaginary parts of the generators have well-defined parity. From (15) it is clear that the real and imaginary parts of S and $\vec{V}$ form a set of eight operators fulfilling (2). In particular a natural choice for Q is $Q = \text{Re}S$, which is a scalar under both "rotations" and "parity".

The simplest theory having $N = 2$ supersymmetry is the $N = 2$ Yang-Mills^{9),12)} which consists of a gauge field, two Weyl fields $\chi^{(1)}, \chi^{(2)}$, a scalar field A and a pseudoscalar field B . All the fields are in the adjoint representation of the gauge group. Decomposing the fermion fields in analogy to (14):

$$\chi_{\alpha}^{(i)} = (\sigma_2)_{\alpha j} \psi + (\vec{\sigma} \sigma_2)_{\alpha j} \vec{V} \quad (16)$$

Separating the real and imaginary parts which have well-defined parity

$$\psi = \chi + \frac{i}{2} \varepsilon_{jkl} \chi_{jkl} \quad V_j = \chi_j + \frac{i}{2} \varepsilon_{jkl} \chi_{kl}$$

and discretizing the continuum generator Q we obtain:

$$Q = \int -\pi_A \chi + \frac{1}{6} \varepsilon_{ijk} \pi_B \chi_{ijk} - (E_i + \Delta_i^+ A) \chi_i - \frac{1}{2} \varepsilon_{ijk} (H_i - \Delta_i^- B) \chi_{jk} \quad (17)$$

The appearance of $\Delta^\pm$ is natural⁶⁾ if we think about χ_{ijk} as a three-form defined on a cube, χ_{ij} as a two-form defined on a plaquette, etc.

The operator Q then couples very naturally each component of the "geometric fermion" χ with the corresponding "geometric boson" degree of freedom (Fig. 2). Therefore, in the direct sum of the real and internal space Q has the full cubic symmetry.

Squaring (17) we get the Hamiltonian

$$H = Q^2 = \sum_x \frac{1}{2} \pi_A^2(x) + \frac{1}{2} \pi_B^2(x) + \frac{1}{2} (\Delta^+ A)^2 + \frac{1}{2} (\Delta^- B)^2 + \frac{1}{2} E^2 + \frac{1}{2} H^2 - i (\chi \Delta_i^- \chi_i + \chi_i \Delta_i^+ \chi_i + \frac{1}{6} \chi_{ijk} \Delta_i^+ \chi_{jk}) + E \cdot \Delta^+ A - H \cdot \Delta^- B \quad (18)$$

Consider the fermionic part of H . Treating χ as an eight-component spinor it reads

$$-\frac{i}{2} \chi^T \left[(r^1 + i r^4) \Delta_1^- + (r^2 + i r^5) \Delta_2^- + (r^3 + i r^6) \Delta_3^- \right] \chi \quad (19)$$

with Γ^i being 8×8 dimensional matrices fulfilling the $O(6)$ Clifford algebra. $\Gamma^{1,2,3}$ are symmetric and $\Gamma^{4,5,6}$ are antisymmetric. Therefore one may write (19) as

$$-\frac{i}{2} \sum_{i=1}^3 \chi^T (r^i \Delta_i^+ + i r^{3+i} \Delta_i^-) \chi \quad (20)$$

with Δ^a and Δ^s of (11). In momentum space it takes the form

$$\int d^3h \tilde{\chi}^T(-h) \left[\sum_{i=1}^3 (\sin h_i r^i + (1 - \cos h_i) r^{3+i}) \right] \tilde{\chi}(h) \quad (21)$$

Near $p = 0$ the $\sin p$ terms dominate over the $1 - \cos p$ terms and one gets the right continuum behaviour. The $1 - \cos p$ terms serve as Wilson terms to prevent other soft modes in momentum space away from the origin. To see that (21)

has indeed the right boson-like spectrum, note that by an $O(6)$ rotation in the $O(6)$ representation generated by the Γ^1 one can bring the matrix

$$\sum_i \left(\sin h_i \Gamma^i + (1 - \cos h_i) \Gamma^{3+i} \right) \text{ into the form, say } \left[\sum 4 \sin^2 \frac{h_i}{2} \right]^{1/2} \Gamma^1$$

Γ^1 has four eigenvalues of 1 and four -1, this shows that the dispersion law of (20) is exactly that required by supersymmetry. In the Susskind picture of single component fermions $\chi(x)$ defined on the above-mentioned thin lattice the fermionic kinetic part takes the standard form

$$-i \sum_x \left[\bar{\chi}(x) \chi(x+e_1) + (-1)^{x_1} \bar{\chi}(x) \chi(x+e_2) + (-1)^{x_1+x_2} \bar{\chi}(x) \chi(x+e_3) \right] \quad (22)$$

The last two terms in (18) are total derivatives on the lattice and may be dropped leaving a completely free lattice theory.

We can have an Abelian theory which is still non-trivial if we take the gauge field $\phi_i(x)$ to be a compact angle variable. The theory is still defined by the generator Q of (17); only the definition of H is changed into

$$H_i = \frac{1}{2} \epsilon_{ijk} \sin y_{jk} \quad (23)$$

with

$$y_{jk} = \Delta_j^+ y_k - \Delta_k^+ y_j$$

Since ϕ_i has a natural scale, one can introduce a coupling constant into the model, multiplying E in Q of (17) by g and H by $1/g$. The Hamiltonian becomes

$$H = \sum_x \left[\frac{1}{2} \pi_A^2(x) + \frac{1}{2} \pi_B^2(x) + \frac{1}{2} \sum_i (\Delta_i^+ A)^2 + \frac{1}{2} \sum_i (\bar{\Delta}_i^- B)^2 + \frac{g^2}{2} \sum_i E_i^2 + \frac{1}{2g^2} \sum_{ij} \sin^2 y_{ij} \right] \quad (24)$$

$$-i \left(\chi \Delta_i^- \chi_i + \cos y_{ij}(x) \chi_{ij} \Delta_i^+ \chi_j + \frac{1}{6} \chi_{ijk} \Delta_i^+ \chi_{jk} \right) + E \Delta^+ A - \frac{1}{2} \epsilon_{ijk} \sin y_{jk} \bar{\Delta}_i^- B$$

There is here a coupling of the fermions to the gauge field through the $\cos \phi_{ij}$ term in (24). In the continuum this term has the form of $F_{\mu\nu}^2 \chi \partial \chi$ which is expected to be irrelevant and to give a free continuum limit. The form of the H^2 term which is $\sin^2 \phi_{ij}$ causes serious trouble however. Since it has a double minimum at $\phi_{ij} = 0, \pi$, trying to take the continuum limit by sending g to zero does not force ϕ_{ij} to be very close to 0 and so to give a continuous distribution of gauge field $\phi_i(x)$. Rather many plaquettes may take the value of

$\varphi_{ij} = \pi$. We cannot integrate this out as a redundant degree of freedom since the $\cos \varphi_{ij} \chi_{ij} \Delta \chi_j$ term is sensitive to whether φ_{ij} is 0 or π . Here in the Abelian case this can be cured in an ad hoc way. One can add to Q a coupling of some components of χ to $1 - \cos \varphi_{ij}$. For example, add to (17) terms like

$$\sum_{x,ij} (1 - \cos \varphi_{ij}(x)) (\chi(x) + \chi(x+e_i) + \chi(x+e_i+e_j) + \chi(x+e_j)) \quad (25)$$

This will make the magnetic term in the Hamiltonian to be

$$\frac{1}{g^2} \sum_x \sum_{ij} [\sin^2 \varphi_{ij}(x) + [4 - \cos \varphi_{ij}(x) - \cos \varphi_{ij}(x-e_i) - \cos \varphi_{ij}(x-e_j) - \cos \varphi_{ij}(x-e_i-e_j)]] \quad (26)$$

and no multiple vacua persist. Of course, the change (25) of Q will modify other terms as well, terms like $1/g \pi_A (1 - \cos \varphi_{ij})$ or $\sin \varphi_{ij} \chi_{ij} \Delta \chi_i$ will be present. At very small coupling, however, the term (26) dominates, forcing φ_{ij} to be close to zero and $\varphi_i(x)$ to be smooth. These strange terms then become irrelevant and may not spoil the continuum limit.

One may also try to deal with a non-Abelian system. All the variables in (17) become gauge matrices. A term

$$\frac{i}{6} [A, B] \epsilon_{ijk} \chi_{ijk}$$

has to be added to Q and the derivatives in (17) have to be replaced by appropriate lattice covariant derivatives. The square of Q is indeed an expression with the right formal continuum limit. Unfortunately the magnetic action has here also non-trivial minima which spoil the smoothness of the field in the weak coupling limit. A trick like (25) is much harder to perform here since terms like constant times χ are not gauge invariant. This may be related to the fact that the interaction term between the fermion and gauge plaquettes is not obviously cubic invariant.

Another problem is that of the Lorentz invariance of the continuum limit as we have remarked, the Abelian system has full cubic symmetry in the diagonal combination of the real and internal spaces. This symmetry may enable the system to recover full Lorentz invariance in the continuum limit. This, in turn, will guarantee the emergence of the full supersymmetry algebra in that limit.

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REFERENCES

- 1) K. Wilson, Phys. Rev. D10 (1974) 2445.
- 2) P. Dondi and H. Nicolai, Nuovo Cimento 41A (1977).
- 3) T. Banks and P. Windey, Tel Aviv University preprint TAUP 982-81 (1981).
- 4) V. Rittenberg and S. Yankielowicz, CERN preprint TH-3263 (1982).
- 5) L. Susskind, Phys. Rev. D16 (1977) 3031.
- 6) E. Kähler, Rendiconti di Matematica (3-4) 21 (1962) 425;
J.M. Rabin, Yale University preprint (YTP81-29) (1981);
T. Banks, Y. Dothan and D. Horn, Tel Aviv University preprint TAUP 1037-82 (1982);
(1982);
P. Becher and H. Joos, DESY preprint DESY 82-031 (1982).
- 7) K. Wilson, Erice Lectures (1975).
- 8) H.B. Nielsen and M. Ninomiya, Nucl. Phys. B193 (1981) 173.
- 9) E. Witten and D. Olive, Phys. Lett. 78B (1978) 97.
- 10) E. Witten, Nucl. Phys. B188 (1981) 513.
- 11) L. Alvarez-Gaumé, D.Z. Freedman and M. Grisaru, Harvard University preprint HUTHP 81/B111 (1981).
- 12) P. Fayet, Nucl. Phys. B113 (1976) 135;
L. Brink, J.H. Schwarz and J. Scherk, Nucl. Phys. B121 (1977) 77.

FIGURE CAPTIONS

- Figure 1 : Associating fermionic components with lattice simplexes, and the corresponding bosonic supersymmetric partners - one-dimension.
- Figure 2 : The same construction in three dimensions.

Fig. 1

Fig. 2