Ref.TH.3403-CERN

THE QUANTUM MECHANICAL SUPERSYMMETRIC
O(3) and O(2,1) MODELS

A.C. Davis^{*)}, A.J. Macfarlane^{**)} and J.W. van Holten^{***)}
CERN - Geneva

A B S T R A C T

We state and solve the problem of the quantum mechanical O(3) and O(2,1) supersymmetric σ -models. Such a study is relevant to the analysis of the corresponding field theories in higher dimensions from which the quantum mechanical models are obtained by dimensional reduction. In particular our results show that supersymmetry is not broken in these field theories. Another interesting result is that in the non-compact model the inclusion of fermions gives rise to the appearance of an infinite multiplet of discrete zero energy states, which are absent in the purely bosonic model. This implies that the O(2,1) symmetry is spontaneously broken.

^{*)} Address after September 1982: Institute for Advanced Study, Princeton, N.J., U.S.A.

^{**)} On leave of absence until 1 Oct. 1982 from DAMTP and St. John's College, Cambridge, U.K.

^{***)} Address after Oct. 1, 1982: Inst. für Theor. Physik, Universität Wuppertal, 5600 Wuppertal 1, Germany.

1. - INTRODUCTION

Non-linear σ -models constitute an interesting and widely studied class of field theories. They have been proposed as low-energy effective field theories¹⁾; as a starting point for investigations on the formation of bound states, chiral symmetry breaking, confinement, tumbling and other non-perturbative aspects of field theory-like instantons and large N behaviour²⁾. Supersymmetric extensions of these models seem even more interesting: the inclusion of fermions gives rise to a much richer bound state spectrum^{3),4)} and softening or even absence of divergences^{4),5)}. Also, non-compact σ -models have emerged in extended supergravity, where they describe the scalar sector of the rigidly $O(N)$ invariant Lagrangians for $N \geq 4$. Such non-compact σ -models with G/H symmetry, where G is a rigid non-compact invariance group and H a local symmetry isomorphic to the maximal compact subgroup of G , have been shown to be ghost-free and have positive energy, at least on the classical level⁶⁾.

At the quantum mechanical level, one ought to ask what are the particle interpretations of such theories and how they are related to the infinite dimensional unitary representations of G . In search of some clues as to what might yield the answers to such difficult questions as these, we turn from the field theories in $(d+1)$ space-time dimensions, $d \geq 1$, to the $d = 0$ case of ordinary quantum mechanical models with non-compact symmetries. The case of the bosonic model with $O(3,1)$ as its non-compact symmetry was completely solved more than twelve years ago by Velo and Wess⁷⁾. In this paper, we study and likewise solve the reductions to $(0+1)$ space-time dimensions of the $O(2,1)$ and $O(3)$ supersymmetric σ -models. As was shown in Ref. 8), we can learn from such an analysis whether supersymmetry is realized manifestly in the spectrum of the corresponding field theories. This turns out to be the case both for the compact and the non-compact model, although they have completely different ground state structures. The compact model has two fermionic zero-energy states which are singlet representations of $O(3)$, while the non-compact model has two infinite dimensional $O(2,1)$ multiplets of bosonic zero-energy states, implying spontaneous breakdown of the internal $O(2,1)$ symmetry.

2. - THE MODELS

The models of interest here can be defined in terms of three real scalar superfields χ_A ($A = 1, 2, 3$), subject to a constraint

$$\eta^{AB} \chi_A \chi_B = -\frac{1}{g} \quad (1)$$

where $\eta^{AB} = \text{diag.}(+, +, \pm)$ while $g > 0$ for the non-compact $O(2,1)$ model and $g < 0$ for the compact $O(3)$ case. The superfield Lagrangian is

$$\mathcal{L} = \frac{1}{2} \eta^{AB} \overline{D\chi_A} D\chi_B. \quad (2)$$

D denotes the supersymmetric derivative.

The constraint (1) can be solved in a standard way in terms of two real scalar superfields

$$\Phi_i = \varphi_i + \bar{\theta}\psi_i + \bar{\theta}\theta F_i, \quad i=1,2. \quad (3)$$

Here θ and ψ_i are two-component Majorana spinors, transforming under an internal $SO(2)$. This reflects the fact that the superfields (3) form a representation of $N=2$ supersymmetry in $d=0,1$. With the definitions

$$\chi_i = \frac{\Phi_i}{(1-g\Phi_k^2)}, \quad \chi_3 = \frac{1}{2\sqrt{|g|}} \frac{(1+g\Phi_j^2)}{(1-g\Phi_k^2)}. \quad (4)$$

The Lagrangian (2) becomes

$$\mathcal{L} = \frac{1}{2} \frac{\overline{D\Phi_i} D\Phi_i}{(1-g\Phi_k^2)^2}. \quad (5)$$

This Lagrangian is invariant under the following non-linear infinitesimal transformations

$$\begin{aligned} \delta_1 \Phi_1 &= -2\lambda_1 g \Phi_1 \Phi_2, & \delta_1 \Phi_2 &= \lambda_1 (1+g(\Phi_1^2 - \Phi_2^2)), \\ \delta_2 \Phi_1 &= -\lambda_2 (1-g(\Phi_1^2 - \Phi_2^2)), & \delta_2 \Phi_2 &= 2\lambda_2 g \Phi_1 \Phi_2, \\ \delta_3 \Phi_1 &= 2\lambda_3 \Phi_2, & \delta_3 \Phi_2 &= -2\lambda_3 \Phi_1. \end{aligned} \quad (6)$$

The transformations commute with supersymmetry. We can rewrite the Lagrangian in terms of the components. Upon elimination of the auxiliary fields F_i , one obtains

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (1-g\tilde{\varphi}^2)^2 (\partial_t \varphi_i \partial_t \varphi_i - i \psi_i \partial_t \psi_i) \\ &\quad - 2i (1-g\tilde{\varphi}^2)^3 (\varphi_j \partial_t \varphi_i) \psi_i \psi_j \\ &\quad + \frac{1}{2} g (1-g\tilde{\varphi}^2)^4 \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \psi_{i\alpha} \psi_{i\beta} \psi_{j\gamma} \psi_{j\delta}. \end{aligned} \quad (7)$$

The canonical quantization of this theory requires care because the conjugate momenta of the Majorana spinors cannot be taken as independent variables and the theory is subject to constraints.

The Dirac formulation of a field theory with constraints in which fermionic co-ordinates are involved⁹⁾ leads to the following canonical equations

$$\begin{aligned} \{ p_i, \varphi_j \}^* &= -\delta_{ij} , \\ \{ p_i, \psi_j \}^* &= 2g(1-g\bar{\varphi}^2)\varphi_i\psi_j , \\ \{ \psi_i, \psi_j \}^* &= i(1-g\bar{\varphi}^2)^2\delta_{ij} , \end{aligned} \quad (8)$$

where all non-trivial Dirac brackets have been displayed. To see that these provide a consistent description of the classical theory, we must calculate the Hamiltonian equations

$$\partial_t A = \{ A, H \}^*$$

that follow from (8) and the Hamiltonian H corresponding to (7). For consistency, these must (and do) coincide with the Euler-Lagrange equations of (7), and obey the appropriate Jacobi identities.

To quantize the theory, we must give the equal-time commutation or anti-commutation relations for the basic field variables, and appropriately solve the operator ordering problem in specifying how the interesting quantities in the theory depend on them. Thus, firstly we must replace the Dirac brackets in (8) by the corresponding commutators divided by i , except for the third one for which an anticommutator is needed as it involves two fermionic variables. The form of (8) suggests that it is useful to define

$$\gamma_i = \frac{-i\sqrt{2}\psi_i}{1-g\bar{\varphi}^2} . \quad (9)$$

It is also convenient to use complex quantities

$$\begin{aligned} K &= \sqrt{|g|} (\varphi_1 + i\varphi_2) , \\ \Pi &= (p_1 - ip_2)/2\sqrt{|g|} , \\ \gamma &= (\gamma_1 + i\gamma_2)/\sqrt{2} , \end{aligned} \quad (10)$$

then the following list shows all the non-trivial commutation or anticommutation relations

$$\begin{aligned} [\pi, K] &= [\pi^*, K^*] = -i, \\ \{\gamma, \gamma^*\} &= 2\mathbf{1}. \end{aligned} \quad (11)$$

In addition to H , interesting variables in the theory are the supercharge Q_α , $\alpha = 1, 2$, and the symmetry group generators. Classical expressions for these quantities are available via Noether's procedure, and the Dirac bracket algebra they satisfy classically follows (8). We must deal with the operator order problems which arise in passing from classical to quantum expressions in such a way as to preserve this algebraic structure when Dirac brackets are replaced by commutation and anticommutation relations. For this part of the work, we first discuss the non-compact model. The compact case is to a large extent analogous and is discussed afterwards.

3. - THE $O(2,1)$ MODEL

From the classical expression, using (10) and setting $|g| = 1$, we obtain the quantum Hamiltonian

$$\begin{aligned} H &= (1-K^*K)\pi^*\pi(1-K^*K) + \frac{i}{4}(1-K^*K)(K^*\pi^* - K\pi)[\gamma_\alpha, \gamma_\alpha^*] \\ &\quad - \frac{1}{16}(1-K^*K)[\gamma_\alpha, \gamma_\alpha^*]^2, \end{aligned} \quad (12)$$

whose terms are ordered in such a way that its Heisenberg equations computed using (11) agree with the classical Hamiltonian equations. The supersymmetry generators are

$$Q_\alpha = Q_\alpha^\dagger = (\gamma_\alpha\pi + \gamma_\alpha^*\pi^*)(1-K^*K) - \frac{i}{4}(\gamma_\alpha^*K - \gamma_\alpha K^*)[\gamma_\beta, \gamma_\beta^*]. \quad (13)$$

They commute with the Hamiltonian and satisfy the anticommutation relation

$$\frac{1}{4}\{Q_\alpha, Q_\alpha\} = H. \quad (14)$$

Because of the hermiticity of Q_α , this relation implies semi-positivity of the energy eigenvalues. The generators corresponding to the $O(2,1)$ symmetry transformations (2) are

$$\begin{aligned} q_1 &= \frac{1}{2}i(\pi + K\pi K) + \frac{1}{8}K[\gamma_\alpha, \gamma_\alpha^*] + h.c., \\ q_2 &= \frac{1}{2}(-\pi + K\pi K) - \frac{1}{8}iK[\gamma_\alpha, \gamma_\alpha^*] + h.c., \\ q_0 = -iq_3 &= i(K\pi - K^*\pi^*) + \frac{1}{4}[\gamma_\alpha, \gamma_\alpha^*]. \end{aligned} \quad (15)$$

They obey the $O(2,1)$ Lie algebra

$$[q_i, q_j] = -\epsilon_{ijk} q_k, \quad (i, j, k = 1, 2, 3). \quad (16)$$

Moreover the Hamiltonian is the Casimir operator of this Lie algebra

$$H = q_i^2 = q_1^2 + q_2^2 - q_0^2. \quad (17)$$

Relations (14), (16) and (17) allow one to determine the spectrum of the theory⁷⁾.

To begin, we define operators $P = P_1 + iP_2$ and P_0 as follows

$$P = \frac{2K}{\sqrt{g}(1-K^*K)}, \quad P_0 = \frac{1}{\sqrt{g}} \frac{(1+K^*K)}{(1-K^*K)}. \quad (18)$$

They have the following algebraic properties

$$\begin{aligned} [P_i, P_j] &= 0, \\ [q_i, P_j] &= -\epsilon_{ijk} P_k, \end{aligned} \quad (i, j, k = 1, 2, 3). \quad (19)$$

Therefore P_i and q_i together form the $(2+1)$ dimensional Poincaré algebra, which acts as a spectrum generating algebra for the model. The two invariants which characterize representations of this algebra are: $m^2 \equiv -P_1^2 = 1/g$, the square of the "mass", and $\Sigma \equiv -\vec{q} \cdot \vec{P}/m = (1/4)[\gamma_\alpha, \gamma_\alpha^*]$, the "helicity". Clearly, all states in the Hilbert space are characterized by one single value of m^2 . The possible eigenvalues of Σ can be found using the anticommutation relation (11) for γ, γ^* . Namely,

$$\Sigma |\alpha\rangle = \alpha |\alpha\rangle, \quad (20)$$

where $\alpha = -1$, for the state $|0\rangle$, annihilated by γ_α ;

$\alpha = 0$, for the states $\gamma_{1,2}^* |0\rangle$;

$\alpha = +1$, for the state $\gamma_1^* \gamma_2^* |0\rangle$.

We are interested here in representations in which the Casimir of the Lorentz group $O(2,1)$ is diagonalized. Simultaneously we can diagonalize q_0 , the angular momentum operator, which equals Σ in the rest frame: $P_i = (0, 0, im)$

Let us introduce

$$N_\pm = \frac{1}{\sqrt{2}} (\mp q_1 - i q_2) = -N_\mp^\dagger,$$

which act as angular momentum raising and lowering operators:

$$[q_0, N_{\pm}] = \pm N_{\pm} \quad , [N_+, N_-] = q_0 . \quad (21)$$

Then

$$H = -q_0(q_0+1) - 2N_-N_+ = -q_0(q_0-1) - 2N_+N_- . \quad (22)$$

As is well known, all the non-trivial unitary representations of $O(2,1)$ are infinite-dimensional. From (22) it follows that there are the following possibilities¹⁰⁾:

- a. There is a state $|k\rangle$ with $q_0|k\rangle = k|k\rangle$, k a positive integer, and $N_-|k\rangle = 0$. Then $H|k\rangle = -k(k-1)|k\rangle$, and because we know that the eigenvalues should be zero or positive, $k=1$ and $k(k-1)=0$. By applying N_+ an arbitrary number of times we obtain zero-energy states with any positive integer value of q_0 .
- b. Similarly, if there is a state $|k\rangle$ with k a negative integer and $N_+|k\rangle = 0$, we have $H|k\rangle = -k(k+1)|k\rangle$ and semi-positivity of H requires $k=-1$, $k(k+1)=0$. Application of N_- leads to an infinite dimensional multiplet of zero energy states with an arbitrary negative integer value of q_0 .
- c. There could be a singlet state $|0\rangle$, with $N_{\pm}|0\rangle = 0$ and consequently zero energy; however, no such state seems to be present.
- d. There can be infinite multiplets of states with no bound above or below for q_0 and positive, non-zero energies. Actually, we will see that normalizability requirements lead to the restriction of energy eigenvalues to $\epsilon \geq 1/4$.

These statements can be corroborated by an explicit solution of the Schrödinger equation, as we now show.

4. - THE $O(2,1)$ SCHRÖDINGER EQUATION

In order to write down and solve the Schrödinger equation we must implement the algebra (11) in a Hilbert space of wave functions $\psi(K, K^*)$. Such a Hilbert space consists of the functions which are normalizable or δ -function normalizable with respect to the invariant inner-product

$$(\psi, \varphi) = \frac{1}{2ig} \text{Tr} \int \frac{dK dK^*}{|K| < 1} \frac{1}{(1-K^*K)^2} \psi^* \varphi , \quad (23)$$

where

$$\frac{1}{2ig} dK dK^* = d\varphi_1 d\varphi_2,$$

and Tr stands for a summation over fermionic variables (see below).

The following representations of π , π^* are Hermitian with respect to this inner-product:

$$\pi = -i(1-K^*K) \frac{\partial}{\partial K} \frac{1}{(1-K^*K)}, \quad \pi^* = -i(1-K^*K) \frac{\partial}{\partial K^*} \frac{1}{(1-K^*K)}. \quad (24)$$

Next we have to introduce a representation for the fermion operators γ , γ^* . There are two possibilities¹¹⁾. We can use a 4×4 dimensional matrix representation, such as

$$\gamma_1 = i\sqrt{2} \begin{pmatrix} \ominus & \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & \ominus \end{pmatrix}, \quad \gamma_2 = i\sqrt{2} \begin{pmatrix} \ominus & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix} & \ominus \end{pmatrix}. \quad (25)$$

Then the wave functions are four-component column vectors, the components of which can be labelled by the eigenvalues of Σ :

$$\Sigma = \frac{1}{4} [\gamma_\alpha, \gamma_\alpha^*] = \begin{pmatrix} 1 & & & \ominus \\ & -1 & & \\ & & 0 & \\ \ominus & & & 0 \end{pmatrix}. \quad (26)$$

Another possibility is to represent $\gamma_{1,2}$ by anticommuting c numbers $\sqrt{2}\zeta_{1,2}$ and define

$$\gamma_{1,2}^* = \sqrt{2} \frac{\delta}{\delta \zeta_{1,2}}. \quad (27)$$

In this case the wave function is an element of the Grassmann algebra:

$$\psi = c_0 + c_1 \zeta_1 + c_2 \zeta_2 + c_3 \zeta_1 \zeta_2, \quad (28)$$

and

$$\frac{1}{4} [\gamma_\alpha, \gamma_\alpha^*] \psi = -c_0 + c_3 \zeta_1 \zeta_2.$$

Thus c_0 represents the component with $\alpha = -1$, c_3 with $\alpha = +1$, and $c_{1,2}$ have $\alpha = 0$. Below we choose to work with the matrix representation (25). Also by convention we call the components c_0, c_3 bosonic and c_1, c_2 fermionic.

It is straightforward to construct the supercharges Q_α , the Hamiltonian and q_0 . The results are:

$$\begin{aligned} \frac{1}{\sqrt{2}} Q_1 &= \begin{pmatrix} \ominus & & & -(1-K^*K) \frac{\partial}{\partial K} \\ & -(1-K^*K) \frac{\partial}{\partial K^*} & & \\ & (1-K^*K)^2 \frac{\partial}{\partial K} \frac{1}{(1-K^*K)} & & \\ (1-K^*K)^2 \frac{\partial}{\partial K^*} \frac{1}{(1-K^*K)} & & \ominus & \end{pmatrix} \\ \frac{1}{\sqrt{2}} Q_2 &= \begin{pmatrix} \ominus & & -(1-K^*K) \frac{\partial}{\partial K} & 0 \\ & 0 & (1-K^*K) \frac{\partial}{\partial K^*} & 0 \\ (1-K^*K)^2 \frac{\partial}{\partial K^*} \frac{1}{(1-K^*K)} & 0 & & \ominus \\ 0 & -(1-K^*K)^2 \frac{\partial}{\partial K} \frac{1}{(1-K^*K)} & & \end{pmatrix} \quad (29) \\ H &= \frac{1}{2} Q_1^2 = \frac{1}{2} Q_2^2 = \begin{pmatrix} H_1 & & \ominus & \\ & H_{-1} & & \\ \ominus & & H_0 & \\ & & & H_0 \end{pmatrix}, \end{aligned}$$

where

$$H_\alpha = -(1-K^*K) \frac{\partial^2}{\partial K^* \partial K} + \alpha (1-K^*K) \left(K^* \frac{\partial}{\partial K^*} - K \frac{\partial}{\partial K} - \alpha \right), \quad (30)$$

and similarly

$$q_{0,\alpha} = K \frac{\partial}{\partial K} - K^* \frac{\partial}{\partial K^*} + \alpha. \quad (31)$$

Using this representation it is easy to find all zero-energy solutions. One has only to solve the first order differential equations

$$Q_1 \psi = Q_2 \psi = 0, \quad (32)$$

and

$$q_0 \psi = m \psi \quad (33)$$

The only solutions are

$$\psi = \begin{pmatrix} \psi_1^{(m)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 \\ \psi_{-1}^{(m)} \\ 0 \\ 0 \end{pmatrix},$$

with

$$\psi_1^{(m)} = (1 - K^* K) K^{m-1}, \quad m \geq 1,$$

and

$$\psi_{-1}^{(m)} = (1 - K^* K) K^{*-m-1}, \quad m \leq -1.$$

(34)

These solutions form an orthogonal set:

$$(\psi_\alpha^{(m)}, \psi_{\alpha'}^{(m')}) = \frac{\pi}{|m|} \delta_{\alpha\alpha'} \delta_{mm'}. \quad (35)$$

Because all these solutions are bosonic [corresponding to c_0 and c_3 in (28)], Witten's index⁸⁾ is:

$$\Delta = (n_B - n_F)_{\epsilon=0} = \infty, \quad (36)$$

proving that supersymmetry is unbroken in the corresponding field theories, the $O(2,1)$ σ -models in higher space-time dimensions. On the other hand, the internal $SO(2,1)$ symmetry is completely broken, the various zero-energy states being connected by the raising and lowering operators $N_\pm$.

The solutions inside the unit disc are sketched in Fig. 1. They are to be projected onto the hyperboloid, moving up higher as m increases. It remains to find the solutions with non-zero energy. For this it is actually more convenient to change to co-ordinates on the hyperboloid, rather than its projection on the unit disc. The hyperboloid

$$p_1^2 + p_2^2 - p_3^2 = -1$$

can be parametrized as

$$\begin{aligned} p_1 &= \cos \varphi \sinh \vartheta, & 0 \leq \varphi < 2\pi, \\ p_2 &= \sin \varphi \sinh \vartheta, & 0 \leq \vartheta < \infty, \\ p_3 &= \cosh \vartheta. \end{aligned} \quad (37)$$

The projection of the point (p_1, p_2, p_3) , with co-ordinates $(x, y, 0)$ in the unit disc (Fig. 2) is:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi,$$

with

$$\rho = \frac{\sinh \theta}{1 + \cosh \theta}.$$

Using these co-ordinates we get

$$\frac{1}{2ig} \frac{dk dk^*}{(1 - k^* k)^2} = d\varphi d\theta \sinh \theta \quad (38)$$

and

$$H_\alpha = - \left(\frac{1}{\sinh \theta} \frac{\partial}{\partial \theta} \sinh \theta \frac{\partial}{\partial \theta} + \frac{1}{(\sinh \theta)^2} \frac{\partial^2}{\partial \varphi^2} \right) + \frac{2\alpha}{1 + \cosh \theta} \left(i \frac{\partial}{\partial \varphi} - \alpha \right),$$

$$q_{0,\alpha} = -i \frac{\partial}{\partial \varphi} + \alpha. \quad (39)$$

The Schrödinger equation

$$H_\alpha \psi = \epsilon \psi$$

has solutions

$$\psi_{m,\alpha}(\varphi, \theta) = e^{-i(m+\alpha)\varphi} Y_m(\cosh \theta), \quad (40)$$

where $Y_m(\xi)$ is a solution of the generalized Legendre equation:

$$\left[(\xi^2 - 1) \frac{\partial^2}{\partial \xi^2} + 2\xi \frac{\partial}{\partial \xi} - j(j-1) - \frac{(m+\alpha)^2}{\xi^2 - 1} - \frac{2\alpha m}{\xi + 1} \right] Y_m(\xi) = 0, \quad (41)$$

defined for $\xi \geq 1$. The energy $-j(j-1)$ is real and positive if

- 1) $0 < j < 1$, j real; in this case $0 < \epsilon \leq 1/4$, but the solutions to (37) are not normalizable;
- 2) $j = 1/2 + i\lambda$; in this case $\epsilon = 1/4 + \lambda^2$, the solutions to (41) form a continuum and are δ -function normalizable¹⁰⁾.

Hence the normalizable solutions (40) correspond to the continuum $\epsilon \geq 1/4$ and allow all values $m = 0, \pm 1, \pm 2, \dots$. A representation of these solutions in terms of a double integral is possible. Namely, the solution with $m = 0, \alpha = 0$ can be written

$$Y_0(\xi) = \int_0^\infty ds \int_0^\infty dt e^{-\xi st - \frac{1}{2}(s^2+t^2)} s^{-\frac{1}{2}+i\lambda} t^{-\frac{1}{2}-i\lambda} \quad (42)$$

Solutions with $m \neq 0$ are then obtained by applying the step operators $N_\pm$, leading to

$$Y_m(\xi) = (\xi^2 - 1)^{\frac{m}{2}} \frac{\partial^m}{\partial \xi^m} Y_0(\xi),$$

and solutions with $\alpha \neq 0$ by applying a supersymmetry transformation:

$$\psi_{m,\alpha} = e^{-i(m+\alpha)\varphi} \left(\sqrt{\xi^2 - 1} \frac{\partial}{\partial \xi} + \frac{m\alpha}{\sqrt{\xi^2 - 1}} \right) Y_m(\xi), \alpha = \pm 1.$$

We have found that the spectrum of the $O(2,1)$ Schrödinger equation consists of a discrete value 0, plus a continuum $1/4 + \lambda^2$, $0 \leq \lambda < \infty$. This result can easily be understood from the point of view of the completeness properties of $O(2,1)$ or $SL(2R)$ representations¹²⁾ as stated in terms of the solutions of (41). The continuum corresponds to the contributions of principal series unitary representations, whereas the discrete point corresponds to the occurrence in our problem of a single discrete series representation. That only one such occurs stems from the fact that $\alpha = 1, -1, 0$ only arise in the solutions (40) of (41). This completes the analysis of the non-compact model.

5. - THE $O(3)$ MODEL

The quantum mechanics of the $O(3)$ model is very similar to the $O(2,1)$ case and we can be brief about it here. The Hamiltonian is

$$\begin{aligned} H &= q_1^2 + q_2^2 + q_3^2 = \frac{1}{4} \{ Q_\alpha, Q_\alpha \} = \\ &= (1 + K^* K) \pi^* \pi (1 + K^* K) + \frac{i}{4} (1 + K^* K) (K^* \pi^* - K \pi) [\gamma_\alpha, \gamma_\alpha^*] \\ &\quad + \frac{1}{16} (1 + K^* K) [\gamma_\alpha, \gamma_\alpha^*]^2, \end{aligned} \quad (43)$$

with

$$\begin{aligned} q_1 &= \frac{i}{2} (\pi - K \pi K) + \frac{1}{8} K [\gamma_\alpha, \gamma_\alpha^*] + h.c., \\ q_2 &= -\frac{1}{2} (\pi + K \pi K) - \frac{i}{8} K [\gamma_\alpha, \gamma_\alpha^*] + h.c., \\ q_3 &= i (K \pi - K^* \pi^*) - \frac{1}{4} [\gamma_\alpha, \gamma_\alpha^*]. \end{aligned} \quad (44)$$

They generate the $O(3)$ algebra

$$[q_i, q_j] = i \epsilon_{ijk} q_k . \quad (45)$$

The Hilbert space consists of the wave functions $\psi(K, K^*)$, normalizable with respect to the inner product

$$(\psi, \varphi) = \frac{1}{2ig} \text{Tr} \int \frac{dK dK^*}{(1+K^*K)^2} \psi^* \varphi , \quad (46)$$

where the integration is now to be done over the whole complex plane. The Hermitian form of π , π^* is

$$\pi = -i(1+K^*K) \frac{\partial}{\partial K} \frac{1}{(1+K^*K)} \quad \pi^* = -i(1+K^*K) \frac{\partial}{\partial K^*} \frac{1}{(1+K^*K)} , \quad (47)$$

while the representation of the fermions can be taken to be identical to (25). The analogue of the Hamiltonian (30) reads

$$H_\alpha = -(1+K^*K)^2 \frac{\partial^2}{\partial K^* \partial K} + \alpha(1+K^*K) \left(K^* \frac{\partial}{\partial K^*} - K \frac{\partial}{\partial K} + \alpha \right) , \quad (48)$$

while the supercharges are

$$\frac{1}{\sqrt{2}} Q_1 = \begin{pmatrix} \ominus & & & -(1+K^*K) \frac{\partial}{\partial K^*} \\ & \ominus & & -(1+K^*K) \frac{\partial}{\partial K} \\ & & (1+K^*K)^2 \frac{\partial}{\partial K} \frac{1}{(1+K^*K)} & \\ (1+K^*K)^2 \frac{\partial}{\partial K} \frac{1}{(1+K^*K)} & & & \ominus \end{pmatrix} , \quad (49)$$

$$\frac{1}{\sqrt{2}} Q_2 = \begin{pmatrix} \ominus & & & -(1+K^*K) \frac{\partial}{\partial K^*} \\ & \ominus & & \circ \\ & & \circ & (1+K^*K) \frac{\partial}{\partial K} \\ (1+K^*K)^2 \frac{\partial}{\partial K} \frac{1}{(1+K^*K)} & & \circ & \\ \circ & & -(1+K^*K)^2 \frac{\partial}{\partial K^*} \frac{1}{(1+K^*K)} & \ominus \end{pmatrix} .$$

In contrast to the non-compact case, the bosonic zero energy solutions $(1 + K^*K)K^m$, $(1 + K^*K)K^{*m}$ are not normalizable and must be rejected. However, there are two normalizable singlet zero-energy fermionic solutions

$$\psi = \begin{pmatrix} 0 \\ 0 \\ c_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_1 \end{pmatrix}, \quad (50)$$

with $c_{1,2}$ constants.

The non-zero energy solutions are again more conveniently obtained in spherical co-ordinates, instead of their projections. We parametrize the unit sphere as usual by co-ordinates

$$\begin{aligned} p_1 &= \cos\varphi \sin\vartheta, & 0 \leq \varphi < 2\pi, \\ p_2 &= \sin\varphi \sin\vartheta, & 0 \leq \vartheta \leq \pi, \\ p_3 &= \cos\vartheta. \end{aligned} \quad (51)$$

Its projection from the south pole into the equatorial plane (Fig. 3) is

$$\begin{aligned} x &= \rho \cos\varphi, \quad y = \rho \sin\varphi \\ \rho &= \frac{\sin\vartheta}{1 + \cos\vartheta}. \end{aligned} \quad (52)$$

This leads to

$$\frac{1}{2ig} \frac{dK^* dK}{(1 + K^*K)^2} = d\varphi d\vartheta \sin\vartheta \quad (53)$$

and the Hamiltonian

$$H_\alpha = - \left(\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \sin\vartheta \frac{\partial}{\partial\vartheta} \right) - \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} + \frac{2\alpha}{1 + \cos\vartheta} \left(i \frac{\partial}{\partial\varphi} + \alpha \right). \quad (54)$$

The solutions of the Schrödinger equation are of the form

$$\psi_{m,\alpha}(\vartheta, \varphi) = e^{-i(m-\alpha)\varphi} F_{m,\alpha}(\cos\vartheta), \quad (55)$$

where $F_{m,\alpha}(\cos\vartheta)$ obeys

$$\left[-\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{m^2 + \alpha^2 - 2m\alpha \cos\theta}{\sin^2\theta} \right] F_{m,\alpha}(\cos\theta) = \epsilon F_{m,\alpha}(\cos\theta) \quad (56)$$

For $\alpha = 0$, the solutions are the standard spherical harmonics representing energy eigenvalues

$$\epsilon = j(j+1), \quad j = 0, 1, 2, \dots, \quad (57)$$

while $-j \leq m \leq j$. In particular for $j = 0$, we have the zero energy state (46). One way to obtain the $\alpha \neq 0$ solutions of (54) is by performing supersymmetry transformations on the $\alpha = 0$ solutions. Of course, (54) is well known in the quantum theory of angular momentum. Its solutions are the Wigner d -functions¹³⁾, i.e.,

$$F_{m,\alpha}(\cos\theta) = d_{m\alpha}^j(\theta)$$

so that $\epsilon = j(j+1)$, independent of the allowed m, α values in the range $-j \leq m, \alpha \leq j$. Here $\alpha = 1, -1, 0$ only occur. Also all the energy levels are discrete and described by normalizable wave functions forming finite $(2j+1)$ dimensional $O(3)$ multiplets.

From the above results it follows that in this case Witten's index has the value $\Delta = -2$. This value is in agreement with the elegant derivation in Ref. 8). It may be inferred that supersymmetry is unbroken in the corresponding field theories.

6. - SUMMARY

In this paper we have analyzed the quantum mechanical $O(2,1)$ and $O(3)$ supersymmetric σ -models. By explicitly solving their Schrödinger equation we have calculated the spectra for the models.

For the non-compact case there is a discrete, infinite dimensional multiplet of zero-energy states. The very existence of these states shows that the $O(2,1)$ symmetry is completely broken, the zero-energy states being connected by raising and lowering operators. Not even the $O(2)$ subgroup remains. By definition, the zero-energy states here are bosonic. It follows that the Witten index⁸⁾ is infinite, showing that supersymmetry is unbroken in the model and, hence, in the corresponding field theory. In addition to the zero-energy states there is a continuum of bosonic and fermionic states starting at an energy eigenvalue of $1/4$.

In contrast to the situation for $O(2,1)$ the spectrum for the $O(3)$ model is discrete starting at zero, each level corresponding to a finite dimensional representation of $O(3)$. At zero energy there are two singlet fermionic states. The existence of these zero-energy states shows the Witten index to be -2 , in agreement with Ref. 8). Again, it follows that supersymmetry is unbroken.

The results for the supersymmetric $O(2,1)$ model can be contrasted with those obtained for the ordinary $O(3,1)$ model of Ref. 7). In Ref. 7) only a continuous spectrum was found. Surprisingly, the inclusion of supersymmetry gives rise to a new, discrete energy level of infinite multiplicity at zero energy.

Our results could have implications for attempts at obtaining a unified field theory from supergravity¹⁴⁾. The scenario suggested in Ref. 14) for obtaining the $SU(5)$ grand unified theory from $N = 8$ supergravity required that the non-compact E_7 symmetry be broken to the maximally compact subgroup, $SU(8)$. The infinite dimensional unitary representations of E_7 would then be pushed to the Planck mass. However, our quantum mechanical $O(2,1)$ example shows that, whilst one does obtain infinite dimensional representations of $O(2,1)$, even the $O(2)$ subgroup is broken for the zero-energy states.

REFERENCES

- 1) For a review see B.W. Lee, Chiral Dynamics, [Gordon and Breach, New York, (1972)].
P. Di Vecchia, Proc. Schlading Inst., Acta Physica Austriaca Suppl. XXII (1980) 341.
- 2) W.A. Bardeen, B.W. Lee and R.E. Shrock, Phys. Rev. D14 (1976) 985;
A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B146 (1978) 63 and Nucl. Phys. B152 (1979) 125;
E. Witten, Nucl. Phys. B149 (1979) 285;
P. Di Vecchia, Phys. Lett. 85B (1979) 357;
A. D'Adda, A.C. Davis and P. Di Vecchia, Nucl. Phys. B198 (1982) 209.
- 3) P. Di Vecchia and S. Ferrara, Nucl. Phys. B130 (1977) 93;
E. Witten, Phys. Rev. D16 (1977) 2991;
O. Alvarez, Phys. Rev. D17 (1978) 1123.
- 4) A.C. Davis, P. Salomonson and J.W. van Holten, Phys. Lett. 113B (1982) 472 and CERN preprint TH.3268 (1982), (to be published in Nucl. Phys. B).
- 5) T.L. Curtright and D.Z. Freedman, Phys. Lett. 90B (1980) 71;
L. Alvarez-Gaumé and D.Z. Freedman, Phys. Lett. 94B (1980) 171 and Phys. Rev. D22 (1980) 846;
M. Roček and P.K. Townsend, Phys. Lett. 96B (1980) 71.
- 6) E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141;
B. Julia and F. Luciani, Phys. Lett. 90B (1980) 270;
M.K. Gaillard and B. Zumino, Nucl. Phys. B193 (1981) 221.
- 7) G. Velo and J. Wess, Nuovo Cimento 1 (1971) 177.
- 8) E. Witten, Nucl. Phys. B202 (1982) 253.
- 9) P.A.M. Dirac, Lectures on Quantum Mechanics, Beller Graduate School of Science, Monograph No. 2, New York (1964);
P. Casalbuoni, Nuovo Cimento 33A (1976) 115 and 34A (1976) 389.
- 10) V. Bargmann, Ann. Math. 48 (1947) 568;
A.O. Barut and C. Fronsdal, Proc. Roy. Soc. 287A (1965) 532;
W.J. Holman and L.C. Biedenharn, Ann. Phys. 39 (1966) 1.
- 11) P. Salomonson and J.W. van Holten, Nucl. Phys. B196 (1981) 509.
- 12) V. Bargmann, op. cit.;
M. Andrews and J. Gunson, J. Math. Phys. 5 (1964) 1391, especially Section 14.
- 13) A.R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton, NJ (1960) p.65.
- 14) J. Ellis, M.K. Gaillard, L. Maiani and B. Zumino, Proc. Europhysics Study Conference on Unification of the Fundamental Interactions, Eds. S. Ferrara, J. Ellis and P. van Nieuwenhuizen (Plenum Press, N.Y., 1980);
J. Ellis, M.K. Gaillard and B. Zumino, Phys. Lett. 94B (1980) 343.

Fig.1

Fig.2

Fig.3