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SUPERSYMMETRY AND THE DIVISION ALGEBRAS +)

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A B S T R A C T

We show how spinors in space-times of dimension $D = t + s$, where t is the time dimension, are associated for $s - t = 1,2,4,8$ (and if $t = 0,1,2$) with the number systems (division algebras), $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. For $t = 1$ and $s - t = 1,2,4$ this association is "realized" by the sequence of Lorentz groups $S\ell(2,\mathbb{R})$, $S\ell(2;\mathbb{C})$, $S\ell(2;\mathbb{H})$ for $D = 3,4,6$ respectively. We discuss how octonions may be related to $D = 10$. For $D = 6$ we give details of $S\ell(2;\mathbb{H})$ spinors and construct supersymmetric models with them. These results explain various "empirical" observations in the literature relating quaternions and supersymmetry.

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1. - INTRODUCTION

Let $\mathbb{A}$ be a linear algebra with an identity and an inverse for every element except zero. Suppose also that there is a norm N satisfying

$$N(xy) = N(x)\,N(y)\,, \quad x, y \in \mathbb{A}\,, \quad N \in \mathbb{R}\,,$$
$$N \geq 0\,, \quad N(x) = 0 \Rightarrow x = 0\,. \tag{1.1}$$

We will refer to such a structure as a division algebra; there seems to be no standard simple name. A theorem of Hurwitz[1)] states that there are only four such algebras whose elements can be identified respectively with the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ (after Hamilton), and the octonions $\mathbb{O}$ (Cayley numbers). This article is an attempt to link the existence and properties of the division algebras with the existence and properties of supersymmetric field theories in various space-time dimensions. We have in mind, principally, field theories with rigid supersymmetry. The relevance of the division algebras to supergravity theories appears to be more complicated.

Complex numbers and quaternions have already appeared in connection with supersymmetry in various contexts. Notably, supersymmetric σ models in two, or three, space-time dimensions have a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2} g_{mn} (\partial_\mu A^m)(\partial^\mu A^n) + \text{supersymmetrization}, \tag{1.2}$$

with $g_{mn}(A)$ the metric of a Riemannian manifold whose co-ordinates are the $\{A^m\}$. If there is only one supersymmetry $(N = 1)$ then the σ model exists for any Riemannian manifold. If we require $N = 2$ supersymmetry then the manifold must be Kähler[2)], and if we further require $N = 4$ supersymmetry then it must be hyper-Kähler[3)]. The former implies a complex structure and the latter a quaternionic one. This association of complex numbers and quaternions with $N = 2$ and $N = 4$ supersymmetry respectively, in two or three space-time dimensions, is also found in the construction of supersymmetric models with spins $\leq 1/2$ by stochastic methods[4)].

For gauge theories the rôle of complex numbers and quaternions is less clear. However, Ivanov has recently shown[5)] how the prepotential, V, of $N = 1$ super-Yang-Mills (YM) theory in four dimensions can be interpreted as the imaginary part of a complex group co-ordinate ϕ; $V = \mathrm{Im}\,\phi$. He has also conjectured that the SU(2) triplet prepotential, V_{ij}, of $N = 2$ YM theory in four dimensions can be interpreted as the imaginary part of a quaternion group co-ordinate. Ivanov's construction is analogous to that of Ogievetsky and Sokatchev[6)] for $N = 1$ supergravity in four dimensions in which the supergravity prepotential h_μ is interpreted as the imaginary part of a complex space-time co-ordinate X_μ.

These "empirical" associations of $\mathbb{C}$ and $\mathbb{H}$ with supersymmetric theories for various N and space-time dimension D can be seen to be in accord with each other if we refer any given theory to its "maximal dimension". This is the maximal dimension in which it could be formulated, in principle, consistent with supersymmetry. For example, an $N = 2$ supersymmetric model can usually be written as a simple supersymmetric model in four dimensions (the converse is always true). The Kähler σ models, in particular, do have a natural formulation in four dimensions. Similarly the hyper-Kähler σ models can be expected to have a natural formulation in their maximal dimension, which is six. The $N = 2$ YM theory can also be written in a maximum of six dimensions. The above noted associations of complex numbers and quaternions with supersymmetry can now be seen to form the following pattern:

$$\begin{aligned} D = 3 &\quad \leftrightarrow \quad \mathbb{R} \\ D = 4 &\quad \leftrightarrow \quad \mathbb{C} \\ D = 6 &\quad \leftrightarrow \quad \mathbb{H} \quad . \end{aligned} \tag{1.3}$$

Notice that the "transverse dimension" $D - 2$ equals the dimension of the associated division algebra. This leads us to anticipate the further association

$$D = 10 \quad \leftrightarrow \quad \mathbb{O} \quad . \tag{1.4}$$

This is an exciting possibility because it could lead to a deeper understanding of the many remarkable properties of 10-dimensional supersymmetric theories, for instance that $D = 10$ is the critical dimension of the spinning string model, or that the β function of the $D = 10$ YM theory, reduced to $D = 4$, vanishes.

In this article we will show how this, and other patterns can be explained in terms of algebraic properties of spinors. For space-times $\eta(t,s)$, i.e., pseudo-Euclidean spaces of t time dimensions and s space dimensions, if the transverse dimension $s - t$ equals 1, 2, 4, 8 mod 8, there is a formal way of associating the space-time with the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ respectively. The association (1.3) and (1.4) is a special case of this rule. In the general case this formal association is probably without much significance, but in certain special cases it may be "realized" in a way that makes it relevant to the construction of field theories. For $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ this comes about (if $t = 0$, 1 or 2) because of isomorphisms of the "Lorentz" groups $SO(s,t)$, or rather of their universal covering groups $\overline{SO}(s,t)$, to classical groups over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively. The most striking sequence occurs for $t = 1$ (Minkowski space) and is

$$\overline{SO}(2,1) \cong S\ell(2;\mathbb{R})$$
$$\overline{SO}(3,1) \cong S\ell(2;\mathbb{C}) \qquad (1.5)$$
$$\overline{SO}(5,1) \cong S\ell(2;\mathbb{H}) \quad .$$

The association of (1.3) now becomes evident. The isomorphism $\overline{SO}(3,1) \cong S\ell(2;\mathbb{C})$ is well known and is the basis for the two-component spinor notation of $D = 4$ The isomorphism $\overline{SO}(5,1) \cong S\ell(2;\mathbb{H})$ means that there is also a two-component spinor notation in $D = 6$, which is developed in this article. Since octonions are non-associative we cannot expect them to play quite the same rôle for $s - t = 8$ as the other division algebras do for $s - t = 1, 2, 4$. At the moment we have only a few suggestive observations to make about this case.

The works cited above are not the only ones to have previously mentioned a connection between one or more of the division algebras and supersymmetry. An article of Lukierski and Nowicki[7] contains some observations similar to ours. In particular, they arrive at an association of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ with $D = 3, 4, 5$ respectively $(t = 1)$. As mentioned above we believe that $D = 6$ is most naturally associated with $\mathbb{H}$. These authors also introduce the group $S\ell(2;\mathbb{H})$ but as the "Euclidean conformal group", a fact that has also been remarked on by Julia[8]. The isomorphisms of Clifford algebras to matrix algebras over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ have been discussed in an article by Coquereaux[9]. Clifford algebras play a rôle in our work, but as yet we do not see any connection between our results and those of Ref. 9). Galperin, Ivanov and Ogievetsky[10] have considered the division algebras in connection with invariant subspaces of superspace, but again we do not yet see any connection between their work and ours.

The organization of this article is as follows. We first classify the kinds of spinors that can be defined in arbitrary space-times $\eta(t,s)$. Scherk[11] has discussed the conditions under which Majorana spinors can be defined. This has been extended to what we call pseudo-Majorana spinors by a number of other authors[12]. There is another kind of spinor (or rather two kinds) known in the context of five-dimensional supersymmetric theories as a "symplectic spinor"[13]. We give what we believe to be the first complete treatment of this question (at least in the Physics literature). Using these results we develop, in Section 3, the association of $s - t = 1, 2, 4, 8$ with $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, respectively, and show how this association is realized in specific cases. In Section 4 we turn to the special case of $S\ell(2;\mathbb{H})$ spinors in six-dimensional Minkowski space and the corresponding two-component spinor formalism. We then use these

results to construct free supersymmetric models in six dimensions. In Section 6, we discuss octonions and their association with ten-dimensional Minkowski space, as well as the question of how the division algebras are related to supergravity.

2. - SPINORS

The Dirac equation is

$$(i\not{\partial}+m)\psi=0 \quad , \qquad \not{\partial}=\Gamma^{\mu}\partial_{\mu} \tag{2.1}$$

where the Γ^{μ} satisfy the Clifford algebra

$$\begin{aligned} \{\Gamma^{\mu},\Gamma^{\nu}\} &= 2\eta^{\mu\nu} , \\ \eta^{\mu\nu} &= \mathrm{diag}(\underbrace{++\cdots+}_{t},\underbrace{--\cdots-}_{s}) \end{aligned} \tag{2.2}$$

The time dimension is t, the space dimension s, and the space-time dimension $D = t + s$. For dimensions, D, $D + 1$, D even, the Γ^{μ} are represented by $2^{D/2} \times 2^{D/2}$ complex matrices which may be chosen such that

$$\begin{aligned} \Gamma^{\dagger}_{(t)} &= \Gamma_{(t)} \quad , \quad (t)=\{1,2,\cdots,t\} \\ \Gamma^{\dagger}_{(s)} &= \Gamma_{(s)} \quad , \quad (s)=\{t+1,t+2,\cdots,D\} . \end{aligned} \tag{2.3}$$

It follows that

$$\Gamma^{\mu\dagger} = -(-1)^{t} A\,\Gamma^{\mu}A^{-1} , \qquad A=\Gamma_{1}\Gamma_{2}\cdots\Gamma_{t} \tag{2.4}$$

Since $\pm\Gamma^{\mu *}$ form an equivalent representation of the Clifford algebra there exists an invertible matrix B for which

$$\Gamma^{\mu} = \eta\, B^{-1}\Gamma^{\mu *}B \quad , \qquad \eta=\pm 1 , \tag{2.5}$$

for either choice of η. It can be shown that B is unitary and satisfies

$$B^{*}B = \epsilon(\eta,s,t) \quad , \qquad \epsilon=\pm 1 . \tag{2.6}$$

The sign ϵ is a function of η, s and t that can be calculated, e.g., by Scherk's method[11]. One introduces the charge conjugation matrix C by

$$B^T = C A^{-1} \quad . \tag{2.7}$$

The properties of A and B imply for C that

$$\begin{aligned} \Gamma^{\mu T} &= (-1)^{t+1} \eta \, C \Gamma^\mu C^{-1} \quad , \\ C^\dagger C &= 1 \quad , \\ C^T &= \epsilon \, \eta^t \, (-1)^{\frac{t(t-1)}{2}} \, C \quad . \end{aligned} \tag{2.8}$$

Consider first D even. The complete set of $2^{D/2} \times 2^{D/2}$ matrices is spanned by the $\{\Gamma^{(n)}\}$, $n = 0, 1, \ldots, D$, with

$$\Gamma^{(n)} = \Gamma^{\mu_1 \mu_2 \cdots \mu_n} = \Gamma^{[\mu_1} \Gamma^{\mu_2} \cdots \Gamma^{\mu_n]} \tag{2.9}$$

(the square bracket indicates antisymmetrization with "strength one"). From (2.8) one sees that the $C\Gamma^{(n)}$ are either symmetric or antisymmetric, and one counts the number of, say, antisymmetric matrices. This number depends on η, t and D. On the other hand, we know that this number is $(1/2)2^{D/2}(2^{D/2} - 1)$. Equating the two results yields

$$\epsilon = -\sqrt{2}\, \eta^t \, (-1)^{\frac{t(t-1)}{2}} \cos\left[\frac{\pi}{4}\left(\eta (-1)^{t+1} D + 3\right)\right] \quad . \tag{2.10}$$

With some algebraic manipulation (or by enumerating all cases) this can be reduced to

$$\epsilon = \cos\frac{\pi}{4}(s-t) - \eta \sin\frac{\pi}{4}(s-t) \quad . \tag{2.11}$$

Since D is even we can introduce another matrix, Γ^{D+1}, that anticommutes with all Γ^μ;

$$\Gamma^{D+1} = \Gamma_1 \Gamma_2 \cdots \Gamma_D \quad . \tag{2.12}$$

This has the properties

$$
\begin{aligned}
(\Gamma^{D+1})^2 &= (-1)^{\frac{s-t}{2}} \,, \\
\Gamma^{D+1} &= B^{-1}\,\Gamma^{D+1\,*}\,B \,.
\end{aligned}
\tag{2.13}
$$

To extend the Γ matrices to dimension $D + 1 = (s + 1) + t$ we require that the additional Γ matrix be Γ^{D+1} or $i\Gamma^{D+1}$ according to whether $(s - t)/2$ is odd or even respectively. [We need not consider $D + 1 = s + (t + 1)$ because this is taken care of by $t \to t + 1$.] So we will have $D + 1$ Γ matrices $\{\Gamma^\mu,\Gamma^{D+1}\}$ for $s - t = 2 \bmod 4$ and $\{\Gamma^\mu, i\Gamma^{D+1}\}$ for $s - t = 0 \bmod 4$. Then requiring that Γ^μ and Γ^{D+1} or $i\Gamma^{D+1}$, according to the case, are transformed identically under B conjugation, we discover that η is fixed by

$$
\eta = -(-1)^{\frac{s-t}{2}} \qquad \text{for } D \to D+1 \,,
\tag{2.14}
$$

which in turn fixes ε. In even dimensions we always have a choice of $\eta = \pm 1$ and ε depends on this choice according to (2.11). Putting these results together we find the space-times $\eta(t,s)$ allowing the four possible combinations of (ε,η):

$$
\begin{aligned}
\epsilon = +1 \,,\ \eta = -1 \quad &: \quad s-t = 1,2,8 \,, \bmod 8 \,, \\
\epsilon = +1 \,,\ \eta = +1 \quad &: \quad s-t = 6,7,8 \,, \bmod 8 \,, \\
\epsilon = -1 \,,\ \eta = -1 \quad &: \quad s-t = 4,5,6 \,, \bmod 8 \,, \\
\epsilon = -1 \,,\ \eta = +1 \quad &: \quad s-t = 2,3,4 \,, \bmod 8 \,.
\end{aligned}
\tag{2.15}
$$

For even dimension D we can construct the chiral projection operators

$$
P_\pm = \frac{1}{2}\left(1 \pm (-1)^{\frac{s-t}{4}}\,\Gamma^{D+1}\right) .
\tag{2.16}
$$

The projections $P_\pm\Gamma^\mu$ of Γ^μ will separately transform under B conjugation as in (2.5) only if

$$
(-1)^{\frac{s-t}{2}} = 1 \qquad \Longrightarrow \qquad s-t = 0 \ \bmod 4 \,.
\tag{2.17}
$$

This is the last of the properties of Γ matrices that we will need to classify the possible reality/chirality constraints on the spinor ψ.

By taking the complex conjugate of the Dirac equation (2.1) we find

$$(i \not\partial + (-\eta) m) B^{-1} \psi^* = 0 \quad . \tag{2.18}$$

If $(-\eta)m = m$ then $B^{-1}\psi^*$ satisfies the same equation as ψ. If in addition $\varepsilon = +1$ then we can equate $B^{-1}\psi^*$ with ψ

$$\psi^* = B\psi \ , \qquad (\epsilon = +1) \quad . \tag{2.19}$$

This is possible only if $\varepsilon = +1$ because (2.19) implies $B^*B = 1$. If $\varepsilon = -1$ and we have two spinors ψ_i, $i = 1, 2$, then we can impose instead an "SU(2) reality" condition

$$\psi^{*i} = (\psi_i)^* = \epsilon^{ij} B \psi_j \ , \qquad (\epsilon = -1) \ , \tag{2.20}$$

where ε^{ij} is the SU(2) invariant alternating tensor. In both cases the condition $(-\eta)m = m$ means that if $\eta = +1$ we must have $m = 0$.

Spinors satisfying (2.19) will be called Majorana if $\eta = -1$ and pseudo-Majorana if $\eta = +1$. Spinors satisfying (2.20) will be called SU(2) Majorana if $\eta = -1$ and SU(2) pseudo-Majorana if $\eta = +1$. [SU(2)] pseudo-Majorana spinors do not allow a mass. This generalizes the result of Wetterich[12]. The space-times that allow these four kinds of spinor can now be read off from (2.15). For even dimension D one can always impose a chirality condition on ψ using the projection operators of (2.16)

$$\begin{aligned} \mathcal{P}_\pm \psi_\pm &= \pm \psi_\pm \ , \\ \mathcal{P}_\pm \psi_\mp &= 0 \quad . \end{aligned} \tag{2.21}$$

This will be compatible with an [SU(2)] reality condition only if Eq. (2.17) is satisfied, i.e., for $s - t = 0 \bmod 4$. A chiral spinor ψ_+ that also satisfies an [SU(2)] reality condition will be called an [SU(2)] (pseudo)-Majorana-Weyl spinor. The results for all these types of spinor are displayed in the Table.

There are two caveats that we must mention concerning the above analysis. The first is that an SU(2) pseudo-Majorana spinor does not allow a mass consistent with SU(2) invariance, but if SU(2) is broken to O(2) then a mass is allowed. In fact, the essential difference between a Dirac spinor and an SU(2) (pseudo)-Majorana spinor or a Weyl spinor and an SU(2) (pseudo)-Majorana-Weyl spinor is that the former has only O(2) symmetry while the latter

has SU(2) symmetry. The second caveat is that we have considered only the Dirac equation, but not the action. Whether the Dirac equation can be obtained from an action is another matter, and indeed in some cases it cannot ! Consider, for example, $t = 0$, $D = 8$ for which $\varepsilon = +1$, and choose $\eta = -1$. The reality condition (2.19) is consistent with the massive Dirac equation but as C is symmetric the putative mass term $m\psi^T C\psi$ vanishes. This is a fermionic analogue of certain boson field equations for which no Lorentz invariant action exists[14]. For $t = 1$, (Minkowski space), the Dirac equation is always derivable from an action.

In certain cases, viz. $s - t = 2 \bmod 4$, one has a choice of $\varepsilon = \pm 1$. The spinor for $\varepsilon = -1$ will have twice as many components as that for $\varepsilon = +1$. An alternative, and complementary, statement of the difference is that for an equal number of components the latter will allow a larger internal symmetry group, just as an SU(2) Majorana spinor allows a larger symmetry group than a Dirac spinor. This suggests a classification of spinors according to their maximal symmetry. Given several spinors ψ_i the reality conditions (2.19) and (2.20) can be generalized to

$$(\psi_i)^* = \psi^{*i} = M^{ij} B \psi_j \quad , \qquad (\epsilon = +1) , \tag{2.22}$$

and

$$(\psi_i)^* = \psi^{*i} = \Omega^{ij} B \psi_j \quad , \qquad (\epsilon = -1) , \tag{2.23}$$

respectively. M is a symmetric real metric of $O(n)$ for $i = 1, \ldots, n$ and Ω is an antisymmetric real metric of $USp(2n)$ for $i = 1, \ldots, 2n$. If $s - t = 2 \bmod 4$, so that we must choose between a reality condition and a chirality condition, the maximal symmetry principle favours the latter because ψ_+ can be chosen to transform under $U(n)$ rather than $O(n)$.

Selecting a (massless) spinor according to the maximal symmetry principle yields the following associations:

$$\begin{aligned} s-t &= 1 \quad \longleftrightarrow \quad O(n) \cong U(n;\mathbb{R}) \quad , \\ s-t &= 2 \quad \longleftrightarrow \quad U(n) \cong U(n;\mathbb{C}) \quad , \\ s-t &= 4 \quad \longleftrightarrow \quad USp(2n) \cong U(n;\mathbb{H}) \; . \end{aligned} \tag{2.24}$$

This is the first indication that the transverse dimension of space-time is linked with the dimension of an associated division algebra. The association of (2.24) is through the maximal internal symmetry of a spinor in $D = s + t$ dimensions. We will next see how this association is also present for the space-time symmetry.

3. - $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$

In D dimensions the number of (real) components of a minimal spinor is $2^{D/2+1} \times r$ where $r = 1, 1/2$ or $1/4$ according to the case, and can be read off from the Table. Let us concentrate on the cases $s - t = 1, 2, 4, 8$ mod 8, i.e.,

$$s-t = \dim \mathbb{A} + 8(n-1) , \qquad \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \quad n \geq 1 . \tag{3.1}$$

The integer n serves to add multiples of 8. For these cases the number of components of the minimal spinor is

$$2^t \, 2^{4(n-1)} \dim \mathbb{A} . \tag{3.2}$$

So, formally, a spinor in $D = s + t$ dimensions with $s - t = \dim \mathbb{A}$ mod 8 can be represented by a $2^{t+4(n-1)}$ component spinor over $\mathbb{A}$. Notice that as we reduce $s - t$ from 8 mod 8 the number of components of the minimal spinor does not change until we reach $s - t = 4$ mod 8. Reducing $s - t$ further the number of components of the minimal spinor again does not change until we reach $s - t = 2$ mod 8.

For this formal representation to have any meaning it must be possible to represent the "Lorentz" group on such a spinor. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ this means that the "Lorentz" group must be isomorphic to a $2^{t+4(n-1)} \times 2^{t+4(n-1)}$ matrix group over $\mathbb{A}$. In the Table all the isomorphisms of the orthogonal groups are given and we see that this condition is satisfied only if $n = 1$, and then only for $t = 0, 1$ or 2. For $t = 0$ we have the sequence of isomorphisms

$$\begin{aligned} SO(1) &\cong S\ell(1;\mathbb{R}) , \\ SO(2) &\cong S\ell_2(1;\mathbb{C}) , \\ \overline{SO}(4) &\cong S\ell(1;\mathbb{H}) \otimes S\ell(1;\mathbb{H}) . \end{aligned} \tag{3.3}$$

The first member of this sequence is, of course, trivial. The chiral spinor of $D = 4$ transforms under only one of the $S\ell(1;\mathbb{H})$ factors of $\overline{SO}(4)$, so the "spinor Lorentz group" is really $S\ell(1;\mathbb{H})$. The subscript 2 on $S\ell_2(1;\mathbb{C})$ indicates that this is the subgroup of $G\ell(1;\mathbb{C})$ whose complex determinant has modulus unity. We remind the reader that $S\ell_1(n;\mathbb{C})$ is the subgroup of $G\ell(n;\mathbb{C})$ whose determinant is real while $S\ell(n;\mathbb{C})$ has simply a unit determinant[15]. Over $\mathbb{H}$ there is only one possible restriction of $G\ell$ to $S\ell$; $S\ell(n;\mathbb{H})$ is the subgroup of $G\ell(n;\mathbb{H})$ with unit determinant where the determinant is defined by

$$\det \mathbb{M} \stackrel{def.}{=} \exp \operatorname{Tr} \ln \mathbb{M} \; ,$$
$$\operatorname{Tr} \mathbb{M} \stackrel{def.}{=} \operatorname{Re} (\operatorname{tr} \mathbb{M}) \; , \tag{3.4}$$

with tr the ordinary matrix trace. For a single quaternion q,

$$q = q_0 + \mathbf{i} q_1 + \mathbf{j} q_2 + \mathbb{k} q_3 \tag{3.5}$$

the definitions (3.4) reduce to

$$\det q = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2} \; ,$$
$$\operatorname{Tr} q = q_0 \; . \tag{3.6}$$

For $t = 1$ we have sequences mentioned in the Introduction:

$$\overline{SO}(2,1) \cong Sl(2;\mathbb{R}) \; ,$$
$$\overline{SO}(3,1) \cong Sl(2;\mathbb{C}) \; , \tag{3.7}$$
$$\overline{SO}(5,1) \cong Sl(2;\mathbb{H}) \; ,$$

while for $t = 2$ we have the sequence:

$$\overline{SO}(3,2) \cong Sp(4;\mathbb{R}) \; ,$$
$$\overline{SO}(4,2) \cong SU(2,2;\mathbb{C}) \; , \tag{3.8}$$
$$\overline{SO}(6,2) \cong SO(4;\mathbb{H}) \; .$$

By $Sp(4;\mathbb{R})$ we mean the group of 4×4 real matrices preserving an antisymmetric 4×4 metric [this is sometimes referred to as $Sp(2;\mathbb{R})$ in mathematical literature]. The case $t = 2$ is interesting because the sequence of groups (3.8) constitute the conformal groups of the previous $t = 1$ sequence. Moreover there is a direct extension to a sequence of supergroups

$$OSp(n/4;\mathbb{R}) \supset SO(n) \otimes Sp(4;\mathbb{R}) \; , \tag{3.9}$$
$$SU(2,2/n;\mathbb{C}) \supset SU(2,2;\mathbb{C}) \otimes \begin{cases} U(n) & n \neq 4 \\ SU(n) & n = 4 \end{cases} \; ,$$

$$OSp(4/2n;\mathbb{H}) \supset SO(4;\mathbb{H}) \otimes USp(2n). \tag{3.9 contd.}$$

The supergroup $OSp(1/4;\mathbb{R})$ is the superconformal group for $D = 3$ considered by van Holten and Van Proeyen[16], while $SU(2,2/1;\mathbb{C})$ is the gauge group of conformal supergravity[17]. The last member of the sequence has been constructed as a matrix supergroup over $\mathbb{H}$ in Ref. 7). We defer a discussion of $s - t = 8$ and the octonions to the final section; since octonions are non-associative there is no such thing as a matrix group over $\mathbb{O}$.

We will now show how the isomorphisms of the $\overline{SO}(s,t)$ groups arise as a consequence of the properties of Γ matrices derived in the previous section. To this end we list the isomorphisms as follows:

$$\begin{array}{lll}
s-t=0: & SO(1,1) \cong Gl(1;\mathbb{R}) \\
 & \overline{SO}(2,2) \cong Sl(2;\mathbb{R}) \otimes Sl(2;\mathbb{R}) \\
 & \overline{SO}(3,3) \cong Sl(4;\mathbb{R}) \\
 & \\
s-t=1: & SO(1) \cong Sl(1;\mathbb{R}) \\
 & \overline{SO}(2,1) \cong Sl(2;\mathbb{R}) \\
 & \overline{SO}(3,2) \cong Sp(4;\mathbb{R}) \\
 & \\
s-t=2: & SO(2) \cong Sl_2(1;\mathbb{C}) \\
 & \overline{SO}(3,1) \cong Sl(2;\mathbb{C}) \\
 & \overline{SO}(4,2) \cong SU(2,2;\mathbb{C}) \\
 & \\
s-t=3: & \overline{SO}(3) \cong Sl(1;\mathbb{H}) \\
 & \overline{SO}(4,1) \cong U(1,1;\mathbb{H}) \\
 & \\
s-t=4: & \overline{SO}(4) \cong Sl(1;\mathbb{H}) \otimes Sl(1;\mathbb{H}) \\
 & \overline{SO}(5,1) \cong Sl(2;\mathbb{H}) \\
 & \overline{SO}(6,2) \cong SO(4;\mathbb{H})
\end{array}$$

Notice first that those isomorphisms for $s - t = 1, 2, 4$ are those occurring in (3.3), (3.7) and (3.8) that we would most like to understand. These occurring for $s - t = 3$ are straghtforwardly explained in terms of those for $s - t = 4$. We will see how this works for $\overline{SO}(4,1) \cong U(1,1;\mathbb{H})$ in the next section. The isomorphisms occurring for $s - t = 0$ form an interesting pattern but one that is different from $s - t = 1,2, 4$. For instance, the dimension of the group is no longer $2^t \dim \mathbb{A}$ for $s - t = 0$ but $2^{t-1} \dim \mathbb{R}$. Notice also that $SO(1,1)$ is the only orthogonal group that is isomorphic to another matrix group for which the determinant is not unity, despite the fact that the determinant of an element of $SO(1,1)$ is unity, by definition ! This can be explained as follows. $SO(1,1)$ acts on both left- and right-handed spinors in an equal but opposite way whereas $G\ell(1;\mathbb{R})$ acts only on the left- or right-handed spinor hence its determinant need not be unity. An isomorphism of an orthogonal group with a $G\ell(n;\mathbb{A})$ group is only possible for $n = 1$ and $\mathbb{A} = \mathbb{R}$ because only in this case is the latter not a product of different factor groups.

To explain the isomorphisms for $s - t = 0$ we observe that in this case $\varepsilon = +1$ so that the B matrix may be chosen to be 1, in which case the Lorentz generators are real. We may also divide them into left and right handed Lorentz generators $\Sigma_+^{\mu\nu}$, $\Sigma_-^{\mu\nu}$. The $\Sigma_+^{\mu\nu}$ are elements of $S\ell(2^{D/2-1};\mathbb{R})$ so the group generated by them, the "spinor Lorentz group", must be a subgroup of $S\ell(2^{D/2-1};\mathbb{R})$. The exception to this, as we have seen, is if $D = 2$ in which case the spinor Lorentz group must be a subgroup of $G\ell(1;\mathbb{R})$ and obviously the subgroup must be $G\ell(1;\mathbb{R})$ itself. We also know that the spinor Lorentz group is isomorphic to $\overline{SO}(s,t)$ except possibly in those cases for which $\overline{SO}(s,t)$ is a direct product. In this case $\Sigma_+^{\mu\nu}$ may generate one factor and $\Sigma_-^{\mu\nu}$ another factor of $\overline{SO}(s,t)$. Obviously there can be only two factors, and they must be identical. Moreover, if $\dim S\ell(2^{D/2-1};\mathbb{R}) < \dim \overline{SO}(s,t)$ for $D > 2$ then this factorization must occur. This is seen to happen for $D = 4$ in which case the factor group must be $S\ell(2;\mathbb{R})$. If $\dim S\ell(2^{D/2-1};\mathbb{R}) = \dim \overline{SO}(s,t)$ then these groups will be isomorphic. This happens for $D = 6$ and therefore $\overline{SO}(3,3) = S\ell(4;\mathbb{R})$.

To explain the isomorphisms for $s - t = 1$ we observe that $\varepsilon = 1$ and B may be chosen to be 1. The $\Sigma^{\mu\nu}$ are then real, and are therefore elements of $S\ell(2^{(D-1)/2};\mathbb{R})$ except in the case $D = 1$ for which the $\Sigma^{\mu\nu}$ do not exist. If $\dim S\ell(2^{(D-1)/2};\mathbb{R}) = \dim \overline{SO}(s,t)$ then these two groups must be isomorphic. This occurs for $D = 3$, hence $\overline{SO}(2,1) = S\ell(2,\mathbb{R})$. For $D > 3$ it is still possible that $\overline{SO}(s,t)$ is isomorphic to a metric preserving subgroup of $S\ell(2^{(D-1)/2};\mathbb{R})$. In fact for $D = 5$ this must be the case because the isomorphism $\overline{SO}(3,3) \cong S\ell(4;\mathbb{R})$ can be "restricted" to $\overline{SO}(3,2) \cong Sp(4;\mathbb{R})$ just as the $\overline{SO}(4,1) \cong U(1,1;\mathbb{H})$ isomorphism is a restriction of $\overline{SO}(5,1) \cong S\ell(2;\mathbb{H})$.

For $s - t = 2$ we again have $\varepsilon = +1$ so that B can be chosen to be 1, but the left- (right-) handed Lorentz generators $\Sigma_+^{\mu\nu}(\Sigma_-^{\mu\nu})$ are not real but are related by

$$(\Sigma_+^{\mu\nu})^* = \Sigma_-^{\mu\nu} \quad . \tag{3.10}$$

The $\Sigma_+^{\mu\nu}$ are therefore elements of $S\ell_2(2^{D/2-1};\mathbb{C})$. This choice of $S\ell$ group is what corresponds to the unit determinant of the Lorentz transformation. If $\dim S\ell_2(2^{D/2-1};\mathbb{C}) = \dim \overline{SO}(s,t)$ then these groups will be isomorphic. This occurs for $D = 2$. For $D > 2$ we can still have an isomorphism to the further restricted group $S\ell(2^{D/2-1};\mathbb{C})$, and if the dimensions match this isomorphism will occur. Indeed the dimensions do match for $D = 4$ so $\overline{SO}(3,1) = S\ell(2;\mathbb{C})$. We may still have a further isomorphism for $D > 4$ to a metric preserving subgroup of $S\ell(2^{D/2-1};\mathbb{C})$. The only candidate, with the right dimension and signature, is $SU(2,2;\mathbb{C})$ for $D = 6$). But for 4×4 complex matrices $(\Sigma_+^{\mu\nu})_{ij}$ we may relate $\Sigma_+^{\mu\nu}$ to $(\Sigma_+^{\mu\nu})^*$ by means of $\varepsilon_{ijk\ell}$. The preservation of this reality condition is equivalent to the restriction of $S\ell(4;\mathbb{C})$ to $SU(2,2;\mathbb{C})$ hence we must have $\overline{SO}(4,2) \cong SU(2,2;\mathbb{C})$.

Finally for $s - t = 4$ we have $\varepsilon = -1$ so B cannot be chosen to be 1. However, the projections $P_\pm \Gamma^\mu$ transform separately under B conjugation in the same way as Γ^μ so there is a reduced $2^{D/2-1} \times 2^{D/2-1}$ B matrix transforming the left-handed Lorentz generators as

$$\Sigma_+^{\mu\nu} = B^{-1}\Sigma_+^{\mu\nu *} B \quad , \qquad B^\dagger B = 1 \, , \quad B^* B = -1 \, . \tag{3.11}$$

The $\Sigma_+^{\mu\nu}$ will be elements of $S\ell(2^{D/2-1};\mathbb{C})$, but those elements of this group satisfying (3.11) are, by definition, elements of $SU^*(2^{D/2-1})$. Hence the group generated by the $\Sigma_+^{\mu\nu}$ must be a subgroup of $SU^*(2^{D/2-1})$. If $\dim SU^*(2^{D/2-1}) < \dim \overline{SO}(s,t)$ then $\overline{SO}(s,t)$ must be a product of two equal factors. This happens for $D = 4$ and the factor group must be $SU^*(2) \cong SU(2)$. If $\dim SU^*(2^{D/2-1}) = \dim \overline{SO}(s,t)$ then these groups will be isomorphic. This happens for $D = 6$, hence

$$\overline{SO}(5,1) \cong SU^*(4) \, . \tag{3.12}$$

This explains why the four-component spinor Lorentz group in six-dimensional Minkowski space is $SU^*(4)$ [18]. If the $\Sigma_+^{\mu\nu}$ could be chosen to be real then they would generate a subgroup of $SO^*(2^{D/2-1})$. This can happen only when the dimension of $SO^*(2^{D/2-1})$ is sufficiently large, i.e., $D \geq 8$ and the dimensions match for $D = 8$. Hence

$$\overline{SO}(6,2) \cong SO^*(8) . \tag{3.13}$$

We see that associated with $s - t = 4$ and Majorana-Weyl spinors are the star groups, which are isomorphic to quaternion groups[15]

$$\begin{aligned} SU^*(2n) &\cong S\ell(n;\mathbb{H}) , \\ SO^*(2n) &\cong SO(n;\mathbb{H}) . \end{aligned} \tag{3.14}$$

The particular case of $\overline{SO}(5,1) \cong S\ell(2;\mathbb{H})$ is of the most immediate interest so we now turn to a detailed discussion of it.

4. - <u>$S\ell(2;\mathbb{H})$</u>

An $S\ell(2;\mathbb{H})$ spinor is a two-component quaternionic object;

$$\psi = \begin{pmatrix} \mu \\ \nu \end{pmatrix} \qquad \mu, \nu \in \mathbb{H} . \tag{4.1}$$

Its quaternion Hermitian conjugate is

$$\psi^\dagger = (\mu^* , \nu^*) \tag{4.2}$$

where μ^* is the quaternion conjugate of μ. We denote the two components of ψ by ψ_α, $\alpha = 1, 2$ and the two components of ψ^+ by $\overline{\psi}_{\dot\alpha}$ $(=(\psi_\alpha)^*)$, $\dot\alpha = 1,2$. The transformation properties of ψ and ψ^+ under $S\ell(2;\mathbb{H})$ are as follows:

$$\begin{aligned} &\psi'_\alpha = a_\alpha{}^\beta \psi_\beta , \qquad \overline{\psi}'_{\dot\alpha} = \overline{\psi}_{\dot\beta} (a^\dagger)^{\dot\beta}{}_{\dot\alpha} , \\ &a \in S\ell(2;\mathbb{H}) , \qquad \det a = 1 . \end{aligned} \tag{4.3}$$

The determinant of a is defined as in (3.4). We refer to ψ as a "chiral" spinor for a reason to become clear later. There is also an antichiral spinor χ

$$\chi = \begin{pmatrix} \rho \\ \lambda \end{pmatrix} \qquad \rho, \lambda \in \mathbb{H} \tag{4.4}$$

and its Hermitian conjugate

$$\chi^\dagger = (\rho^* , \lambda^*) . \tag{4.5}$$

We denote the two components of χ by $\chi^{\dot\alpha}$ and the two components of χ^{+} by $\bar\chi^{\alpha}$ $(=(\chi^{\dot\alpha})^{*})$. χ transforms under $S\ell(2;\mathbb{H})$ as

$$\chi'^{\dot\alpha} = (a^{\dagger-1})^{\dot\alpha}{}_{\dot\beta}\,\chi^{\dot\beta} \quad , \qquad \bar\chi'^{\alpha} = \bar\chi^{\beta}\,(a^{-1})_{\beta}{}^{\alpha} \ . \tag{4.6}$$

The placement of the spinor index, down for ψ, up for χ, serves to distinguish a chiral from an antichiral spinor. Unlike $S\ell(2,\mathbb{C})$ spinors, a chiral spinor is not transformed by Hermitian conjugation into an antichiral one, and there is no $S\ell(2;\mathbb{H})$ invariant operation that can effect such a transformation.

In addition to Lorentz transformations ψ and χ also transform under an additional $S\ell(1;\mathbb{H})$ group which acts by multiplication by a unit-determinant quaternion from the opposite side to the $S\ell(2;\mathbb{H})$ transformation. For example, for ψ,

$$\psi' = \psi\,u \quad , \qquad \psi^{\dagger\prime} = u^{*}\,\psi^{\dagger} \quad , \qquad u \in S\ell(1;\mathbb{H}) \ . \tag{4.7}$$

The quantities

$$\chi^{\dagger}\psi = \bar\chi^{\alpha}\psi_{\alpha} \quad , \qquad \psi^{\dagger}\chi = \bar\psi_{\dot\alpha}\chi^{\dot\alpha} \tag{4.8}$$

are Lorentz scalars but transform under $S\ell(1;\mathbb{H})\otimes S\ell(1;\mathbb{H})$ as a $\underset{\sim}{4}$;

$$(\chi^{\dagger}\psi)' = u\,\chi^{\dagger}\psi\,v \quad ; \qquad u,v \in S\ell(1;\mathbb{H}) \ . \tag{4.9}$$

An $S\ell(1;\mathbb{H})$ scalar can be obtained by taking the quaternion trace defined in (3.4).

The following quantities are $S\ell(1;\mathbb{H})$ scalars but Lorentz vectors:

$$\psi\psi^{\dagger} = \psi_{\alpha}\bar\psi_{\dot\beta} \sim \mathbb{V}_{\alpha\dot\beta} = \begin{pmatrix} V_0 + V_5 & \underset{\sim}{\mathbb{V}}^{*} \\ \underset{\sim}{\mathbb{V}} & V_0 - V_5 \end{pmatrix}$$

$$\chi\chi^{\dagger} = \chi^{\dot\alpha}\bar\chi^{\beta} \sim \tilde{\mathbb{V}}^{\dot\alpha\beta} = \begin{pmatrix} V_0 - V_5 & -\underset{\sim}{\mathbb{V}}^{*} \\ -\underset{\sim}{\mathbb{V}} & V_0 + V_5 \end{pmatrix} \tag{4.10}$$

$V_\pm = V_0 \pm V_5$ are the timelike and longitudinal components of V_μ while the transverse components are contained in the quaternion $\underset{\sim}{\mathbb{V}}$,

$$\underset{\sim}{\mathbb{V}} = V_1 + \mathbf{i} V_2 + \mathbf{j} V_3 + \mathbf{k} V_4 , \tag{4.11}$$

or its quaternion conjugate

$$\underset{\sim}{\mathbb{V}}^* = V_1 - \mathbf{i} V_2 - \mathbf{j} V_3 - \mathbf{k} V_4 . \tag{4.12}$$

$\mathbb{V}$ and $\tilde{\mathbb{V}}$ are apparently inequivalent representations of $S\ell(2;\mathbb{H})$ but in fact are related by

$$\tilde{\mathbb{V}} = \mathbb{V}^{-1} \det \mathbb{V} . \tag{4.13}$$

This can be verified directly or it may be deduced as follows. By inspection we observe that

$$\tilde{\mathbb{V}} = \epsilon \mathbb{V}^T \epsilon^T , \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{4.14}$$

But this is not a manifestly covariant relation because ε is not an invariant tensor of $S\ell(2;\mathbb{H})$ [although it is of $S\ell(2,\mathbb{C})$]. Nor is the transpose of a quaternionic matrix an invariant operation because $(\mathbb{V}\mathbb{a})^T \neq \mathbb{a}^T\mathbb{V}^T$ as a result of the non-commutativity of quaternions. The non-covariance of ε is cancelled by that of $\mathbb{V}^T$, so (4.13) is actually covariant, but not manifestly so. Now recall that $\varepsilon M^T \varepsilon^T$ gives a cofactor matrix of M for any 2×2 matrix over $\mathbb{R}$ or $\mathbb{C}$. This remains true for $\mathbb{M}$ over $\mathbb{H}$ <u>if and only if</u> the four elements of $\mathbb{M}$ commute among themselves. This condition is satisfied by $\mathbb{V}$ [recall that $\mathbb{q}\mathbb{q}^* = \mathbb{q}^*\mathbb{q} = (\det \mathbb{q})^2$ for $\mathbb{q} \in \mathbb{H}$]. Hence

$$\mathbb{V} \epsilon \mathbb{V}^T \epsilon^T = \epsilon \mathbb{V}^T \epsilon^T \mathbb{V} = (\det \mathbb{V}) \mathbb{1} = V^\mu V_\mu \mathbb{1} , \tag{4.15}$$

and the equivalence of (4.14) to (4.13) is established.

Notice that $\mathbb{V}$ or $\tilde{\mathbb{V}}$ is characterized by the single requirement that it be a Hermitian 2×2 quaternionic matrix (as is true for $D = 4$ and $D = 3$ with $\mathbb{H} \to \mathbb{C}$ and $\mathbb{H} \to \mathbb{R}$ respectively). The components $\{V_\mu\}$ of $\mathbb{V}$ can be expanded on a basis $\{\mathbb{\Gamma}^\mu; \mu = 0,1,\ldots,5\}$ of 2×2 Hermitian quaternion matrices, and similarly for $\tilde{\mathbb{V}}$ on the conjugate basis $\{\tilde{\mathbb{\Gamma}}^\mu\}$

$$\mathbb{V} = \Gamma^\mu V_\mu \quad , \quad \tilde{\mathbb{V}} = \tilde{\Gamma}^\mu V_\mu \ . \tag{4.16}$$

The Γ^μ, $\tilde{\Gamma}^\mu$ satisfy the relation

$$\Gamma^\mu \tilde{\Gamma}_\nu + \Gamma_\nu \tilde{\Gamma}^\mu = 2\,\delta^\mu_\nu\, \mathbb{1} \ . \tag{4.17}$$

A second rank antisymmetric tensor $F_{\mu\nu}$ is represented in $S\ell(2;\mathbb{H})$ notation by a traceless bi-spinor

$$\begin{aligned} &\mathbb{F}_\alpha{}^\beta \quad , \quad Tr\,\mathbb{F} = 0 \ , \\ &(\mathbb{F}_\alpha{}^\beta)^\dagger = \mathbb{F}^{\dot\beta}{}_{\dot\alpha} \ . \end{aligned} \tag{4.18}$$

Its components can be expanded on the basis $\Sigma^{\mu\nu}$

$$\mathbb{F} = \Sigma^{\mu\nu} F_{\mu\nu} \ , \tag{4.19}$$

where,

$$(\Sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{2}\,(\Gamma^\mu \tilde{\Gamma}^\nu - \Gamma^\nu \tilde{\Gamma}^\mu)_\alpha{}^\beta \ . \tag{4.20}$$

The derivative ∂_μ is represented in $S\ell(2;\mathbb{H})$ notation as for V_μ:

$$\boldsymbol{\partial} = \begin{pmatrix} \partial_+ & \underset{\sim}{\boldsymbol{\partial}}^* \\ \underset{\sim}{\boldsymbol{\partial}} & \partial_- \end{pmatrix} = \Gamma^\mu \partial_\mu \ . \tag{4.21}$$

However, as an operator $\boldsymbol{\partial}$ is not self-adjoint. To have a self-adjoint operator we need to introduce an additional imaginary unit i that commutes with the quaternion imaginary units $\mathbb{i}$, $\mathbb{j}$, $\mathbb{k}$. Then $i\boldsymbol{\partial}$ is self-adjoint if Hermitian conjutation is taken to send $i \to -i$ simultaneously with $\mathbb{i} \to -\mathbb{i}$, etc. It is this i that will play the rôle of the imaginary unit in the quantum theory so it should be clear that the $S\ell(2;\mathbb{H})$ spinor formalism will not require an extension of the quantum theory to "quaternionic quantum mechanics".

To make the transition from $S\ell(2;\mathbb{H})$ notation to $SU^*(4)$ notation we make use of the 2×2 complex matrix representation of the imaginary quaternion units

$$\mathbb{i} \leftrightarrow i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbb{j} \leftrightarrow i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{k} \leftrightarrow i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{4.22}$$

The spinor ψ becomes

$$\psi \leftrightarrow \begin{pmatrix} \mu_0 + i\mu_3 & i\mu_1 + \mu_2 \\ i\mu_1 - \mu_2 & \mu_0 - i\mu_3 \\ \nu_0 + i\nu_3 & i\nu_1 + \nu_2 \\ i\nu_1 - \nu_2 & \nu_0 - i\nu_3 \end{pmatrix} = \psi_i \quad . \tag{4.23}$$

The index $i = 1, 2$ of ψ_i indicates the column, first or second, of (4.23). This index is of $SU(2) \cong S\ell(1;\mathbb{H})$ because the $S\ell(1;\mathbb{H})$ transformation of (4.7) effects a rotation of the two columns into each other. By the substitution (4.22) an element of $S\ell(2;\mathbb{H})$ or $S\ell(1;\mathbb{H})$ becomes an element of $SU^*(4)$ or $SU(2)$ respectively;

$$\begin{aligned} \mathbb{a} &\longrightarrow a \in SU^*(4), \\ \mathbb{u} &\longrightarrow u \in SU(2). \end{aligned} \tag{4.24}$$

[Notice that, in general $\mathbb{M} \to M$ under this substitution and $\det M$ is real positive. In fact $\det \mathbb{M} = (\det M)^{1/2}$ and this could serve as an equivalent definition of the determinant of $\mathbb{M}$].

By inspection of (4.23) we see that the two columns are related by

$$\begin{aligned} (\psi_2)^* &= -B\,\psi_1 \\ B &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \end{aligned} \tag{4.25}$$

whence we recognize ψ_i as an $SU(2)$ Majorana-Weyl spinor satisfying

$$(\psi_{\alpha i})^* = \psi^{*}{}_{\dot\alpha}{}^{i} = \epsilon^{ij} B_{\dot\alpha}{}^{\beta} \psi_{\beta j} \tag{4.26}$$

with B the matrix characterizing the pseudo-unitarity of $SU^*(4)$; $a^{*-1}Ba = B$. $\psi_{\alpha i}$ must be a Weyl spinor because the spinor index α takes four values. The

four-(complex) component spinor $\chi^{\dot{\alpha}i}$, obtained in a similar way is an anti-Weyl spinor; hence the nomenclature "chiral" and "antichiral" for ψ and χ. The relation (4.26) does have an $S\ell(2;\mathbb{H})$ analogue; if one introduces the conjugate $\mathbb{q}^{**}$ of $\mathbb{q} \in \mathbb{H}$, defined as

$$\mathbb{q}^{**} = -\mathbb{j}\,\mathbb{q}\,\mathbb{j} \tag{4.27}$$

which is equivalent to the complex conjugate of $\mathbb{q}$ regarded as a 2×2 complex matrix, then the $SU(2)$ reality condition (4.26) is equivalent to the relation $\psi^{**} = -\mathbb{j}\,\psi\,\mathbb{j}$. The left and right imaginary units $\mathbb{j}$ correspond to B and ε in (4.26) respectively.

Since $\psi_{\alpha i}$ is a Weyl spinor it can be considered as the upper four components of an eight-component spinor Ψ_i,

$$\Psi_i = \begin{pmatrix} \psi_i \\ \chi^i \end{pmatrix} , \tag{4.28}$$

in the basis for which Γ_7 is diagonal

$$\Gamma_7 = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix} , \quad \mathbb{1}_4 = 4\times4 \text{ unit matrix.} \tag{4.29}$$

The remaining Γ matrices are off-diagonal

$$\Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)_{\alpha\dot{\beta}} \\ (\tilde{\gamma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} . \tag{4.30}$$

A useful representation for γ^μ and $\tilde{\gamma}^\mu$ is obtained by making the substitution (4.22) into $\mathbb{\Gamma}^\mu$ and $\tilde{\mathbb{\Gamma}}^\mu$. This gives

$$\gamma^\mu = (\gamma^0, \gamma^i) , \quad \tilde{\gamma}^\mu = (\gamma^0, -\gamma^i) ,$$

where $\gamma^0 = \mathbb{1}_4$ and the five γ^i matrices are

$$\gamma^1 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma^5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} , \quad \gamma^k = \begin{pmatrix} 0 & -i\sigma_{k-1} \\ i\sigma_{k-1} & 0 \end{pmatrix} \text{ for } k = 2,3,4. \tag{4.31}$$

As in the $S\ell(2;\mathbb{C})$ case the γ^μ, $\tilde{\gamma}^\mu$ satisfy

$$\gamma_\mu \tilde{\gamma}_\nu + \gamma_\nu \tilde{\gamma}_\mu = \tilde{\gamma}_\mu \gamma_\nu + \tilde{\gamma}_\nu \gamma_\mu = 2\eta_{\mu\nu} \quad , \qquad (4.32)$$
$$\frac{1}{4} \mathrm{tr}\,(\gamma_\mu \tilde{\gamma}_\nu) = \eta_{\mu\nu} \quad ,$$

which is the $SU^*(4)$ version of (4.17). The Lorentz generators are

$$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\tilde{\gamma}_\nu - \gamma_\nu\tilde{\gamma}_\mu) \, , \quad (\sigma_{\mu\nu})^\dagger = -\tilde{\sigma}_{\mu\nu} = -\frac{1}{2}(\tilde{\gamma}_\mu\gamma_\nu - \tilde{\gamma}_\nu\gamma_\mu) \, . \qquad (4.33)$$

The placement of the spinor indices is $(\gamma_\mu)_{\alpha\dot{\beta}}$, $(\tilde{\gamma}^\mu)^{\dot{\alpha}\beta}$, $(\sigma_{\mu\nu})_\alpha{}^\beta$ and $(\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$. Unlike the $S\ell(2;\mathbb{C})$ case there are no quantities such as $(\gamma_\mu)_{\dot{\alpha}}{}^\beta$ or $(\sigma_{\mu\nu})_{\alpha\beta}$. Single spinor indices cannot be raised or lowered. The invariant tensor of $SU^*(4)$ is $\varepsilon_{\alpha\beta\gamma\delta}$ rather than $\varepsilon_{\alpha\beta}$ and so a chiral spinor $\psi_{\alpha i}$ cannot be converted into an antichiral one. However, dotted indices may be freely converted into undotted indices by means of the matrix B. For example

$$(\gamma_\mu)_{\alpha\dot{\beta}} = (\gamma_\mu)_{\alpha\beta}\, B_{\dot{\beta}}{}^\beta \qquad (4.34)$$

and the $(\gamma_\mu)_{\alpha\beta}$ constitute the set of six 4×4 antisymmetric matrices. Pairs of antisymmetric spinor indices may be raised or lowered with $\varepsilon_{\alpha\beta\gamma\delta}$. In particular we have the relations

$$(\gamma_\mu)_{\alpha\beta} = \frac{1}{2}\, \epsilon_{\alpha\beta\gamma\delta}\, (\tilde{\gamma}_\mu)^{\gamma\delta} \qquad (4.35)$$

and

$$(\gamma_\mu)_{\alpha\beta}\, (\gamma^\mu)_{\gamma\delta} = 2\, \epsilon_{\alpha\beta\gamma\delta} \, . \qquad (4.36)$$

This completes the transition from two-component to four-component spinor notations.

We add a few words on the relation from $D = 6$ to $D = 5$. The bispinor representation of V^μ, $\mu = 0,1,\dots,4$, is obtained from that of $D = 6$ by setting $V_5 = 0$ in (4.10);

$$V = \begin{pmatrix} V_0 & \underset{\sim}{V}^* \\ \underset{\sim}{V} & V_0 \end{pmatrix} = \Gamma^\mu V_\mu \quad , \qquad D = 5 \, . \qquad (4.37)$$

This $\mathbb{V}$ is characterized by its Hermiticity and its η tracelessness;

$$\mathrm{Tr}\,(\eta \mathbb{V}) = 0 \quad , \qquad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.38}$$

The elements $\mathbf{a}$ of $S\ell(2;\mathbb{H})$ that preserve this property satisfy

$$\mathbf{a}^\dagger \eta\, \mathbf{a} = \eta \tag{4.39}$$

and are therefore the elements of the group $U(1,1;\mathbb{H})$. The two-component spinors of $S\ell(2;\mathbb{H})$ become those of $U(1,1;\mathbb{H})$ as they stand, but spinors with upper and lower indices are now equivalent because single spinor indices may be raised (or lowered) with the metric $\eta^{\dot{\alpha}\beta}$ (or its inverse). Notice the parallel between the reduction $S\ell(2;\mathbb{H}) \to U(1,1;\mathbb{H})$ and the more familiar one $S\ell(2;\mathbb{C}) \to SU(1,1;\mathbb{C})$.

5. - QUATERNIONS AND SUPERSYMMETRY

As mentioned in the Introduction, supersymmetric models whose maximal dimension is six are known, or conjectured, to have some kind of quaternionic structure. This is manifest in $S\ell(2;\mathbb{H})$ notation, so we now turn to the application of the results of the previous section to supersymmetry. We will construct the free on-shell supersymmetric hyper-Kähler model and the free off-shell supersymmetric Maxwell theory in six dimensions. We leave to the future the quaternion form of the interacting theories.

For the hyper-Kähler model the fields are a scalar quaternion $\mathbb{A}$ and a $S\ell(2;\mathbb{H})$ chiral spinor ψ. The action is

$$I = \int d^6x\ \mathrm{Tr}\left[-\mathbb{A}^* \Box \mathbb{A} + \psi^\dagger i \tilde{\partial} \psi \right] \tag{5.1}$$

where Tr denotes the quaternion trace of (3.4) and i is the imaginary unit needed to make $i\tilde{\partial}$ self-adjoint. The differential operators ∂ and $\Box$ are related by

$$\partial \tilde{\partial} = \tilde{\partial} \partial = \partial_\mu \partial^\mu \mathbb{1} = \Box \mathbb{1} . \tag{5.2}$$

The supersymmetry transformations are

$$\begin{aligned} \delta \mathbb{A} &= \epsilon^\dagger \psi \\ \delta \psi &= -i\, \Gamma^\mu \epsilon\, \partial_\mu \mathbb{A} \end{aligned} \tag{5.3}$$

where ϵ is an antichiral, constant, $S\ell(2;\mathbb{H})$ spinor. Its components, in an expansion on a quaternion basis, are, like those of ψ, anticommuting. If the requirement that ϵ be constant is relaxed the action has the variation

$$\delta I = \int d^6x \, \mathrm{Tr}\left[\mathbb{J}^{\mu\dagger} \partial_\mu \epsilon + (\partial_\mu \epsilon^\dagger) \mathbb{J}^\mu \right] \tag{5.4}$$

with

$$\mathbb{J}^\mu = \Gamma^\nu \tilde{\Gamma}^\mu \psi \, \partial_\nu \mathbb{A}^* , \tag{5.5}$$

the conserved supersymmetry current. Since $\tilde{\Gamma}^0 = 1$ the supersymmetry charge is

$$\mathbb{Q} = \int d^5x \; \Gamma^\nu \psi \, \partial_\nu \mathbb{A}^* \tag{5.6}$$

which is a constant $S\ell(2;\mathbb{H})$ spinor functional of ψ and $\mathbb{A}^*$.

Because we are dealing with a free theory, invariance of the action is not sufficient to establish supersymmetry. We must also demonstrate that the algebra closes, on-shell, in the required way. For $\mathbb{A}$ we find

$$[\delta_1, \delta_2] \, \mathbb{A} = 2 \, \xi^\mu \partial_\mu \mathbb{A} \tag{5.7}$$

with ξ^μ the real number $(\mu = 0,\ldots,5)$,

$$\xi^\mu = \frac{i}{2} \left(\epsilon_1^\dagger \Gamma^\mu \epsilon_2 - \epsilon_2^\dagger \Gamma^\mu \epsilon_1 \right) . \tag{5.8}$$

It is easily seen that $\xi^\mu = (\xi^\mu)^\dagger$ but since it is the factor of i that guarantees hermiticity, rather than antihermiticity, it is not immediately obvious that this implies that ξ^μ is a real number, i.e., not a quaternion. We have verified explicitly that ξ^μ is indeed a real number. The antihermiticity of $-i\xi^\mu$ is caused by the anticommutativity of the components (in an expansion on 1, $\mathbb{i}$, $\mathbb{j}$, $\mathbb{k}$) of ϵ_1 and ϵ_2. The factor of i in (5.8) compensates for this.

For ψ we find

$$[\delta_1, \delta_2] \, \psi = i \, \Gamma^\mu \left(\epsilon_1 \epsilon_2^\dagger - \epsilon_2 \epsilon_1^\dagger \right) \partial_\mu \psi . \tag{5.9}$$

Using the identity

$$\epsilon_1 \epsilon_2^\dagger - \epsilon_2 \epsilon_1^\dagger = - i \, \xi^\nu \tilde{\Gamma}_\nu \tag{5.10}$$

and the relation (4.17) we are able to rewrite (5.9) as

$$[\delta_1, \delta_2]\,\psi = 2\,\xi^\mu \partial_\mu \psi - \xi^\nu \Gamma_\nu \tilde{\partial}\psi \ . \tag{5.11}$$

The second term in the commutator is a field equation so that on-shell we obtain the uniform commutator

$$[\delta_1, \delta_2] = 2\,\xi^\mu \partial_\mu \tag{5.12}$$

as required for supersymmetry. How this algebra is realized in terms of the charge $\mathbb{Q}$ is not clear, although in SU*(4) notation it is simply

$$\{ Q^i_\alpha , Q^j_\beta \} = \epsilon^{ij}\, i\, \partial_{\alpha\beta} \quad . \tag{5.13}$$

It is of interest to reduce this model to two dimensions because, firstly, that is the dimension in which the hyper-Kähler σ model is finite, and secondly, because in two dimensions the Lorentz structure decouples from the quaternion structure. That is, in two dimensions ψ can be expanded Lorentz covariantly on a quaternion basis. The four components of ϵ in this expansion are the four parameters of an $N = 4$ supersymmetry in two dimensions. The Lorentz group splits into the product of $G\ell(1;\mathbb{R}) \cong SO(1,1)$ and $S\ell(1;\mathbb{H})_R \otimes S\ell(1;\mathbb{H})_L \cong \overline{SO}(4)$ corresponding to the following elements, $\mathbb{a}$, of $S\ell(2;\mathbb{H})$

$$G\ell(1;\mathbb{R}) : \quad \mathbb{a} = \begin{pmatrix} e^{\omega} & 0 \\ 0 & e^{-\omega} \end{pmatrix} , \quad \omega \in \mathbb{R}$$

$$S\ell(1;\mathbb{H})_R : \quad \mathbb{a} = \begin{pmatrix} q_R & 0 \\ 0 & 1 \end{pmatrix} , \quad \begin{array}{l} q_R \in \mathbb{H} , \\ \det q_R = 1 , \end{array} \tag{5.14}$$

$$S\ell(1;\mathbb{H})_L : \quad \mathbb{a} = \begin{pmatrix} 1 & 0 \\ 0 & q_L \end{pmatrix} , \quad \begin{array}{l} q_L \in \mathbb{H} , \\ \det q_L = 1 \ . \end{array}$$

The reduction is achieved simply by setting $\underset{\sim}{\partial} = 0$ so that

$$\partial \longrightarrow \begin{pmatrix} \partial_+ & 0 \\ 0 & \partial_- \end{pmatrix} . \tag{5.15}$$

Other vector fields $\mathbb{V}$ maintain the form of (4.10) but (V_0, V_5) become the components of a two-vector V_μ while $\underset{\sim}{\mathbb{V}}$ becomes a scalar quaternion, transforming as a $\underset{\sim}{4}$ under $\overline{SO}(4)$.

For the Maxwell theory we introduce a vector gauge potential $\mathbb{V}$ and its field strength $\mathbb{F}$

$$\mathbb{F} = \partial\tilde{\mathbb{V}} - \mathbb{V}\overleftarrow{\tilde{\partial}} \quad , \qquad \mathrm{Tr}\,\mathbb{F} = 0 \quad . \tag{5.16}$$

We need also an <u>antichiral</u> spinor λ and a <u>traceless</u> auxiliary quaternion field $\mathbb{X}$. The action is

$$I = \int d^6x \,\mathrm{Tr}\left[\tfrac{1}{4}\mathbb{F}\mathbb{F} + \lambda^\dagger i\partial\lambda - \tfrac{1}{2}\mathbb{X}\mathbb{X}\right] . \tag{5.17}$$

The transformation laws are

$$\begin{aligned} \delta\tilde{\mathbb{V}} &= \tfrac{i}{\sqrt{2}}\left(\epsilon\lambda^\dagger - \lambda\epsilon^\dagger\right) \\ \delta\lambda &= -\tfrac{1}{\sqrt{2}}\mathbb{F}^\dagger\epsilon - \tfrac{1}{\sqrt{2}}\epsilon\mathbb{X} \\ \delta\mathbb{X} &= \tfrac{1}{\sqrt{2}}\left(\lambda^\dagger i\overleftarrow{\tilde{\partial}}\epsilon + \epsilon^\dagger i\partial\lambda\right) \quad . \end{aligned} \tag{5.18}$$

We have verified that the transformations close off-shell. The algebra is tedious and requires various identities involving $S\ell(2;\mathbb{H})$ spinors, of which the most useful are the following:

$$\begin{aligned} \chi^\dagger\epsilon_1 \mathbb{A}\,\epsilon_2^\dagger\psi - (1\leftrightarrow 2) &= -i\chi^\dagger\tilde{\Gamma}^\mu\psi\,\xi'_\mu - i\chi^\dagger\tilde{\Gamma}^{\mu\nu\rho}\psi\, t_{\mu\nu\rho} \,, \\ \xi'_\mu &= \tfrac{i}{2}\left(\epsilon_1^\dagger\Gamma^\mu\epsilon_2 - (1\leftrightarrow 2)\right)\,\mathrm{Tr}\,\mathbb{A} \quad , \\ t_{\mu\nu\rho} &= \tfrac{i}{2}\,\mathrm{Tr}\left[\left(\epsilon_1^\dagger\Gamma_{\rho\nu\mu}\epsilon_2 - (1\leftrightarrow 2)\right)\mathbb{A}\right] \equiv \mathrm{Tr}\left(T_{\mu\nu\rho}\mathbb{A}\right), \end{aligned} \tag{5.19}$$

and $\tilde{\Gamma}^{\mu\nu\rho}$ is the antisymmetrized product of 3 Γ's with unit weight:

$$\tilde{\Gamma}^{\mu\nu\rho} = \tilde{\Gamma}^{[\mu}\,\Gamma^\nu\,\tilde{\Gamma}^{\rho]} \, . \tag{5.20}$$

Another, for three arbitrary quaternions $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, is

$$\not{A}\, \epsilon_1^\dagger \mathbb{C}\, \epsilon_2 B \; - \; (1 \leftrightarrow 2)$$
$$= -i\, \mathrm{Tr}\,(\mathbb{C}\, \tilde{\Gamma}^{\mu})\, \xi_\mu \not{A} \not{B} \; - \; i\, \mathrm{Tr}\,(\mathbb{C}\, \tilde{\Gamma}^{\mu\nu\rho}) \not{A}\, T_{\mu\nu\rho} \not{B} \; . \tag{5.21}$$

The fully interacting SU*(4) superspace treatment of the models discussed here will be the topic of another article[19].

6. - OCTONIONS

For $t = 0$, Euclidean space, a spinor in $D = 8$ may be represented by a one-component octonion ($2^t = 1$ in this case)

$$\psi = \psi_0 + \psi_i e^i \; . \tag{6.1}$$

The $\{e^i\}$ are the seven octonion imaginary units and ψ_0, ψ_i are anticommuting components. The group SO(8) is realized on ψ by the following multiplication by four unit octonions[20] a, b, c, d:

$$\psi' = c\,(\, b^{-1} a^{-1} \,(\, b\,(a\, \psi\, a^{-1})\, b^{-1})\; a b\,)\, d \; . \tag{6.2}$$

For $t = 1$, Minkowski space, a spinor in $D = 10$ may be represented by a two-component octonion ($2^t = 2$ in this case)

$$\psi = \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \qquad \mu, \nu \in \mathbb{O} \; . \tag{6.3}$$

A vector is obtained from

$$\psi \psi^\dagger \sim V = \begin{pmatrix} V_+ & \underset{\sim}{V}^* \\ \underset{\sim}{V} & V_- \end{pmatrix} \tag{6.4}$$

exactly as in the quaternion case. $\underset{\sim}{V}$ is now an octonion, and $\underset{\sim}{V}^*$ is octonion conjugate $[(e^i)^* = -e^i,\ i = 1, \ldots, 7]$. As the result (6.2) suggests we cannot expect a simple action of the Lorentz group $\overline{SO}(9,1)$ on ψ or V. It seems likely that the correct framework for an octonion description of supersymmetric theories in $D = 10$ is that of Jordan algebras as $\overline{SO}(9,1)$ is the structure group of the Jordan algebra of 2×2 Hermitian matrices over $\mathbb{O}$[21].

A hint of the importance that octonions might have for an understanding of the $D = 10$ super-Yang-Mills theory is provided by recent work of Curtright[22]. In this approach to understanding the vanishing of the β function of this model, when reduced to four dimensions, he makes crucial use of the triality properties of the transverse rotation group $SO(8)$, which in turn are closely connected to properties of the octonions.

It is tempting to speculate that the non-existence of rigid supersymmetric theories beyond $D = 10$ is connected to the non-existence of division algebras beyond the octonions, although, of course, the former fact can be explained in other ways. When local supersymmetry is introduced, i.e. supergravity, we can go beyond $D = 10$ to $D = 11$ so it seems clear that the relationship of the division algebras to supergravity theories, if there is any, will be more complicated than that discussed in this work. Nevertheless, there is already some indication that octonions are relevant to 11-dimensional supergravity[23], and quaternions to 10-dimensional supergravity[24].

The observation that $D = 11$ is the maximal dimension for a supergravity theory is based on the fact that, for Minkowski space $(t = 1)$, a spinor in $D = 12$ would have at least 64 components leading to $N = 16$ supersymmetry in four dimensions and spins greater than 2. We mention here the fact, apparently well known, that for $t = 2$ a spinor in $D = 12$ can have only 32 compoments (as can be seen from the Table). Conceivably this could allow an extension of 11-dimensional supergravity to $D = 12$.

We conclude with another observation (taken from the Table) on 10- and 11-dimensional Minkowski space. Only for $D = 10, 11 \bmod 8$ does a Euclidean spinor necessarily have double the number of components of a Minkowski spinor. It would be interesting if this were relevant to the problem of defining the Euclidean version of the Minkowski space field theory, but according to our current understanding of Euclidean field theory it is not.

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REFERENCES

1) See, for example:
"The Algebraic Theory of Spinors", by C.C. Chevally, (Columbia University Press, 1954).

2) B. Zumino, Phys. Lett. B87 (1979) 203.

3) L. Alvarez-Gaumé and D.Z. Freedman, Comm. Math. Phys. 80 (1981) 443.

4) G. Parisi and N. Sourlas, to be published in Nucl. Phys. B.;
S. Cecotti and L. Girardello, to be published in Ann. Phys.

5) E.A. Ivanov, Dubna preprint E2-82-427 (1982).

6) V.I. Ogievetsky and E. Sokatchev, Phys. Lett. B79 (1978) 222.

7) J. Lukierski and A. Nowicki, Fortschritte der Physik 30 (1982) 75.

8) B. Julia, LPTENS preprint 82/14 (1982).

9) R. Coquereaux, Phys. Lett. B115 (1982) 389.

10) A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, Dubna preprint E2-80-750 (1980).

11) J. Scherk in "Recent Developments in Gravitation", eds. M. Lévy and S. Deser, (Plenum Press, N.Y., 1979).

12) P. van Nieuwenhuizen in "Supergravity '81", eds. S. Ferrara and J.G. Taylor (Cambridge University Press, 1982);
G. Chapline and R. Slansky, to be published in Nucl. Phys. B;
C. Wetterich, CERN preprint TH.3310 (1982).

13) See, for example:
E. Cremmer, in "Superspace and Supergravity", eds. S.W. Hawking and M. Rocek (Cambridge University Press, 1981).

14) N. Marcus and J.H. Schwartz, preprint CALT-68-910 (1982).

15) R. Gilmore, "Lie Groups, Lie Algebras and Some of their Applications", (John Wiley and Sons, N.Y., 1974).

16) J.W. van Holten and A. Van Proeyen, CERN preprint TH.3234 (1982).

17) M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179.

18) P.S. Howe, private communication.

19) P.S. Howe, G. Sierra and P.K. Townsend, in preparation.

20) See, for example:
P. Ramond, preprint CALT-68-577 (1976).

21) See, for example:
M. Günaydin, Proceedings of the VIIIth International Colloquium on Group Theoretical Methods in Physics, (1979), Kiriat Anavim, Israel.

22) T.L. Curtright, Phys. Rev. Lett. 48 (1982) 1704.

23) F. Englert, CERN preprint TH.3394 (1982).

24) M.J. Duff, P.K. Townsend and P. van Nieuwenhuizen, CERN preprint TH.3465 (1982).

TABLE CAPTION

Pattern of allowed spinors for space-times of dimension $D = t + s$. The pattern is modulo 8 in $s - t$. Also shown are the various isomorphisms of the orthogonal groups.

Legend:

- ———— Majorana spinor
- -------- pseudo-Majorana spinor
- ════ USp(2n) Majorana spinor
- ======== USp(2n) pseudo-Majorana spinor
- ———— / -------- } (pseudo)-Majorana-Weyl spinor
- ════ / ======== } USp(2n) (pseudo)-Majorana-Weyl spinor

In addition in the following overlapping cases there is the choice of a Weyl spinor:

- ———— / ======== } Weyl spinor
- ════ / -------- } Weyl spinor

t \ D	1	2	3	4	5	6	7	8	9	10	11...
0	$\begin{pmatrix} SO(1) \\ S\ell(1;\mathbb{R}) \end{pmatrix}$	$SO(2)$ $S\ell_2(1;\mathbb{C})$ $U(1)$	$\overline{SO}(3)$ $SU^*(2)$ $SU(2)$ $S\ell(1;\mathbb{H})$	$\overline{SO}(4)$ $(S\ell(1;\mathbb{H}))^2$ $(SU^*(2))^2$ $(SU(2))^2$	$\overline{SO}(5)$ $USp(4)$ $U(2;\mathbb{H})$	$\overline{SO}(6)$ $SU(4)$					
1		$SO(1,1)$ $G\ell(1;\mathbb{R})$	$\overline{SO}(2,1)$ $S\ell(2;\mathbb{R})$ $SU(1,1;\mathbb{C})$ $Sp(2;\mathbb{R})$	$\overline{SO}(3,1)$ $S\ell(2;\mathbb{C})$	$\overline{SO}(4,1)$ $U(1,1;\mathbb{H})$	$\overline{SO}(5,1)$ $S\ell(2;\mathbb{H})$ $SU^*(4)$					
2				$\overline{SO}(2,2)$ $(S\ell(2;\mathbb{R}))^2$	$\overline{SO}(3,2)$ $Sp(4;\mathbb{R})$	$\overline{SO}(4,2)$ $SU(2,2;\mathbb{C})$		$\overline{SO}(6,2)$ $SO(4;\mathbb{H})$ $SO^*(8)$			
3						$\overline{SO}(3,3)$ $S\ell(4;\mathbb{R})$					
4											